Quantization of irregular Riemann-Hilbert maps

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Question: solving linear ODE

- linear ODE for a complex function $f(z)$

$$f^{(n)}(z) + a_{n-1}(z) f^{(n-1)}(z) + \cdots + a_0(z) f(z) = 0$$

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Proposition

Any solution $f(z)$ *is a multivalued meromorphic function on* $\mathbb{C}$ *that can only have singularity at* z_0, *where* z_0 *is a pole of some* $a_i(z)$.

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$${}_1F_1(a, b; z) = \begin{cases} \frac{\Gamma(b)}{\Gamma(a)} e^{-z} z^{b-a}, & \text{as } z \to 0 \text{ from left half plane} \\ \frac{\Gamma(b)}{\Gamma(b-a)} (-z)^a, & \text{as } z \to 0 \text{ from left right plane.} \end{cases}$$

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(2) For a small parameter ε, consider

$$\varepsilon f^{(n)} + a_{n-1}(z) f^{(n-1)} + \cdots + a_0(z) f = 0$$

what is the behaviour of $f(z;\varepsilon)$ as $\varepsilon \to 0$?

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- Idea: to "solve" ODE from a moduli viewpoint, via symplectic geometry and its quantization.

Riemann-Hilbert (RH) maps (regular sigularity)

- Liner system of ODEs with d fixed simple poles at $z_1, ..., z_d$, i.e.,

$$\frac{dF}{dz} = \left(\frac{A_1}{z - z_1} + \cdots + \frac{A_d}{z - z_d} \right) \cdot F(z).$$

Here $A_1, ..., A_d \in \mathfrak{gl}_n$. Thus de Rham space is $\mathfrak{gl}_n \times \cdots \times \mathfrak{gl}_n$.

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- Such correspondence is called a RH map

$$\begin{aligned} \mathfrak{gl}_n \times \cdots \times \mathfrak{gl}_n \to \mathrm{Hom}(\pi_1, GL_n) \cong \mathrm{GL}_n \times \cdots \times \mathrm{GL}_n \\ (A_1, ..., A_d) \mapsto (M_1, ..., M_d) \end{aligned}$$

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Theorem (Hitchin)

The RH map is locally analytic Symplectic/Poisson isomorphism.

Irregular RH maps

- Linear system of ODEs with pole of degree $p_i > 1$ at z_i, i.e.,

$$\frac{dF}{dz} = \left(\sum_{i=1}^{d} \sum_{j=1}^{p_i} \frac{A_{i,j}}{(z - z_i)^j} \right) F,$$

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Example (n=2,d=1,p$_1$ = 2)

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- At each z_i, the ODE has internal monodromy data called Stokes matrices $(S_1^{(i)}, \dots, S_{2p_i}^{(i)}) \in (B_- \times B_+)^{p_i}$.

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- The irregular RH map is an analytic map (fix pole degrees and positions)

$$\mathrm{Mat}_n^{p_1} \times \cdots \times \mathrm{Mat}_n^{p_d} \to (B_- \times B_+)^{p_1} \times \cdots \times (B_- \times B_+)^{p_d}$$

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- Known for $n = 2$, $i = 1$, $p_1 = 2$ (Kummer); for $n = 2$, $i = 2$, $p_1 = p_2 = 1$ (Riemann).

Irregular RH maps at pole of order $k+1$

- Consider the ODE for a function $f(z) \in \mathrm{GL}_n$

$$\frac{df}{dz} = \left(\frac{u}{z^{k+1}} + \frac{A_k}{z^k} + \cdots + \frac{A_2}{z^2} + \frac{A_1}{z} \right) \cdot f,$$

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- For fixed u, the moduli space is the $A(t) \in \mathfrak{gl}_n(\mathbb{C}[t]/t^k)$

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Theorem (Boalch)

For fixed regular u, the irregular Riemann-Hilbert map

$$\nu(u) : \mathfrak{gl}_n(\mathbb{C}[t]/t^k) \to (B_- \times B_+)^k \; ; \; A(t) \mapsto (S_1, ..., S_{2k})$$

is a locally analytic Poisson isomorphism.

- The function algebras $Sym(\mathfrak{gl}_n(\mathbb{C}[t]/t^k))$ and $Fun(B_- \times B_+)^k$ are Poisson algebras.

Definition

(1) A Poisson algebra is a commutative algebra A, together with a bilinear map $\{\cdot,\cdot\}\colon A \times A \to A$, called a Poisson bracket, satisfying that for all $f, g, h \in A$,

$$\{fg, h\} = f\{g, h\} + g\{f, h\},$$
$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.$$

Poisson algebras and quantization

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(2) A quantization of $(A, \{,\})$ is an associative algebra product $\star$ on $A[\![\hbar]\!]$ such that

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Note that as $\hbar \to 0$, the (noncommuative) associative algebra $(A[\![\hbar]\!], \star)$ recovers the commutative algebra A.

Quantum RH maps

- Recall: the RH map $\nu(u) : \mathfrak{gl}_n(\mathbb{C}[t]/t^k) \to (B_- \times B_+)^k$ is a map preserving the Poisson algebra structures.

- Problem: to find a quantization of $Fun((B_- \times B_+)^k)$, i.e., an associative algebra $U_\hbar^{(k)}$ and an associative isomorphism $\nu(u)_\hbar$ such that the following diagram commute

$$\begin{array}{ccc} U_\hbar^{(k)} & \xrightarrow{\nu(u)_\hbar} & U(\mathfrak{gl}_n(\mathbb{C}[t]/t^k))[\![\hbar]\!] \\ {\scriptstyle h \to 0}\downarrow & & \downarrow{\scriptstyle h \to 0} \\ Fun((B_- \times B_+)^k) & \xrightarrow{\nu(u)^*} & Sym(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)) \end{array}$$

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- Application: use the representation theory of $U_\hbar^{(k)}$ to understand the highly transcendental irregular RH maps!

Second order pole case

Quantum group and the Stokes phenomenon at second order pole

Stokes matrices of ODEs with second order poles

- Consider the linear system on z-plane

$$\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{A}{z}\right) F,$$

where $F(z) \in \mathfrak{gl}_n$, $u = \mathrm{diag}(u_1, ..., u_n)$, and $A \in \mathfrak{gl}_n$.

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- Any fundamental solution $F(z) \in \mathrm{GL}_n$ has asymptotics

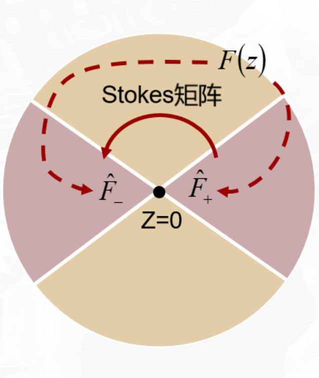
$$e^{\frac{u}{z}} z^{-[A]} \cdot F(z) \sim T_\pm \quad as\ z \to 0\ in\ left/right\ planes\ \mathbb{H}_\pm,$$

for some invertible constant matrices $T_\pm$.

- The different asymptotics of $F(z)$ are measured by the ratio

$$S_+(A, u) = T_+ \cdot T_-^{-1},$$

called Stokes matrix, similarly define $S_-(A, u)$.



Quantum groups

- RH $\nu(u) : \mathfrak{gl}_n \to B_- \times B_+; A \mapsto (S_-(A), S_+(A))$.
- The quantization of the Poisson algebra $Fun(B_- \times B_+)$ is the quantum group (Drinfeld-Jimbo): $U_\hbar(\mathfrak{gl}_n)$ is associative algebra with generators $\{e^{\hbar h_i}, e_i, f_i\}_{i=1,\ldots,n}$

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  - for each $1 \leq i, j \leq n-1$,

$$[e_i, f_j] = \delta_{ij} \frac{e^{\hbar(h_i - h_{i+1})} - e^{\hbar(-h_i + h_{i+1})}}{e^{\hbar} - e^{-\hbar}};$$

  - for $|i - j| = 1$,

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  - for $|i-j| = 1$,
$$e_i^2 e_j - (e^{\hbar} + e^{-\hbar}) e_i e_j e_i + e_j e_i^2 = 0.$$
- As $\hbar \to 0$, the algebra $U_\hbar(\mathfrak{gl}_n)$ becomes the ordinary $U(\mathfrak{gl}_n)$.

Quantization problem

To construct associative algebra isomoprhism $\nu(u)_\hbar$ such that

$$\begin{array}{ccc} U_\hbar(\mathfrak{gl}_n) & \xrightarrow{\nu(u)_\hbar} & U(\mathfrak{gl}_n)[[\hbar]] \\ {\scriptstyle h\to 0}\downarrow & & {\scriptstyle h\to 0}\downarrow \\ Fun(B_- \times B_+) & \xrightarrow{\nu(u)^*} & Fun(\mathfrak{gl}_n) \end{array}$$

Stokes matrices of ODEs in noncommutative rings

- $U(\mathfrak{gl}_n)$: generator $\{e_{ij}\}_{i,j=1,\dots,n}$, relation $[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}$.

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- n by n matrix $T = (T_{ij})$ with entries valued in $U(\mathfrak{gl}_n)$

$$T_{ij} = e_{ij}, \qquad \text{for } 1 \le i, j \le n.$$

- For any n by n diagonal matrix u with distinct eigenvalues, consider

$$\frac{dF}{dz} = \hbar\Big(\frac{u}{z^2} + \frac{T}{z}\Big) \cdot F,$$

for a function $F(z) \in \mathrm{Mat}_n \otimes U(\mathfrak{gl}_n)[\![\hbar]\!]$.

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- The quantum Stokes matrices $S_{\hbar\pm}(u) = (S_{\hbar\pm}(u)_{ij})$, with entries $S_{\hbar\pm}(u)_{ij}$ in $U(\mathfrak{gl}_n)[\![\hbar]\!]$.

Representations of quantum group from Stokes matrices

Theorem (Xu)

For any fixed u, the map

$$\nu_\hbar(u) : U_\hbar(\mathfrak{gl}_n) \to U(\mathfrak{gl}_n) \ ; \ e_i \mapsto S_{\hbar+}(u)_{i,i+1}, \ \ f_i \mapsto S_{\hbar-}(u)_{i+1,i}$$

is an isomorphism between the Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{gl}_n)$ and $U(\mathfrak{gl}_n)$.

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is an isomorphism between the Drinfeld-Jimbo quantum group $U_\hbar(\mathfrak{gl}_n)$ and $U(\mathfrak{gl}_n)$. Furthermore, it leads to the following diagram

$$\begin{array}{ccc} U_\hbar(\mathfrak{gl}_n) & \xrightarrow{\nu(u)_\hbar} & U(\mathfrak{gl}_n)[\![\hbar]\!] \\ h \to 0 \Big\downarrow & & h \to 0 \Big\downarrow \\ Fun(B_- \times B_+) & \xrightarrow{\nu(u)^*} & Fun(\mathfrak{gl}_n) \end{array}$$

Interpretations: Knot invariants, Yang-Baxter equation

- Equivalently, take standard R-matrix in $Mat_n \otimes Mat_n$,

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Then the quantum Stokes matrices at a second order pole satisfy the Yang-Baxter equation

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- (Witten, Reshetikhin-Turaev) Quantum group encodes the quantum invariants of the Chern-Simons theory. In particular, for $U_{\hbar}(\mathfrak{gl}_2)$ we get Jones polynomial. One can say the jump of asymptotics of solutions of linear ODE at a second order pole encodes the knot invariants!

A dictionary

Table: A dictionary

	linear ODE at 2nd order pole	Quantum group $U_\hbar(\mathfrak{gl}_n)$
1	Nonresonant case $\hbar \notin \mathbb{Q}$	Realization of $U_\hbar(\mathfrak{gl}_n)$ at a generic $\hbar$
2	Resonant case $\hbar \in \mathbb{Q}$	Representation at roots of unity
3	WKB approximation as $\hbar \to \infty$	$\mathfrak{gl}_n$-Crystals
4	Wall-crossing in WKB approximation	Cactus group actions on crystals
5	Whitham dynamics	HKRW covers on eigenbasis
6	Analytic branching rules	Braching rules/ Gelfand-Tsetlin theory
7	Asymptotic Riemann-Hilbert problem	An explicit Drinfeld isomorphism
8	Involution of equations	Quantum symmetric pairs
9	Formal power series solutions	Yangians/ Trigonometric R-matrix
10	Semiclassical limit	Dual Poisson Lie groups

Canonical/crystal basis

- Classical problem in representation theory: to find a basis $\mathbb{B}$ of $U(\mathfrak{gl}_n)$ such that for any highest weight representation V, the set

$$\mathbb{B} \cdot v_0 \in V$$

is a basis of V. Very hard. A classical Lie theory problem, but only solved by quantum group.

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$$\left(\mathbb{B}_V, \{\tilde{e}_i, \tilde{f}_i\}_{i=1,\dots,n-1}\right)$$

a finite set $\mathbb{B}_V$ which models a weight basis of V equipped with crystal operators $\tilde{e}_i$ and $\tilde{f}_i$. Encoding representation data.

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- Combinatorial and geometric realizations of crystals.

WKB approximation = crystal limits

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$$\frac{1}{\hbar}\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F$$

• The WKB method, named after Wentzel, Kramers, and Brillouin, is for approximating solutions of a differential equation whose highest derivative is multiplied by a small parameter $1/\hbar$.

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• Indication: crystal structures should arises from the limits of q-Stokes matrices $S_{\hbar\pm}(u) = (S_{\hbar\pm}(u)_{ij})$ as $h \to -\infty$, where

$$S_{\hbar\pm}(u)_{ij} \in U(\mathfrak{gl}_n)[\![\hbar]\!] \to \mathrm{End}(V).$$

WKB analysis and crystals

- The algebraic characterization of the $1/h \to 0$ asymptotics of $S_{\hbar\pm}(u) \in \mathrm{End}(V) \otimes \mathrm{End}(\mathbb{C}^n)$ of $\frac{1}{h}\frac{dF}{dz} = \left(\frac{u}{z^2} + \frac{T}{z}\right) \cdot F$.

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- The action of the off-diagonal entry $S_{\hbar+}(u)_{k,k+1}$ on certain set $\mathbb{B}_V$ of canonical basis $\{v_i(u)\}_{i \in I}$ of V

$$S_{\hbar+}(u)_{k,k+1} \cdot v_i(u) = \sum_{j \in I} e^{h\phi_{ij}^{(k)}(u) + \sqrt{-1} g_{ij}^{(k)}(u,h)} \big(v_j(u) + O(h^{-1})\big),$$

where $\phi_{ij}^{(k)}(u)$, $g_{ij}^{(k)}(u,h)$ are real valued functions for all $1 \le i, j \le k \le n-1$.

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where $\phi_{ij}^{(k)}(u)$, $g_{ij}^{(k)}(u,h)$ are real valued functions for all $1 \leq i,j \leq k \leq n-1$.

- The WKB approximation of $S_{\hbar+}(u)_{k,k+1}$ naturally defines an operator $\tilde{e}_k$ on the set $\mathbb{B}_V$ by picking the leading term

$$\tilde{e}_k(v_i(u)) := v_j(u), \quad \text{if} \quad \phi_{ij}^{(k)}(u) = \max\{\phi_{il}^{(k)}(u) \mid l \in I\}.$$

A transcendental realization of crystals

Conjecture (Xu, Proved under the WKB asmptotic assumption)

For any u, there exists a canonical basis $\{v_I(u)\}$ of V, operators $\tilde{e}_k(u)$ and $\tilde{f}_k(u)$ for $k = 1, ..., n-1$ such that there exists constants c, c'

$$\lim_{q=e^{\pi i h}\to 0} q^c S_{\hbar+}(u)_{k,k+1} \cdot v_I(u) = \tilde{e}_k(v_I(u)),$$

$$\lim_{q=e^{\pi i h}\to 0} q^{c'} S_{\hbar-}(u)_{k+1,k} \cdot v_I(u) = \tilde{f}_k(v_I(u)).$$

Furthermore, the datum $(\{v_I(u)\}, \tilde{e}_k(u), \tilde{f}_k(u))$ is a $\mathfrak{gl}_n-$crystal.

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Theorem (Xu)

The conjecture is true as $u_n \gg u_{n-1} \gg \cdots \gg u_1$. And the WKB datum coincides with the known $\mathfrak{gl}_n$-crystal structure on semistandard Young tableaux.

Спасибо большое

Part II

Arbitrary order pole and quantization of Riemann-Hilbert mpas

Quantum Stokes matrices at pole of order $k+1$

- The universal enveloping algebra $U\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big)$ generated by $\{e_{ij}t^{m-1}\}$ for $i,j=1,...,n$ and $m=1,...,k$ subject to the relation

$$[e_{ij}t^a, e_{kl}t^b] = \begin{cases} \delta_{jk}e_{il}t^{a+b} - \delta_{li}e_{kj}t^{a+b}, & \text{if } a+b\leq k \\ 0, & \text{if } a+b>k. \end{cases}$$

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- Consider the equation

$$\frac{dF}{dz} = h\left(\frac{u}{z^{k+1}} + \frac{T_{[k]}}{z^k} + \cdots + \frac{T_{[2]}}{z^2} + \frac{T_{[1]}}{z}\right)\cdot F,$$

where $u \in \mathfrak{h}_{\text{reg}}$, h is a complex parameter, each $T_{[m]}$ is an $n \times n$ matrix with entries valued in $U\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big)$

$$(T_{[m]})_{ij} = e_{ij}t^{m-1}, \qquad \text{for } 1 \le i,j \le n, \quad 1 \le m \le k.$$

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- $2k$ quantum Stokes matrices

$$S_i(u)\in\widehat{U}\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big)\otimes\mathrm{End}(\mathbb{C}^n) \;\; for\ i=1,...,2k$$

Here S_{2i+1} is upper triangular and S_{2i} is lower triangular.

- Take the standard R-matrix $R \in \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n)$,

$$R = \sum_{i \neq j, i,j=1}^{n} E_{ii} \otimes E_{jj} + e^{\pi i h} \sum_{i=1}^{n} E_{ii} \otimes E_{ii} + (e^{\pi i h} - e^{-\pi i h}) \sum_{1 \leq j < i \leq n} E_{ij} \otimes E_{ji}.$$

- Introduce

$$\mathbb{S}^{(1)}_{[i]} := S^{(1)}_1 S^{(1)}_2 \cdots S^{(1)}_i \in \widehat{U}\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big) \otimes \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n),$$
$$\mathbb{S}^{(2)}_{[i]} := S^{(2)}_{i+1} S^{(2)}_{i+2} \cdots S^{(2)}_{2k} \in \widehat{U}\big(\mathfrak{gl}_n(\mathbb{C}[t]/t^k)\big) \otimes \mathrm{End}(\mathbb{C}^n) \otimes \mathrm{End}(\mathbb{C}^n).$$

Here the indices are taken modulo $2k$.

Theorem (Xu)

For any $u \in \mathfrak{h}_{\mathrm{reg}}$, the quantum Stokes matrices satisfy the algebraic relations ($RL...L = L...LR$)

$$\mathbb{R}^{12} \mathbb{S}^{(1)}_{[i]} \mathbb{S}^{(2)}_{[i]} = \mathbb{S}^{(2)}_{[i]} \mathbb{S}^{(1)}_{[i]} \mathbb{R}^{12}, \quad i = 1, ..., 2k-1.$$