Post-groups, post-groupoids and the Yang-Baxter equation

Yunhe Sheng
(Joint work with Chengming Bai, Li Guo, Rong Tang and Chenchang Zhu)

Department of Mathematics, Jilin University, China

Sino-Russian Interdisciplinary Mathematical Conference-2,
November 27, 2024


Райчихинск
阿尔哈拉站
Архара
比罗比詹
Биробиджан
伯力
Хабаровск
佩列亚斯拉夫卡
Переяславка
维亚泽姆斯基
Вяземский
伊春市
齐齐哈尔市
鹤岗市
Lermontovskoye
Лермонтовское
佳木斯市
比金
Бикин
双鸭山市
大庆市
绥化市
黑龙江
卢切戈尔斯克
Лучегорск
兴安盟
达利涅列琴斯克
Дальнереченск
哈尔滨市
七台河市
白城市
列索扎沃茨克
Лесозаводск
松原市
鸡西市
斯帕斯克达利尼
Спасск-Дальний
牡丹江市
野猪河
阿尔谢尼耶夫
Арсеньев
通辽市
长春市
吉林市
双城子
Уссурийск
吉林
海参崴
Владивосток
阿尔乔姆
Артем
四平市
延边朝鲜族自治州
纳霍德卡
Находка
Andreevka
Андреевка
铁岭市
沈阳市
通化市
清津市
청진시
惠山市
혜산시
鞍山市
辽宁
营口市
丹东市
朝鲜
日本海
(东海)

Outline

Integration Problem

In Lie Theory:

$$\text{Lie groups } G \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Lie algebras } \mathfrak{g}$$

In Poisson Geometry:

$$\text{Poisson Lie groups } (M, \pi) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Lie bialgebras } (\mathfrak{g}, \delta)$$

Integration Problem

In Lie Theory:

$$\text{Lie groups } G \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Lie algebras } \mathfrak{g}$$

In Poisson Geometry:

$$\text{Poisson Lie groups } (M,\pi) \underset{\text{integration}}{\overset{\text{differentiation}}{\rightleftarrows}} \text{Lie bialgebras } (\mathfrak{g},\delta)$$

Integration Problem

Integrability of Lie algebroids, Courant algebroids, Leibniz algbras, L_∞-algebras:

M. Crainic and R. L. Fernandes, Integrability of Lie brackets. *Ann. of Math. (2)* **157** (2003), 575-620.

A. Henriques. Integrating L_∞-algebras. *Compos. Math.* 144(4) (2008), 1017-1045.

Y. Sheng and C. Zhu, Higher Extensions of Lie Algebroids, *Commun. Contemp. Math.* 19 (3) (2017), 1650034, 41 pages.

C. Laurent-Gengoux and F. Wagemann, Lie rackoids integrating Courant algebroids, *Ann. Global Anal. Geom.* 57 (2020), no. 2, 225-256.

Questions:

- What is the integration of post-Lie algebras?
- What is the integration of post-Lie algebroids?

Post-Lie algebras

Definition (Vallette)

A **post-Lie algebra** $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ consists of a Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ and a binary product $\rhd : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ such that

$$\begin{aligned} x \rhd [y,z]_{\mathfrak{g}} &= [x \rhd y, z]_{\mathfrak{g}} + [y, x \rhd z]_{\mathfrak{g}}, \\ [x,y]_{\mathfrak{g}} \rhd z &= a_{\rhd}(x,y,z) - a_{\rhd}(y,x,z), \end{aligned}$$

where $a_{\rhd}(x,y,z) = x \rhd (y \rhd z) - (x \rhd y) \rhd z$.

Post-Lie algebras

Proposition (Vallette)

A post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \triangleright)$ *gives rise to a new Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\triangleright})$*, called the* **sub-adjacent Lie algebra** *and denoted by* $\mathfrak{g}_{\triangleright}$*, where the Lie bracket* $[\cdot,\cdot]_{\triangleright}$ *is defined by*

$$[x, y]_{\triangleright} = [x, y]_{\mathfrak{g}} + x \triangleright y - y \triangleright x, \ \forall x, y \in \mathfrak{g}.$$

Moreover, $L^{\triangleright} : (\mathfrak{g}, [\cdot,\cdot]_{\triangleright}) \to \mathrm{Der}(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *is an action of the sub-adjacent Lie algebra* $\mathfrak{g}_{\triangleright}$ *on the original Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

In a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \triangleright)$, if the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ is trivial, then we obtain a pre-Lie algebra, namely a vector space $\mathfrak{g}$ with a multiplication $\triangleright$ satisfying

$$a_{\triangleright}(x, y, z) - a_{\triangleright}(y, x, z) = 0.$$

Post-Lie algebras

Proposition (Vallette)

A post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \triangleright)$ *gives rise to a new Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\triangleright})$, *called the* **sub-adjacent Lie algebra** *and denoted by* $\mathfrak{g}_{\triangleright}$, *where the Lie bracket* $[\cdot,\cdot]_{\triangleright}$ *is defined by*

$$[x,y]_{\triangleright} = [x,y]_{\mathfrak{g}} + x \triangleright y - y \triangleright x, \ \forall x, y \in \mathfrak{g}.$$

Moreover, $L^{\triangleright} : (\mathfrak{g}, [\cdot,\cdot]_{\triangleright}) \to \mathrm{Der}(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *is an action of the sub-adjacent Lie algebra* $\mathfrak{g}_{\triangleright}$ *on the original Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

In a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \triangleright)$, if the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ is trivial, then we obtain a pre-Lie algebra, namely a vector space $\mathfrak{g}$ with a multiplication $\triangleright$ satisfying

$$a_{\triangleright}(x,y,z) - a_{\triangleright}(y,x,z) = 0.$$

Post-Lie algebras

Proposition (Vallette)

A post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$ *gives rise to a new Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\rhd})$, *called the* **sub-adjacent Lie algebra** *and denoted by* $\mathfrak{g}_{\rhd}$, *where the Lie bracket* $[\cdot,\cdot]_{\rhd}$ *is defined by*

$$[x, y]_{\rhd} = [x, y]_{\mathfrak{g}} + x \rhd y - y \rhd x, \ \forall x, y \in \mathfrak{g}.$$

Moreover, $L^{\rhd} : (\mathfrak{g}, [\cdot,\cdot]_{\rhd}) \to \mathrm{Der}(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ *is an action of the sub-adjacent Lie algebra* $\mathfrak{g}_{\rhd}$ *on the original Lie algebra* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$.

In a post-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \rhd)$, if the Lie bracket $[\cdot,\cdot]_{\mathfrak{g}}$ is trivial, then we obtain a pre-Lie algebra, namely a vector space $\mathfrak{g}$ with a multiplication $\rhd$ satisfying

$$a_{\rhd}(x, y, z) - a_{\rhd}(y, x, z) = 0.$$

Examples

Example

Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}})$ be a Lie algebra, $(A, \cdot)$ a commutative associative algebra and $\rho : \mathfrak{g} \to \mathrm{Der}(A)$ a Lie algebra homomorphism. Then there is a post-Lie algebra structure on $A \otimes \mathfrak{g}$, which is given by

$$\begin{aligned} [a \otimes x, b \otimes y] &= ab \otimes [x, y]_{\mathfrak{g}}, \\ (a \otimes x) \rhd (b \otimes y) &= a \cdot \rho(x)(b) \otimes y, \end{aligned}$$

for all $a, b \in A,\ x, y \in \mathfrak{g}$.

Examples

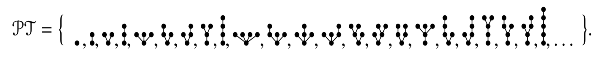
Example

Let $(V, \rhd)$ be a magma algebra. Extend the magma operation $\rhd : V \otimes V \to V$ to the free Lie algebra $(Lie(V), [\cdot, \cdot])$. Then $(Lie(V), [\cdot, \cdot], \rhd)$ is a post-Lie algebra.

Examples

Example (Free post-Lie algebra)

Let $\mathcal{PT}$ be the set of isomorphism classes of planar rooted trees:



Let $\mathbf{k}\{\mathcal{PT}\}$ be the free $\mathbf{k}$-vector space generated by $\mathcal{PT}$. The **left grafting operator** $\triangleright : \mathbf{k}\{\mathcal{PT}\} \otimes \mathbf{k}\{\mathcal{PT}\} \to \mathbf{k}\{\mathcal{PT}\}$ is defined by

$$\tau \triangleright \omega = \sum_{s \in \mathrm{Nodes}(\omega)} \tau \circ_s \omega, \quad \forall \tau, \omega \in \mathcal{PT},$$

where $\tau \circ_s \omega$ is the planar rooted tree resulting from attaching the root of τ to the node s of the tree ω from the left. Consider the free Lie algebra $Lie(\mathbf{k}\{\mathcal{PT}\})$, and extend the left grafting operator $\triangleright$ on $\mathbf{k}\{\mathcal{PT}\}$ to the free Lie algebra $Lie(\mathbf{k}\{\mathcal{PT}\})$. Then $(Lie(\mathbf{k}\{\mathcal{PT}\}), [\cdot, \cdot], \triangleright)$ is a post-Lie algebra, which is the free post-Lie algebra generated by one generator $\{\bullet\}$.

Examples

Example (flat and torsion free connections)

There is a natural pre-Lie algebra structure on the vector space $\mathfrak{X}(\mathbb{R}^n)$ of smooth vector fields, which is given by

$$\sum_{1\leq i\leq n} X_i\partial_i \triangleright \sum_{1\leq j\leq n} Y_j\partial_j = \sum_{1\leq i,j\leq n} X_i(\partial_i Y_j)\partial_j,$$

for all $X_i, Y_j \in C^\infty(\mathbb{R}^n)$.

Flat and torsion free connection $\dashrightarrow$ pre-Lie algebra

Flat and constant torsion connection $\dashrightarrow$ post-Lie algebra

Examples

Example (flat and torsion free connections)

There is a natural pre-Lie algebra structure on the vector space $\mathfrak{X}(\mathbb{R}^n)$ of smooth vector fields, which is given by

$$\sum_{1\leq i\leq n} X_i\partial_i \rhd \sum_{1\leq j\leq n} Y_j\partial_j = \sum_{1\leq i,j\leq n} X_i(\partial_i Y_j)\partial_j,$$

for all $X_i, Y_j \in C^\infty(\mathbb{R}^n)$.

Flat and torsion free connection $\dashrightarrow$ pre-Lie algebra

Flat and constant torsion connection $\dashrightarrow$ post-Lie algebra

Examples

Example (flat and torsion free connections)

There is a natural pre-Lie algebra structure on the vector space $\mathfrak{X}(\mathbb{R}^n)$ of smooth vector fields, which is given by

$$\sum_{1\le i\le n} X_i\partial_i \triangleright \sum_{1\le j\le n} Y_j\partial_j = \sum_{1\le i,j\le n} X_i(\partial_i Y_j)\partial_j,$$

for all $X_i, Y_j \in C^\infty(\mathbb{R}^n)$.

Flat and torsion free connection $\dashrightarrow$ pre-Lie algebra

Flat and constant torsion connection $\dashrightarrow$ post-Lie algebra

Applications

- Rota-Baxter operators

C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Comm. Math. Phys.* 297 (2010), 553-596.

- Butcher series

J. C. Butcher, An algebraic theory of integration methods, *Math. Comp.* **26** (1972), 79-106.

H. Z. Munthe-Kaas and A. Lundervold, On post-Lie algebras, Lie-Butcher series and moving frames, *Found. Comput. Math.* **13** (2013), 583-613.

Applications

- Rota-Baxter operators

C. Bai, L. Guo and X. Ni, Nonabelian generalized Lax pairs, the classical Yang-Baxter equation and PostLie algebras, *Comm. Math. Phys.* 297 (2010), 553-596.

- Butcher series

J. C. Butcher, An algebraic theory of integration methods, *Math. Comp.* **26** (1972), 79-106.

H. Z. Munthe-Kaas and A. Lundervold, On post-Lie algebras, Lie-Butcher series and moving frames, *Found. Comput. Math.* **13** (2013), 583-613.

Butcher series

We consider systems of differential equations

$$\begin{cases} \frac{\mathrm{d}y_1}{\mathrm{d}x} & = & f_1(y_1, y_2, \cdots, y_n), \\ \frac{\mathrm{d}y_2}{\mathrm{d}x} & = & f_2(y_1, y_2, \cdots, y_n), \\ & \vdots & \\ \frac{\mathrm{d}y_n}{\mathrm{d}x} & = & f_n(y_1, y_2, \cdots, y_n). \end{cases}$$

Use the vector notation as following:

$$y = (y_1, y_2, \cdots, y_n)^T, \;\; f(y) = (f_1(y), f_2(y), \cdots, f_n(y))^T,$$

we can rewrite the above systems of differential equations to $\frac{\mathrm{d}y}{\mathrm{d}x} = f(y)$ with the initial value

$$y(x_0) = y_0 = (y_1(x_0), y_2(x_0), \cdots, y_n(x_0))^T.$$

Butcher series

It is really surprise that one should use the free pre-Lie algebra in the formal Taylor expansion of the solution at $x_0 + h$.

Any map $f : \mathbb{R}^n \to \mathbb{R}^n$ gives a smooth vector field

$$f(y) = \sum_{1 \leq i \leq n} f_i(y)\partial_i \in \mathfrak{X}(\mathbb{R}^n).$$

Moreover, there is a unique pre-Lie algebra homomorphism F from $\mathbb{R}\{\mathcal{T}\}$ to $\mathfrak{X}(\mathbb{R}^n)$ such that

$$F(\bullet) = f.$$

Applications in Butcher series

The formal Taylor expansion of the solution at $x_0 + h$ is

$$y(x_0 + h) = y_0 + \sum_{\tau \in \mathcal{T}} \frac{h^{|\tau|}}{\sigma(\tau)\tau!} F(\tau)(y_0).$$

Set $\mathcal{T}^+ = \mathcal{T} \cup \{\emptyset\}$ and denote by $\mathcal{B}_{\mathbb{R}}$ the set of all maps $\{a : \mathcal{T}^+ \to \mathbb{R} | a(\emptyset) = 1\}$. The formal series defined by

$$B(a, h, f)(y) = a(\emptyset)y + \sum_{\tau \in \mathcal{T}} \frac{h^{|\tau|} a(\tau)}{\sigma(\tau)} F(\tau)(y)$$

is called the **Butcher series**.

J. C. Butcher, An algebraic theory of integration methods, *Math. Comp.* **26** (1972), 79-106.

H. Z. Munthe-Kaas and A. Lundervold, On post-Lie algebras, Lie-Butcher series and moving frames, *Found. Comput. Math.* **13** (2013), 583-613.

Applications

- regularity structures

Y. Bruned and F. Katsetsiadis, Post-Lie algebras in regularity structures. *Forum Math. Sigma* **11** (2023), Paper No. e98.

Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures. *Invent. Math.* **215** (2019), 1039-1156.

- braces and skew-left braces

A. Smoktunowicz, On the passage from finite braces to pre-Lie rings. *Adv. Math.* 409 (2022), 108683.

S. Trappeniers, A Lazard correspondence for post-Lie rings and skew braces. arXiv:2406.02475.

Applications

- regularity structures

Y. Bruned and F. Katsetsiadis, Post-Lie algebras in regularity structures. *Forum Math. Sigma* **11** (2023), Paper No. e98.

Y. Bruned, M. Hairer and L. Zambotti, Algebraic renormalisation of regularity structures. *Invent. Math.* **215** (2019), 1039-1156.

- braces and skew-left braces

A. Smoktunowicz, On the passage from finite braces to pre-Lie rings. *Adv. Math.* 409 (2022), 108683.

S. Trappeniers, A Lazard correspondence for post-Lie rings and skew braces. arXiv:2406.02475.

Outline

Definition (Bai-Guo-S.-Tang)

A **post-group** is a group $(G, \cdot)$ equipped with another binary operation $\triangleright$ on G such that

1. for all $a \in G$, the left multiplication

$$L_a^{\triangleright} : G \to G, \quad L_a^{\triangleright} b = a \triangleright b, \quad \forall b \in G,$$

   is an automorphism of the group $(G, \cdot)$, that is,

$$a \triangleright (b \cdot c) = (a \triangleright b) \cdot (a \triangleright c), \quad \forall a, b, c \in G;$$

2. the following "weighted" associativity for $\triangleright$ holds:

$$a \triangleright (b \triangleright c) = \big(a \cdot (a \triangleright b)\big) \triangleright c, \quad \forall a, b, c \in G.$$

Post-groups

Theorem (Bai-Guo-S.-Tang)

Let $(G, \cdot, \rhd)$ be a post-group. Define $\circ : G \times G \to G$ by

$$a \circ b \quad = \quad a \cdot (a \rhd b), \quad \forall a, b \in G.$$

Then $(G, \circ)$ is a group with e being the unit, and the inverse map $\dagger : G \to G$ given by

$$a^{\dagger} \quad := \quad (L_a^{\rhd})^{-1}(a^{-1}).$$

Moreover, $L^{\rhd} : G \to \mathrm{Aut}(G)$ is an action of the group $(G, \circ)$ on the group $(G, \cdot)$.

The group $G_{\rhd} := (G, \circ)$ is called the **subadjacent group** of the post-group $(G, \cdot, \rhd)$.

Let e the identity of the Lie group G. Let $\mathfrak{g} = T_e G$ be the Lie algebra of G.

Define $\triangleright : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$ by

$$x \triangleright y = L^{\triangleright}_{*e}(x)(y) = \frac{d}{dt}\Big|_{t=0} L^{\triangleright}_{\exp(tx)} y = \frac{d}{dt}\Big|_{t=0} \frac{d}{ds}\Big|_{s=0} L^{\triangleright}_{\exp(tx)} \exp(sy).$$

Theorem (Bai-Guo-S.-Tang)

Let $(G, \cdot, \triangleright)$ be a post-Lie group. Then $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \triangleright)$ is a post-Lie algebra.

Let

$$\exp^{(\cdot)} : \mathfrak{g} \longrightarrow G$$

be the exponential map.

$$
\begin{aligned}
& x \triangleright (y \triangleright z) - y \triangleright (x \triangleright z) \\
= \;& \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}\Big(L^{\triangleright}_{\exp(tx)}L^{\triangleright}_{\exp(sy)}\exp(rz) - L^{\triangleright}_{\exp(sy)}L^{\triangleright}_{\exp(tx)}\exp(rz)\Big) \\
= \;& \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}\Big(L^{\triangleright}_{\exp(tx)\cdot(\exp(tx)\triangleright\exp(sy))}\exp(rz) - L^{\triangleright}_{\exp(sy)\cdot(\exp(sy)\triangleright\exp(tx))}\exp(rz)\Big) \\
= \;& \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}L^{\triangleright}_{\exp(tx)\cdot\exp(sy)}\exp(rz) + \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}L^{\triangleright}_{\exp(tx)\triangleright\exp(sy)}\exp(rz) \\
& -\frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}L^{\triangleright}_{\exp(sy)\cdot\exp(tx)}\exp(rz) - \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}L^{\triangleright}_{\exp(sy)\triangleright\exp(tx)}\exp(rz) \\
= \;& \frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}L^{\triangleright}_{\exp(tx+sy+\frac{1}{2}ts[x,y]_{\mathfrak{g}}+\cdots)}\exp(rz) + (x \triangleright y) \triangleright z \\
& -\frac{d}{dt}\Big|_{t=0}\frac{d}{ds}\Big|_{s=0}\frac{d}{dr}\Big|_{r=0}L^{\triangleright}_{\exp(sy+tx+\frac{1}{2}ts[y,x]_{\mathfrak{g}}+\cdots)}\exp(rz) - (y \triangleright x) \triangleright z \\
= \;& \frac{1}{2}[x,y]_{\mathfrak{g}} \triangleright z + (x \triangleright y) \triangleright z - \frac{1}{2}[y,x]_{\mathfrak{g}} \triangleright z - (y \triangleright x) \triangleright z \\
= \;& [x,y]_{\triangleright} \triangleright z,
\end{aligned}
$$

Outline

Rota-Baxter Lie groups

Definition (Guo-Lang-S.)

A Rota-Baxter operator of weight 1 on a Lie group G is a smooth map $\mathfrak{B} : G \to G$ such that

$$\mathfrak{B}(g)\mathfrak{B}(h) = \mathfrak{B}(g\mathrm{Ad}_{\mathfrak{B}(g)}h), \qquad g, h \in G.$$

Theorem (Bai-Guo-S.-Tang)

Let $\mathfrak{B} : G \longrightarrow G$ be a Rota-Baxter operator on a group $(G, \cdot)$. Define a binary product $\triangleright : G \times G \to G$ as following:

$$g \triangleright h = \mathrm{Ad}_{\mathfrak{B}(g)}h, \quad \forall g, h \in G.$$

Then $(G, \cdot, \triangleright)$ is a post-group.

Rota-Baxter Lie groups

Definition (Guo-Lang-S.)

A Rota-Baxter operator of weight 1 on a Lie group G is a smooth map $\mathfrak{B} : G \to G$ such that

$$\mathfrak{B}(g)\mathfrak{B}(h) = \mathfrak{B}(g\mathrm{Ad}_{\mathfrak{B}(g)}h), \qquad g, h \in G.$$

Theorem (Bai-Guo-S.-Tang)

Let $\mathfrak{B} : G \longrightarrow G$ be a Rota-Baxter operator on a group $(G, \cdot)$. Define a binary product $\triangleright : G \times G \to G$ as following:

$$g \triangleright h = \mathrm{Ad}_{\mathfrak{B}(g)}h, \quad \forall g, h \in G.$$

Then $(G, \cdot, \triangleright)$ is a post-group.

From post-groups to Rota-Baxter operators

Proposition (Bai-Guo-S.-Tang)

Let $(G, \cdot, \triangleright)$ be a post-group. Then the identity map $\mathrm{Id} : G \to G$ is a relative Rota-Baxter operator on the subadjacent group $(G, \circ)$ with respect to the action $L^{\triangleright}$ on the group $(G, \cdot)$.

Skew-left braces

Definition (Rump)

A skew-left brace $(G, \circ, \cdot)$ consists of a group $(G, \cdot)$ and a group $(G, \circ)$ such that

$$a \circ (b \cdot c) = (a \circ b) \cdot a^{\cdot -1} \cdot (a \circ c), \quad \forall a, b, c \in G.$$

W. Rump, A decomposition theorem for square-free unitary solutions of the quantum Yang-Baxter equation. **Adv. Math.** 193 (2005), 40-55.

T. Gateva-Ivanova, Set-theoretic solutions of the Yang-Baxter equation, braces and symmetric groups. **Adv. Math.** 338 (2018), 649-701.

F. Cedó, A. Smoktunowicz and L. Vendramin, Skew left braces of nilpotent type. **Proc. Lond. Math. Soc.** 118 (2019), 1367-1392.

Proposition (Bai-Guo-S.-Tang)

Let $(G, \circ, \cdot)$ be a skew-left brace. Define a binary product $\triangleright : G \times G \to G$ by

$$a \triangleright b = a^{-1} \cdot (a \circ b), \quad \forall a, b \in G,$$

here a^{-1} is the inverse of a in $(G, \cdot)$. Then $(G, \cdot, \triangleright)$ is a post-group.

Skew-left braces

Proposition (Bai-Guo-S.-Tang)

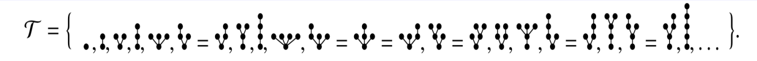
Let $(G, \cdot, \triangleright)$ be a post-group. Then $(G, \circ, \cdot)$ is a skew-left brace.

Theorem (Bai-Guo-S.-Tang)

The category of post-groups is isomorphic to the category of skew-left braces.

Butcher groups

Let $\mathcal{T}$ be the set of isomorphism classes of rooted trees:



We set $\mathcal{T}^+ = \mathcal{T} \cup \{\emptyset\}$ and denote by

$$\mathcal{B}_{\mathbb{R}} = \{a : \mathcal{T}^+ \to \mathbb{R} | a(\emptyset) = 1\}.$$

Theorem (Hairer-Wanner)

$(\mathcal{B}_{\mathbb{R}}, \circ)$ *is a group, which is called* **Butcher group**, *where*

$$(a \circ b)(\tau) \quad = \quad a(\tau) + \sum_{c \in AC(\tau)} a(P^c(\tau)) b(R^c(\tau)).$$

E. Hairer and G. Wanner, On the Butcher group and general multi-value methods, *Computing* 13 (1974), 1-15.

Butcher group

We define an abelian group structure on $\mathcal{B}_{\mathbb{R}}$ by

$$(a \cdot b)(\emptyset) = 1, \ (a \cdot b)(\omega) = a(\omega) + b(\omega), \ \forall \omega \in \mathcal{T},$$

Define the binary product $\triangleright : \mathcal{B}_{\mathbb{R}} \times \mathcal{B}_{\mathbb{R}} \to \mathcal{B}_{\mathbb{R}}$ by

$$\begin{aligned} (a \triangleright b)(\emptyset) &= 1, \\ (a \triangleright b)(\tau) &= \sum_{c \in AC(\tau)} a(P^c(\tau))b(R^c(\tau)), \forall \omega \in \mathcal{T}. \end{aligned}$$

Theorem (Bai-Guo-S.-Tang)

With the above notations, $(\mathcal{B}_{\mathbb{R}}, \cdot, \triangleright)$ is a post-group, whose subadjacent group is exactly the Butcher group $(\mathcal{B}_{\mathbb{R}}, \circ)$.

Outline

The Yang-Baxter equation

We show that a post-group gives rise to a braiding group, and thus lead to a set-theoretical solution of the Yang-Baxter equation.

Definition

Let X be a set. A set-theoretical solution to the **Yang-Baxter equation** on X is a bijective map $R: X \times X \to X \times X$ satisfying:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

V. G. Drinfel'd, On some unsolved problems in quantum group theory. Quantum groups (Leningrad, 1990), 1-8, Lecture Notes in Math., 1510, Springer, Berlin, 1992.

P. Etingof, T. Schedler and A. Soloviev, Set-theoretical solutions to the quantum Yang-Baxter equation. *Duke Math. J.* **100** (1999), 169-209.

J. Lu, M. Yan and Y. Zhu, On the set-theoretical Yang-Baxter equation. *Duke Math. J.* **104** (2000), 1-18.

Yang-Baxter equations

Let $(G, \cdot, \rhd)$ be a post-group. Define $R_G : G \times G \to G \times G$ by

$$R_G(x, y) = (x \rhd y, (x \rhd y)^{\dagger} \circ x \circ y), \ \forall x, y \in G,$$

where $\circ$ is the subadjacent group structure.

Theorem (Bai-Guo-S.-Tang)

Let $(G, \cdot, \rhd)$ be a post-group. Then R_G is a solution of the Yang-Baxter equation on the set G.

Outline

Definition

A **post-Lie algebroid** structure on a vector bundle $A \longrightarrow M$ is a triple that consists of a $C^\infty(M)$-linear Lie algebra structure $[\cdot,\cdot]_A$ on $\Gamma(A)$, a bilinear operation $\triangleright_A : \Gamma(A) \otimes \Gamma(A) \longrightarrow \Gamma(A)$ and a vector bundle morphism $a_A : A \longrightarrow TM$, called the **anchor**, such that $(\Gamma(A), [\cdot,\cdot]_A, \triangleright_A)$ is a post-Lie algebra, and for all $f \in C^\infty(M)$ and $u, v \in \Gamma(A)$, the following relations are satisfied:

(i) $u \triangleright_A (fv) = f(u \triangleright_A v) + a_A(u)(f)v,$

(ii) $(fu) \triangleright_A v = f(u \triangleright_A v).$

H. Z. Munthe-Kaas and A. Lundervold, On post-Lie algebras, Lie-Butcher series and moving frames, *Found. Comput. Math.* **13** (2013), 583-613.

Example

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra and M a manifold. Then the section space $\Gamma(M \times \mathfrak{g})$ of the trivial bundle $A = M \times \mathfrak{g}$ enjoys a $C^\infty(M)$-linear Lie algebra structure $[\cdot,\cdot]_\mathfrak{g}$ given by

$$[fu, gv]_\mathfrak{g} = fg[u,v]_\mathfrak{g}, \quad \forall f, g \in C^\infty(M),\ u, v \in \mathfrak{g}.$$

Let $\phi : \mathfrak{g} \to \mathfrak{X}(M)$ be an action of $\mathfrak{g}$ on M. Then one can define $a_A : M \times \mathfrak{g} \to TM$ by

$$a_A(m,u) = \phi(u)_m, \quad \forall m \in M, u \in \mathfrak{g}$$

and define $\triangleright_A : \Gamma(A) \times \Gamma(A) \longrightarrow \Gamma(A)$ by

$$(fu) \triangleright_A (gv) = f\phi(u)(g)v, \quad \forall f, g \in C^\infty(M),\ u, v \in \mathfrak{g}.$$

Then $(A = M \times \mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \triangleright_A, a_A)$ is a post-Lie algebroid.

Definition 2.5. A **post-(Lie) groupoid** consists of the following data:

- a (Lie) group bundle $(\mathbb{G} \xrightarrow{\pi} M, \cdot)$, where $\cdot$ is the multiplication;
- a surjective map (submersion) $\Phi : \mathbb{G} \to M$ satisfying $\Phi(\iota_m) = m$, for all $m \in M$;
- a (smooth) map $\triangleright : \mathbb{G}_\Phi \times_\pi \mathbb{G} \to \mathbb{G}$ satisfying $\pi(\gamma) = \pi(\gamma \triangleright \delta)$ and the left multiplication

$$L^{\triangleright}_{\gamma} : \mathbb{G}_{\Phi(\gamma)} \to \mathbb{G}_{\pi(\gamma)}, \quad L^{\triangleright}_{\gamma}\delta = \gamma \triangleright \delta \ \text{ for all } \delta \in \mathbb{G}_{\Phi(\gamma)},$$

is invertible for any $\gamma \in \mathbb{G}$, where

$$\mathbb{G}_\Phi \times_\pi \mathbb{G} = \{(\gamma, \delta) \in \mathbb{G} \times \mathbb{G} \text{ such that } \Phi(\gamma) = \pi(\delta)\},$$

such that for all $(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{G}_\Phi \times_\pi \mathbb{G}_\Phi \times_\pi \mathbb{G}$, and $\gamma_2' \in \mathbb{G}$ satisfying $\pi(\gamma_2) = \pi(\gamma_2')$, the following axioms hold:

(i) $\Phi(\gamma_2) = \Phi\big(\gamma_1 \cdot (\gamma_1 \triangleright \gamma_2)\big)$,

(ii) $\gamma_1 \triangleright (\gamma_2 \cdot \gamma_2') = (\gamma_1 \triangleright \gamma_2) \cdot (\gamma_1 \triangleright \gamma_2')$,

(iii) $\gamma_1 \triangleright (\gamma_2 \triangleright \gamma_3) = \big(\gamma_1 \cdot (\gamma_1 \triangleright \gamma_2)\big) \triangleright \gamma_3$.

We will denote a post-(Lie) groupoid by $(\mathbb{G} \xrightarrow{\pi} M, \cdot, \Phi, \triangleright)$.

Example

Let G be a group and M a set. Then $\mathbb{G} = M \times G \xrightarrow{\pi} M$ is a trivial group bundle, where π is the projection to M. Let $\Phi : M \times G \to M$ be an action of G on M. In this case,

$$\mathbb{G}_\Phi \times_\pi \mathbb{G} = \{((m,g),(n,h)) \in \mathbb{G} \times \mathbb{G} \text{ such that } n = \Phi(m,g)\}.$$

Define $\triangleright : \mathbb{G}_\Phi \times_\pi \mathbb{G} \to \mathbb{G}$ by

$$(m,g) \triangleright (n,h) = (m,h).$$

Then $(M \times G \xrightarrow{\pi} M, \cdot, \Phi, \triangleright)$ is a post-groupoid.

Theorem (S. -Tang-Zhu)

Let $(\mathbb{G} \xrightarrow{\pi} M, \cdot, \Phi, \triangleright)$ *be a post-groupoid. Then* $(\mathbb{G} \overset{\alpha=\pi}{\underset{\beta=\Phi}{\rightrightarrows}} M, \star, \iota, \mathrm{inv}_\triangleright)$ *is a groupoid, called the* **sub-adjacent groupoid**, *where the groupoid multiplication* $\star : \mathbb{G}_\Phi \times_\pi \mathbb{G} \to \mathbb{G}$ *is defined by*

$$\gamma_1 \star \gamma_2 = \gamma_1 \cdot (\gamma_1 \triangleright \gamma_2),$$

and the new inverse map $\mathrm{inv}_\triangleright$ *is given by*

$$\mathrm{inv}_\triangleright(\gamma) = (L_\gamma^\triangleright)^{-1}\mathrm{inv}(\gamma).$$

Moreover, $\triangleright$ *gives rise to an action of the sub-adjacent groupoid* $(\mathbb{G} \overset{\alpha=\pi}{\underset{\beta=\Phi}{\rightrightarrows}} M, \star, \iota, \mathrm{inv}_\triangleright)$ *on the group bundle* $\mathbb{G} \xrightarrow{\pi} M$.

Denote the Lie algebroid of the subadjacent Lie groupoid $(\mathbb{G} \overset{\alpha=\pi}{\underset{\beta=\Phi}{\rightrightarrows}} M, \star, \iota, \mathrm{inv}_{\triangleright})$ by $(A \longrightarrow M, [\cdot,\cdot], \Phi_*)$. The Lie groupoid homomorphism $L^{\triangleright}$ induces a Lie algebroid homomorphism $L^{\triangleright}_*$ from $(A \longrightarrow M, [\cdot,\cdot], \Phi_*)$ to $\mathrm{Der}(A)$. In particular, there holds:

$$\Phi_*(u) = \mathfrak{a}(L^{\triangleright}_*(u)), \quad \forall u \in \Gamma(A).$$

Define $\triangleright_A : \Gamma(A) \otimes \Gamma(A) \to \Gamma(A)$ by

$$u \triangleright_A v = L^{\triangleright}_*(u)v, \quad \forall u, v \in \Gamma(A).$$

Theorem (S. -Tang-Zhu)

Let $(\mathbb{G} \xrightarrow{\pi} M, \cdot, \Phi, \triangleright)$ be a post-Lie groupoid. Then $(A \longrightarrow M, [\cdot,\cdot]_A, \Phi_*, \triangleright_A)$ is a post-Lie algebroid, where $(A \longrightarrow M, [\cdot,\cdot]_A)$ is the Lie algebra bundle associated to the Lie group bundle $\mathbb{G} \xrightarrow{\pi} M$.

Definition

A **quiver** over M is a set $\mathbb{A}$ equipped with two maps $\alpha, \beta : \mathbb{A} \to M$, called the source map and the target map.

Definition

Let $\left(\mathbb{A} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} M\right)$ be a quiver. A quiver-theoretical solution of the **Yang-Baxter equation** on the quiver $\mathbb{A}$ is an isomorphism R of quivers from $\mathbb{A}_\beta \times_\alpha \mathbb{A}$ to $\mathbb{A}_\beta \times_\alpha \mathbb{A}$ satisfying:

$$R_{12}R_{23}R_{12} = R_{23}R_{12}R_{23}.$$

N. Andruskiewitsch, On the quiver-theoretical quantum Yang-Baxter equation. *Selecta Math. (N.S.)* **11** (2005), 203-246.

Let $(\mathbb{G} \xrightarrow{\pi} M, \Phi, \triangleright)$ be a post-groupoid. Define $R_{\mathbb{G}} : \mathbb{G}_{\Phi} \times_{\pi} \mathbb{G} \to \mathbb{G}_{\Phi} \times_{\pi} \mathbb{G}$ by

$$R_{\mathbb{G}}(\gamma, \delta) = \big(\gamma \triangleright \delta, \mathrm{inv}_{\triangleright}(\gamma \triangleright \delta) \star \gamma \star \delta\big), \quad \forall (\gamma, \delta) \in \mathbb{G}_{\Phi} \times_{\pi} \mathbb{G}.$$

Theorem (S. -Tang-Zhu)

Let $(\mathbb{G} \xrightarrow{\pi} M, \cdot, \Phi, \triangleright)$ be a post-groupoid. Then $R_{\mathbb{G}}$ is a quiver-theoretical solution of the Yang-Baxter equation on the quiver $\mathbb{G} \overset{\alpha=\pi}{\underset{\beta=\Phi}{\rightrightarrows}} M$.

Example

Example

Consider the post-groupoid $(M \times G \xrightarrow{\pi} M, \cdot, \Phi, \triangleright)$ associated to an action $\Phi : M \times G \to M$. Then $R_{M\times G} : (M \times G)_\Phi \times_\pi (M \times G) \to (M \times G)_\Phi \times_\pi (M \times G)$ defined by

$$R_{M\times G}((m,g),(n,h)) = \big((m,h),(\Phi(m,h),h^{-1}gh)\big),$$

for all $((m,g),(n,h)) \in (M \times G)_\Phi \times_\pi (M \times G)$, is a non-degenerate quiver-theoretical solution of the Yang-Baxter equation on the quiver $M \times G \underset{\Phi}{\overset{\pi}{\rightrightarrows}} M$.

Recent developments

- K. Ebrahimi-Fard and G. S. Venkatesh, A Formal Power Series Approach to Multiplicative Dynamic Feedback Interconnection, arXiv:2301.04949.
- M. Al-Kaabi, K. Ebrahimi-Fard and D. Manchon, Free post-groups, post-groups from group actions, and post-Lie algebras, arXiv:2306.08284.
- K. Ebrahimi-Fard, W. Steven Gray and G. Venkatesh, On the Post-Lie Structure in SISO Affine Feedback Control Systems, arXiv:2311.04070.
- K. Ebrahimi-Fard and T. Ringeard, A post-group theoretic perspective on the operator-valued S-transform in free probability, arXiv:2402.16450
- Davide Ferri, On dynamical skew braces and skew bracoids, arXiv:2410.10717.

The End

Thanks for your attention!