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Two Perspectives on Recursion and Induction: Modal Provability Logic and Fixed-Point Logics

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I

General theme:

Compare two logical approaches to Induction and Recursion, exemplified by: Provability Logic, Modal Mu-Calculus

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Two Approaches to induction and Recursion

Beyond ¬ ∧ ∨ ∃ ∀

Induction and recursion are general logical operations showing the interplay of definability and proof

Approach 1: models with well-founded orders. Define notions by recursion as in set theory: rank(x) = (rank(y) | y ∈ x}

Approach 2: fixed-point operators on arbitrary models. Like in μ–calculus, background: Tarski-Knaster theorem.

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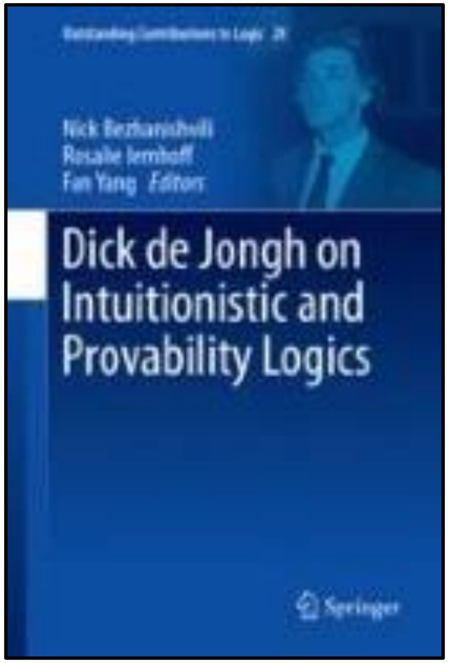

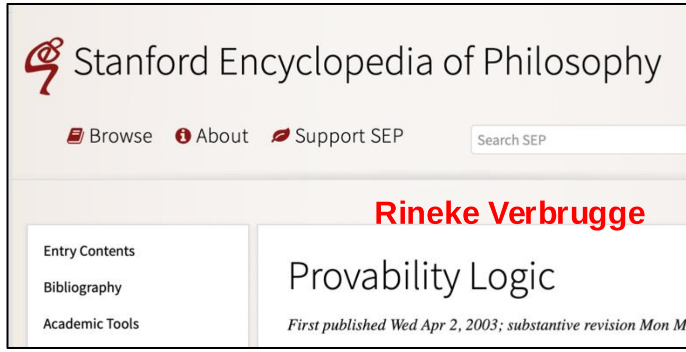

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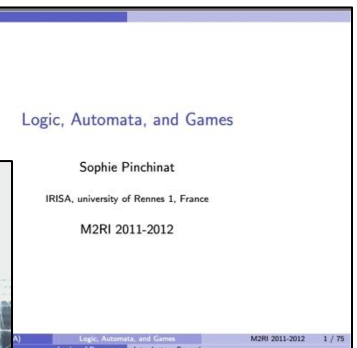

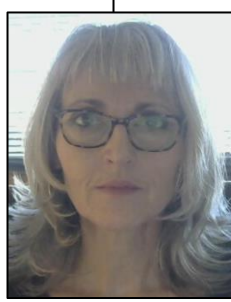

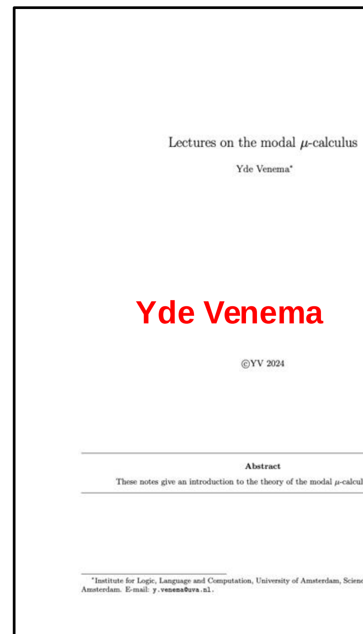

Yde Venema

Two Kinds of Logics for Induction and Recursion

Johan van Benthem, 31 March 2024, ILLC Amsterdam

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Topics for Today

Getting maximal generality: What makes the Fixed-Point Theorem in Provability Logic tick?

Can the two approaches merge:

What are the connections between GL and μ–calculus?

Are the two approaches the same in some sense?

Case study: first-order fixed-point logic μFO.

II

Basic Provability Logic and the Fixed-Point Theorem

Gödel-Löb Provability Logic

Minimal modal logic **K** plus

$\Box(\Box\varphi \to \varphi) \to \Box\varphi$ **Löb's Axiom**

Thm GL is complete for validity on

upward well-founded transitive orders

Correspondence fact. Löb's Axiom is true in a frame (W, R)

iff R is (i) transitive, (ii) upward well-founded (is in μFO)

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Excursion: GL Viewed in General Modal Logic

Fact McKinsey Axiom $\Box\Diamond p \to \Diamond\Box p$ is not in μFO (LÖWS failure).

Löb's Axiom in μFO because of the **PIA syntax** of its **p**-antecedent: "**positive formula implies atom**".

If true, such formulas hold for a minimal set value of **p**.

Generalized Sahlqvist. Even works for $[1]([2]p \to p) \to [3]p$.

September 2005

Minimal predicates, fixed-points, and definability

Johan van Benthem

J. Symbolic Logic 70(3): 696-712 (September 2005). DOI: 10.2178/jsl/1122038910

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The Classical dJS Fixed Point Theorem

Call a modal formula $\varphi = \varphi(p, \mathbf{q})$ *p-modalized* if all occurrences of the proposition letter p in φ are in the scope of at least one modal operator $\Box$, while there is no syntactic constraint on the occurrences of the proposition letters in the sequence $\mathbf{q}$. We state the following result in its semantic version; a proof-theoretic version can be stated using the completeness theorem for Gödel-Löb logic.

Theorem 1 (Fixed-Point Theorem, De Jongh-Sambin-). *For any p-mod ed formula $\varphi(p, \mathbf{q})$, there exists a formula $\alpha = \alpha(\mathbf{q})$ such that $\varphi(\alpha(\mathbf{q}), \mathbf{q}) \leftrightarrow \alpha(\mathbf{q})$ is valid. This fixed-point formula α can be found by an effective procedure. Moreover, the fixed-point is unique up to logical equivalence.*

What are the general features here?

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Examples, Solutions in Base Language

Equation:		Solution:	
	$p \leftrightarrow \Box p$		$p = T$
	$p \leftrightarrow \neg\Box p$		$p = \neg\Box\bot$
	$p \leftrightarrow (\Box p \rightarrow q)$		$p = (\Box q \rightarrow q)$

Striking: Fixed-points inside the base language

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III

Generalizing the Fixed-Point Theorem

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Generalized Quantifier Perspective

A parametrized generalized quantifier (henceforth called a 'quantifier', for short) is a relation between points and sets of points. In any model $\mathbf{M}$, each modal formula $\varphi(p)$ defines a modal generalized quantifier $Q_{\varphi,p}$ as follows:

$$Q_{\varphi,p}sX \text{ iff } \mathbf{M}[p := X], s \models \varphi(p)$$

Modal formulas are invariant for **generated submodels**

Modal quantifiers **QsX** only depend on the intersection of X with {s} U R_s

When p is modalized the dependence is only on R_s *R transitive*

- Quantifiers with the latter property are **future-oriented**

- A quantifier Q is **persistent** if QsX implies QtX for all t with Rst:

  or in modal terms, $Q\varphi \rightarrow \Box Q\varphi$ is valid.

De Jongh's Lemma

Fact *All future-oriented persistent generalized quantifiers Q have the fixed-point $Q\top$, i.e., the equivalence $Q\top \leftrightarrow QQ\top$ is valid on well-founded models.*

Proof. We reason inside models $\mathbf{M}, s$ at some specific point s.

(i) $Q\top \rightarrow QQ\top$. Let $\mathbf{M}, s \models Q\top$. By persistence, we also have $\Box Q\top$, and therefore $\Box(Q\top \leftrightarrow \top)$. Using future-orientation, we get that $\mathbf{M}, s \models QQ\top$.

(ii) $QQ\top \rightarrow Q\top$ is proved using well-foundedness. Assume that $\mathbf{M}, s \models \neg Q\top$, then $\neg Q\top \wedge \Box Q\top$ is true at s or at some t with Rst. Using future-orientation as before, $\neg QQ\top$ is then true at t, and hence it is also true at s: either directly, or using persistence once more. (Note: The contraposition style in this second leg is just for convenience; one can also run it positively via well-founded induction.)

Applies Far Beyond the Basic Modal Language

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Even Further: Just Upward Well-Founded Orders

Drop the transitivity assumption

[Upward] well-foundedness: No infinite ascending R-sequences

Available **Induction Principle**:

each non-empty set X has a maximal element $s \in X$ in the sense that no t with R^*st is in X, where R^* denotes the transitive closure of R.

Future orientation now refers to **generated submodel**

Fact *If Q is future-oriented, then it has a unique fixed-point.*

Aside: What Happens to Provability Logic?

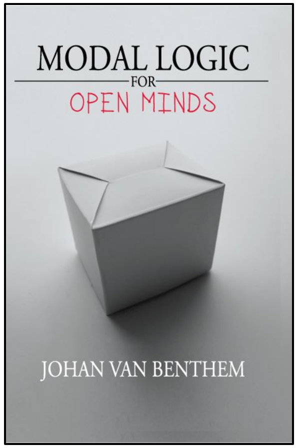
GL now in **PDL** language with transitive closure modality:

Well-foundedness frame-defined by $[*](\Box p \to p) \to [*]p$

Transitivity now automatic. System supports nice formal proofs.

As before: generalized Sahlqvist correspondence into **μFO**

Explained in textbook style in:



Proof of the Fixed Point Fact (recall your set theory)

Proof. (i) Uniqueness follows by a standard well-foundedness argument. Beyond a last point s where two putative fixed-points p, q differ, they coincide on the points reachable from s in the transitive closure R^*. But at that s, by future-orientation, Qp, Qq have the same truth value, and hence so do p, q. (ii) As for existence, the following second-order formula in fact defines the fixed-point:

$$\exists p\,(Qp \wedge [R^*](Qp \leftrightarrow p))$$

Call this formula α: we will prove that $Q\alpha \leftrightarrow \alpha$ by the above form of induction. Assume that $[R^*](Q\alpha \leftrightarrow \alpha)$ is true. We show that $Q\alpha \leftrightarrow \alpha$ is true as well.

(a) Let $Q\alpha$ be true. Then α describes a unary predicate (or set) p such that $Qp \wedge [R^*](Qp \leftrightarrow p)$, and the latter fact means by definition that α holds.

(b) Let α be true. That is, for some p we have that $Qp \wedge [R^*](Qp \leftrightarrow p)$ is true. We also have $[R^*](Q\alpha \leftrightarrow \alpha)$, and so, by unicity of fixed-points, we have $[R^*](\alpha \leftrightarrow p)$. Using future-orientation w.r.t. Qp, we then get $Q\alpha$.

The de Jongh Lemma is a Special Case

Proof. We show that, if the quantifier Q is persistent, then the above formula $\alpha = \exists p\,(Qp \wedge [R^*](Qp \leftrightarrow p))$ is in fact equivalent with $Q\top$.

(i) Suppose that $Q\top$ holds at s, then it holds in the whole generated submodel at s, by repeated applications of persistence. Therefore, $Q\top$ is equivalent to $\top$ in the whole generated submodel, and α is true by taking p to be $\top$.

(ii) Now let α hold at s. Then, for some p, Qp is true, and so by persistence, Qp is equivalent to $\top$ in all reachable successors. Also, we have $[R^*](Qp \leftrightarrow p)$, and so $[R^*](\top \leftrightarrow p)$. By the future-orientation of Qp, $Q\top$ then holds at s.

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A Maybe Not So Interesting Consequence

Fact *The Fixed-Point Theorem holds for second-order modal logic over a well-founded order with arbitrary added future-oriented modalities.*

Unfortunately, second-order logic often trivializes what are significant results for weaker languages

But we can improve to a more interesting result as follows:

Better: Generalized Quantifier Languages

Next, consider a propositional modal language with many different modalities [first-order or higher-order], viewed as generalized quantifiers $Q_1, \ldots, Q_k$ that are all future-oriented in the sense of the preceding section. Fixed-point equations are formulas $p \leftrightarrow \psi(p)$ for possibly complex p-modalized formulas ψ in this language. Each such formula can be viewed as being of the form

$$\varphi[Q_1\alpha_1(p), \ldots, Q_k\alpha_k(p)]$$

where the displayed subformulas list all occurrences of p in the p-free 'skeleton formula' φ. (Incidentally, while such decompositions always exist, they need not be unique.) Clearly, such a formula can also be seen as defining one generalized quantifier. w.r.t. the propositional variable p. Moreover, given the invariance for generated submodels and the future-orientation of all the quantifiers occurring inside the formula, this overall quantifier is future-oriented.

A Very General Fixed Point Theorem

This format applies to many modalities beyond basic ones

Theorem *For any modal language over a well-founded order whose formulas are invariant for generated submodels and whose vocabulary contains arbitrary future-oriented persistent modalities, each fixed-point equation has a unique solution definable inside the language.*

Proof due in essence to Lisa Reidhaar-Olson: may be the most general form of existing proofs for the fixed-point theorems in Provability Logic

Open problem

Multi-modal logics and connecting more well-founded orders AV

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Proof, Part 1

Proof. We follow the proof in (Reidhaar-Olson, 1989), but cutting out the proof-theoretic excursions, and taking a direct route to existence. Moreover, unlike in that proof, we use the earlier abstract solution of Fact 3 at a crucial stage.

Also as before, we reason locally at some arbitrary point s in a model. Consider a formula $\psi(p) = \varphi[Q_1\alpha_1(p), \ldots, Q_k\alpha_k(p)]$ as described above. This yields a unique fixed-point α with a second-order definition as stated earlier so the following formula is true throughout our model:

$$\alpha \leftrightarrow \varphi[Q_1\alpha_1(\alpha), \ldots, Q_k\alpha_k(\alpha)]$$

Here, since combinations $Q_i\alpha_i(p)$ with formulas α_i that are themselves invariant under generated submodels still define future-oriented invariant quantifiers, we can simplify notation in what follows without loss of generality to

$$\psi(p) = \varphi[Q_1p, \ldots, Q_kp]$$

where the unique fixed-point simplifies to:

Note: α is not in our language!

$$\alpha = \varphi[Q_1(\alpha), \ldots, Q_k(\alpha)]$$

Proof, Part 2

Now we prove the definability of this fixed-point inside our language. The base case with $k = 0$ is obvious, as the formula ψ itself is then a fixed-point.

Next, let $k > 0$, and assume that we have the result already for all formulas with fewer than k occurrences of p in the display. Define ψ_i as the formula

$$\varphi[Q_1 p, \ldots, \top, \ldots, Q_k p]$$

where the i-th occurrence $Q_i p$ has been replaced by $\top$. By the inductive assumptions, this has a fixed-point δ_i satisfying

$$\delta_i \leftrightarrow \varphi[Q_1(\delta_i), \ldots, \top, \ldots, Q_k(\delta_i)]$$

Lemma *The following equivalence holds everywhere:* $Q_i(\delta_i) \leftrightarrow Q_i(\alpha)$.

Proof. (i) First assume that $Q_i(\delta_i)$ is true at a point s. By the persistence of Q_i, we have as in earlier arguments that $(Q_i(\delta_i) \leftrightarrow \top) \wedge [R^*](Q_i(\delta_i) \leftrightarrow \top)$ is true at s. By invariance of φ for generated submodels and replacement of equivalents in the generated submodel at s, this implies that we have the following equivalence true throughout this generated submodel:

$$\delta_i \leftrightarrow \varphi[Q_1(\delta_i), \ldots, Q_i(\delta_i), \ldots, Q_k(\delta_i)]$$

Proof, Part 3

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But then δ_i is a fixed-point for the original formula ψ, and as fixed-points are unique, we have $\delta_i \leftrightarrow \alpha$ true in the generated submodel, with α the earlier fixed-point. It then follows from the truth of $Q_i(\delta_i)$ that $Q_i(\alpha)$ is true as well.

(ii) Next let $\neg Q_i(\delta_i)$ be true at a point s. Then by well-foundedness, there is a point t R-reachable from s in 0 or more steps with $\neg Q_i(\delta_i) \wedge [R^*]Q_i(\delta_i)$ true. In this point, we also have $[R^*](Q_i(\delta_i) \leftrightarrow \top)$ true. As before, it follows that

$$[R^*](\delta_i \leftrightarrow \varphi[Q_1(\delta_i), \ldots, Q_i(\delta_i), \ldots, Q_k(\delta_i)])$$

and hence, by uniqueness of fixed-points again, $[R^*](\delta_i \leftrightarrow \alpha)$ is true at t. Then, by the future-orientation of Q_i, $\neg Q_i(\delta_i)$ implies that $\neg Q_i(\alpha)$ is true at t: and, either directly, or using persistence of Q_i as often as necessary on the finite path from s to t, we have that $\neg Q_i(\alpha)$ is true at s.

The rest of the argument is now easy. Using the lemma repeatedly, together with semantic replacement of equivalents, we can replace the successive occurrences of α in $\varphi[Q_1(\alpha), \ldots, Q_k(\alpha)]$ to obtain the equivalence

$$\alpha \leftrightarrow \varphi[Q_1(\delta_1), \ldots, Q_k(\delta_k)]$$

where the formula to the right-hand side is inside our modal language.

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IV

Excursion:

Non-Persistent Modalities

Solving in other modal logics: PDL and MSL

Non-Persistent Modalities and PDL Solutions

Example (de Jongh) The fixed-point formula **p <-> MOST¬p** defines the set of **odd numbers** on the structure **N, >**

Odd positions in N definable using the PDL program s ; (s ; s)*

Fact *Fixed-point equations* $p \leftrightarrow \varphi(p)$ *with p-modalized* $\varphi(p)$ *over finite words in an alphabet* $\{a\}$ *are uniquely solvable with a fixed-point defined in bare* PDL.

This works only in linear 'language models'

More Difficult Example: Solving Games

Zermelo's Theorem Finite-depth two-player zero-sum games are determined. Proof identifies winning positions recursively:

Here $\mathtt{move}$ stands for the union of all moves, $\mathrm{win_i}$ marks winning end positions for player i, and $\mathrm{turn_i}$ marks turns for i:

$$\mathrm{WIN_i} \leftrightarrow$$
$$((\neg\langle\mathtt{move}\rangle\top \wedge \mathrm{win_i}) \vee (\mathrm{turn_i} \wedge \langle\mathtt{move}\rangle\mathrm{WIN_i}) \vee (\mathrm{turn_j} \wedge [\mathtt{move}]\mathrm{WIN_i}))$$

This is a fixed-point equation of the form

$$p \leftrightarrow (q \vee (r \wedge \Diamond p) \vee (s \wedge \Box p))$$

Conjecture Solution not definable in **PDL** with arbitrary tests

New Language Design: Modal Substitution Logic

Still natural fixed points in finite game trees: **keep substituting solutions** at lower levels into higher levels

Iterated substitution of modal formulas for proposition letters

Modal substitution logic. This suggests an extension of the basic modal language to a system MSL with a minimum needed for the preceding to work. We introduce explicit notation for *substitutions* $p := \psi$ that are interpreted semantically as transforming models $\mathbf{M}$ by resetting the valuation for p to the truth set $[[\psi]]^{\mathbf{M}}$.

This is the simple one-step version of the system

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Modal Substitution Logic, Extended

Moreover, we add *regular operations* over these transformations, in particular, iterations $(p := \psi)^*$. Thus, the syntax of MSL is that of PDL, now with formulas plus expressions for model-transforming programs, but it is more powerful.

For instance, the above pattern $(\Diamond\Box)^*\varphi$ is definable as

$$\langle p := \varphi; (p := \Diamond\Box p)^*\rangle\top$$

where p is a fresh proposition letter not occurring in φ. The above fixed-points are all definable in this formalism, using finitary iterations of substitutions.

The language of the system MSL may look unusual, but it is finitary, and it is semantically modal in that its formulas are invariant for bisimulation. One can also show that MSL extends PDL, while, on finite trees, it

seems contained in the modal mu-calculus.

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New sort of finitary syntax for a fragment of infinitary modal logic that can define fixed-points of interest

Recent paper on MSL

ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Annals of Pure and Applied Logic 00 (2024) 1–29

Annals of Pure and Applied Logic

Sujata Ghosh, Dazhu Li, Fenrong Liu, Yaxin Tu

A modal approach towards substitutions

V

Modal Mu-Calculus

Syntax and Semantics

Originally for program/process logics, many later applications

Extend modal logic with operators $\mu p \bullet \varphi(p)$ (smallest fixed-point, **p** only positive in φ), $\nu p \bullet \varphi(p)$ (greatest fixed-point) with respect to the inclusion-monotone approximation map

$$F_\varphi(X) = \{s \in W \mid M[p:=X], s \models \varphi\}$$

These fixed-points exist by the **Tarski-Knaster Theorem**.

Extends **PDL**, $\mu p \bullet \Box p$ defines well-foundedness, much more.

Complete axiomatization: minimal modal logic plus

(i) Pre-fixedpoint Axiom, (ii) Smallest fixed-Point Rule

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Knowns and Unknowns

Fact **μML** is the bisimulation-invariant fragment of **MSO**

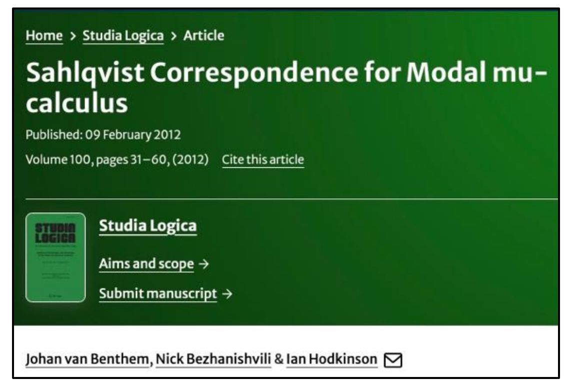
Fast-growing theory connecting **modal logic**, **automata**, **games**

Current lab for interesting new (circular, infinitary) proof theory

Modal correspondence theory

generalizes to the mu-calculus

w.r.t. definability in **μFO**.



Open problem What about classical modal model theory,

for instance, the **Goldblatt-Thomason Theorem**?

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VI

Provability Logic and Mu-Calculus

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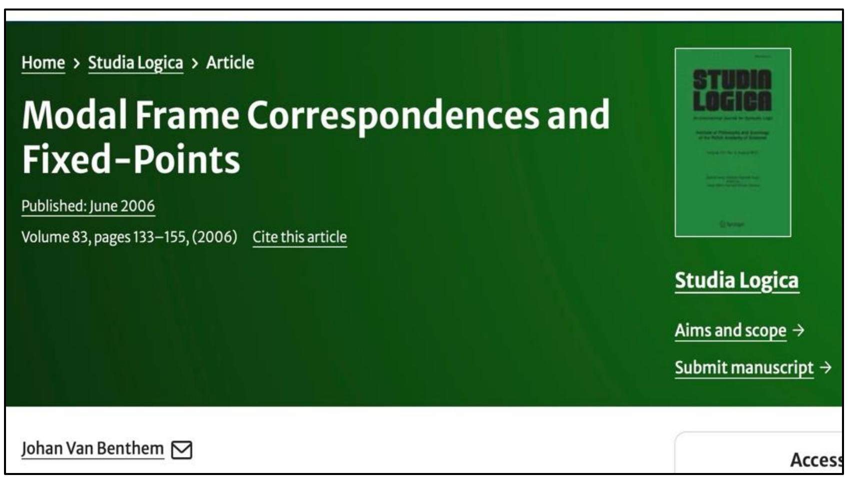
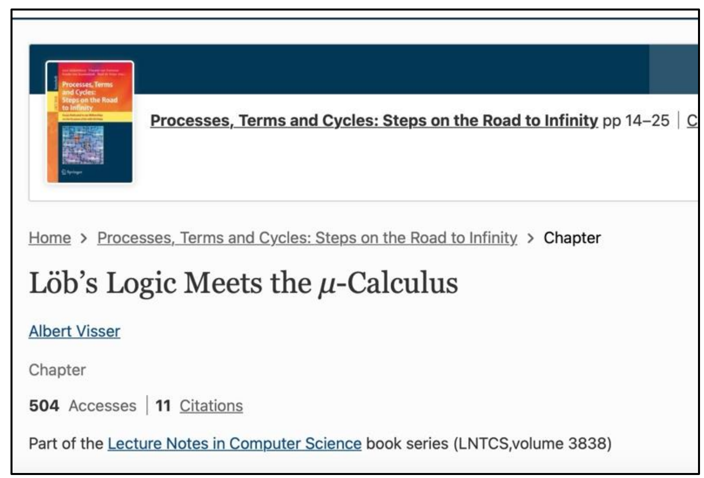
Provability Logic and Modal Mu-Calculus

Fact 1 (JB) There is a faithful validity-preserving embedding of the basic provability logic **GL** into **μML**

Fact 2 (AV) **GL** is a retract of **μML** in a suitable category of logics with abstract interpretation maps.





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Provability Logic and Mu-Calculus

The two areas can profit from being considered in parallel

Just sit down for a moment, and you will see

any number of interesting questions

about generalizing PL and generalizing μML

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VII

The Contrast in a General FO Perspective

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Relating the Two Views of Fixed Points

GL on well-founded orders, boxed syntax for recursion variable,

analogy with general fixed-point theorems in set theory etc.,

solutions inside original language, constructively computable

Mu-Calculus adds descriptions for all fixed-points given by

positive syntax/monotone maps, not just well-founded models

And Yet

The two seem related in deeper ways: ordinal approximation for

the mu-calculus is an essential well-founded structure, etc.

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First-Order Fixed-Point Logic

μFO, the fixed-point extension of first-order logic (since 1970s),

fixed-point operators for predicates of any arity, extends **μML**

First studied by Moschovakis 1974.

Natural system, yet no Lindström characterization known:

Formulas invariant for **potential isomorphisms**

Downward Löwenheim-Skolem theorem holds (Flum)

But we seem to be missing essentials.

Other results: **Zero-One Law** on finite models (Gurevich & Shelah)

Can encode its **inflationary fixed-point** extension (Kreutzer).

Making The Connection in This Setting

Much more complex binding patterns than modal **μ-**calculus:

nested fixed points of any arities, different variables in atoms with

fixed-point predicates, ∃ ∀ quantify into fixed-point subformulas.

Theorem/Claim **μFO** is SAT equivalent with **FO + WFF(R)**,

general first-order logic with one sentence defining

well-foundedness of an arbitrary order R.

Proof inspired by J. Flum, 'On the (Infinite) Model Theory of Fixed-Point Logics'.

X. Caicedo & C. Montenegro, eds., 2021, *Models, Algebras, and Proofs*, 67-75, CRC Press.

Main idea: Coding in First-Order Logic

Abstract two-sorted models: object, 'ordinals'.

Decompose **μFO** formula-layers into **FO skeletons**

Skel(α)(X, Y, Z, y, x) with 'upper' and 'lower' fixed-points.

Replace fixed-point predicates by **atomic predicates**

$X(a_1, z_1, \ldots, a_n, z_n, b, y, s)$ as described on next slide.

Connect these plus their associated fixed-point predicates **X***

by **the standard recursion clauses with ordinals in FO**.

This yields a first-order translation **tr(φ)** for any formula

plus a finite 'set-up formula' **Approx(φ).**

Intended Meaning New Fixed-Point Predicates

$X(a_1, z_1, \ldots, a_n, z_n, b, y, s)$ holds of individual and ordinal objects $\underline{\mathbf{a}}_1, \underline{\mathbf{z}}_1, \ldots, \underline{\mathbf{a}}_n, \underline{\mathbf{z}}_n, \underline{\mathbf{b}}, \underline{\mathbf{y}}, \underline{\mathbf{s}}$ iff the tuple $\underline{\mathbf{s}}$ belongs to the $\underline{\mathbf{b}}$-th approximation of **X** computed with assignments to all free variables of the listed fixed-point predicates encoded by $\underline{\mathbf{z}}_1, \ldots, \underline{\mathbf{z}}_n, \underline{\mathbf{y}}$, with predicate variables $\mathbf{Y}_i$ for the governing fixed-points in the initial formula φ set to their $\underline{\mathbf{a}}_i$-th approximations w.r.t. the relevant part of the listed first-order object assignment $\underline{\mathbf{z}}_1, \ldots, \underline{\mathbf{z}}_n, \underline{\mathbf{y}}$.

Associated smallest fixed-point $\mathbf{X}^*$-predicates: obvious meaning.

Summing Up

Theorem A formula φ of μFO is satisfiable in the standard fixed-point semantics

iff $Tr(\varphi) \wedge Approx(\varphi) \wedge WF(<)$ is satisfiable in $FO + WF(<)$.

There is probably no meaning-preserving translation

from **μFO** into **FO + WFF(R)**.

This opens up many new questions

E.g., returning to our earlier modal formalisms,

can we compress this to smaller fragments of **FO**?

VIII

Conclusion

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Today's main points:

What makes the dJS Theorem tick for GQ languages, and how it generalizes to other languages and models

Suggest profitable combinations GL, PDL and μML [as well as new modal logics such as MSL]

Explore mutual reductions on a first-order logic base

Our analysis was **semantic**. **Is there an illuminating proof-theoretic take on the results presented here?**

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IX

Some Questions and Answers Today

Further comments welcome at:

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Q. The Gödel et al. Fixpoint Lemma has a clear arithmetical content, its modal GL formalization arguably just describes a well-founded order. What is the deeper connection?

A. This has been a vexing question since the 1970s, and no satisfactory answer exists that I know of. Perhaps we can compare the proof of the arithmetical Fixpoint Lemma more in detail with that of the dJS Theorem, and see how appeals to well-foundedness mirror appeals to Gödel encoding?

To me this relates to the following general issue:

Kleene's recursion theorems in Recursion Theory involve a mix of two things: (i) general logical properties of recursion, (ii) the great power of coding (e.g. of Turing machines) in the natural numbers, so that 'programs can also be data'.

Generalized Recursion Theory tried to separate out the special influence of working with IN and see what general logical structure of recursion and induction remains.

This is also what Aczel's chapter does in the Handbook of Mathematical Logic, and so do general fixed-point logics.

Maybe a valid view of what GL does is also in this line?

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Q. Non-transitive frames have also started appearing in PL.

A. Great, and I would be interested in the arithmetical interpretation for this: maybe single accessibility steps in the models now mirror single or partial proof steps?

Another approach might be to look for 'realization theorems' no longer in the natural numbers (like Solovay's), but in set theory where $\in$ is already non-transitive, while generalized forms of coding are present, maybe in the structure $(V_\omega, \in)$?

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Q. What does the μFO to FO + WF(R) reduction say?

A. From right to left there is a meaning-preserving faithful translation, from left to right only a SAT reduction:

φ is satisfiable in a standard model for μFO iff tr(φ) & approx(φ) is satisfiable in a standard model for FO + WF(R).

I also learnt there might already be a similar result where the atom added to FO involves an assertion about games. This might reflect the important role of positional parity games in the semantic study of the modal μ-calculus.

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Q. What is the complexity of μFO?

A. My guess: Π^1_1-complete, but this must be in the literature. Same answer for results about how μFO behaves on just finite models: see the literature on Finite Model Theory.

Finite model restrictions also make sense for the further issues raised today, e.g., generalizing the dJS Theorem.

Can we lower these complexities? E.g., μGF, the fixed-point extension of the Guarded Fragment is decidable. Or could a 'generalized assignment semantics' make μFO decidable?