Long-time behaviour of trajectories for dynamical systems with random forces: non-Markov case

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In my lecture I will present some results, recently obtained jointly with Armen Shirikyan. They deal with discrete time or continuous time dynamical systems with random force for unknown $\{u_k\}$ or $u(t)$ in some phase space H:

$$u_k = S(u_{k-1}, \eta_k^\omega), \quad k = 1, 2, \dots, \qquad u_0 = v; \tag{1}$$

or

$$\dot{u}(t) = F(u(t), \eta^\omega(t)) \quad t \geq 0, \qquad u(0) = v. \tag{2}$$

Here $\omega \in (\Omega, \mathcal{F}, \mathbf{P})$, and $\{\eta_k^\omega, k \in \mathbb{Z}\}$ and $\{\eta^\omega(t), t \geq 0\}$ are stationary in time bounded random processes, valued in some other space E; so $S : H \times E \to H$. The phase space H may be of infinite dimension. Thus equation (2) may be a nonlinear PDE, perturbed by a random force. Results for equations (1) and (2) are similar, and I will talk only about the discrete time case (1), where the statements are easier. Moreover, I will restrict myself to the case when spaces H and E are finite-dimensional:

$$H = \mathbb{R}^N, \quad E = \mathbb{R}^M, \quad M \geq N.$$

Then due to the boundedness, $\eta_k^\omega \in Y$ for all k, ω, for some compact subset Y of E. We assume that system (1) is dissipative in the sense that there exists a compact set $X \Subset H$ such that if $v \in X$, then $u_k \in X$ for all k and ω. Moreover, for any $v \in H$ there is $k(v)$ such that $u_k \in X$ for $k \geq k(v)$ and all ω. Our goal is to show that under certain restrictions system (1) is mixing. That is, in H there is a measure μ such that for any initial data v, $\mathcal{D}(u_k) \rightharpoonup \mu$ as $k \to \infty$ (here and below $\mathcal{D}$ signifies distribution and $\rightharpoonup$ stands for the weak convergence of measures).

The mixing in systems (1) and (2) was examined in a lot of works. In vast majority of them for discrete-time systems (1) it was assumed that the random variables $\eta_1, \eta_2, \dots$ are independent (and identically distributed). While for continuous-time systems (2) it was assumed that the random process $\eta(t)$ is a white noise, so (2) is a stochastic differential equation, or a stochastic PDE. The reason is that in this case solutions $\{u_k\}$ for systems (1) and solutions $u(t)$ for systems (2) are Markov processes in H, so the whole huge arsenal of Markov tools applies to study them. There are just a few works on mixing in systems (1) and (2) where processes η are not independent at different moments of time. There it is assumed that the processes are Gaussian, and the maps S and F which define these systems are very special. The goal of the research which I am presenting is to drop the independence assumption for a large class of systems (1), without assuming the Gaussianity. To state the result I impose some restrictions on the random force and on the equation. Concerning the random process η we assume that:

(RP 1) (mixing). The process $\{\eta_k\}$ is strongly exponentially mixing. (Roughly, it means that in some sense "correlations" of random variables η_k^ω and η_{k+t}^ω decays exponentially as $t \to \infty$.)

(RP 2) (Lipschitz regularity). For $l \in \mathbb{Z}$ denote by $\vec{\eta}_l$ the vector of the past of process $\{\eta_k\}$, $\vec{\eta}_l = (\dots, \eta_{l-1}, \eta_l) \in Y^\infty \subset E^\infty$. For a vector $\vec{\xi} \in Y^\infty$ consider the conditional distribution $Q(\vec{\xi})(\cdot)$ of random variable η_1^ω, given that the past $\vec{\eta}_0$ equals $\vec{\xi}$. This is a measure on the space $E = \mathbb{R}^M$, depending on $\vec{\xi}$. Assume that it has a density: $Q(\vec{\xi})(\cdot) = p_{\vec{\xi}}(x)dx$, where $\operatorname{supp} p_{\vec{\xi}}(\cdot) \subset Y$ and p is a Lipschitz function of $\vec{\xi}$ and x.

(RP 3) (recursiveness to the origin). For any given number $n \in \mathbb{N}$ and any given past $\vec{\xi} \in Y^\infty$, process $\{\eta_k, k \geq 1\}$ conditioned to the past $\vec{\eta}_0 = \vec{\xi}$, with a positive probability will sooner or later return to the small vicinity of the origin and will stay there time n.

Concerning the C^2-mapping S, apart from the dissipativity, we assume that:

(S 1) (linearised controllability). For any $u \in X \Subset H$ and $\eta \in Y \Subset E$

$$\text{linear mapping} \quad D_\eta S(u, \eta) : E \to H \quad \text{has dense image.}$$

(S 2) (dissipativity to the origin). $S(0, 0) = 0$, and if in (1) $\eta_k = 0$ for all k, then for any initial data $v \in H$ we have $u_k \to 0$ as $k \to \infty$.

Theorem 1. Suppose that hypothesis (RP 1)-(RP 3) and (S 1)-(S 2) hold. Then system (1) is exponentially mixing: in H exists a measure μ such that for any initial data $v \in H$, $\mathcal{D}u_k \rightharpoonup \mu$ exponentially fast.

The prove the result we first extend system (1) to a Markov process for vectors $U_k = (u_k, \vec{\eta}_k)$ and next prove the mixing for the extended system, using the Doeblin coupling and the method of Kantorovich functional, suggested in [KS12, Section 3.1.1]. The proof uses the

Newton method of quadratic convergence in a form, similar to that used by A.N.Kolmogorov to prove his celebrated theorem which originated the KAM theory. Cf. paper [KNS20] where a similar approach is used to handle related stochastic systems, also see [Kuk18, Section 5] for an informal presentation of the results in [KNS20].

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[KNS20] S. Kuksin, V. Nersesyan, and A. Shirikyan, *Exponential mixing for a class of dissipative pdes with bounded degenerate noise*, GAFA **30** (2020), pp. 126–187.

[KS12] S. Kuksin and A. Shirikyan, *Mathematics of two-dimensional turbulence*, CUP, 2012.

[Kuk18] S. Kuksin, *Ergodicity, mixing and KAM*, Séminaire Laurent Schwartz EDP et applications, Exp. No. 8, 2018, p. 9, DOI: 10.5802/slsedp.128.