p-adic Riemann-Hilbert

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Plan

1. Hilbert's 21st problem
2. Classical Riemann-Hilbert correspondence
3. Introduction to p-adic numbers
4. A p-adic Riemann-Hilbert functor
5. Applications

Hilbert's 21st problem

- Consider the ODE:

$$(*) \qquad f^{(n)} + a_1(z) f^{(n-1)} + \cdots + a_n(z) = 0, \qquad f^{(i)} = \frac{d^i}{dz^i} f$$

$$U \subset \mathbb{C} \text{ open domain}, \quad a_i(z) \in \mathcal{O}(U)$$

- E.g. hypergeometric differential equation:

$$z(1-z)f^{(2)} + \Big(c - (b+b+1)z\Big) f^{(1)} - abf = 0$$

$$U = \mathbb{C} - \{0, 1\}, \quad a, b, c \in \mathbb{C}$$

Bessel differential equation:

$$\left(2\frac{d}{dz}\right)^2 f - 2f = 0, \quad U = \mathbb{C} - \{0\}$$

Hilbert's 21st problem

- Rewrite (*) as

$$(**) \qquad \frac{d}{dz}\vec{f}(z) = A(z)\vec{f}(z)$$

$$\vec{f}(z) = \begin{pmatrix} 1 \\ f(z) \\ f^{(1)}(z) \\ \vdots \\ f^{(n-1)}(z) \end{pmatrix}, \quad A(z) = \begin{pmatrix} 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & 1 \\ -a_n(z) & \cdots & -a_2(z) & -a_1(z) \end{pmatrix}$$

- In general, consider

$$\vec{f} : U \to \mathbb{C}^n, \quad A \in M_{n\times n}(\mathcal{O}(U)), \quad \frac{d}{dz}\vec{f}(z) = A(z)\vec{f}(z)$$

- For $D_\varepsilon = \{z \mid |z - z_0| < \varepsilon\} \subset U$,

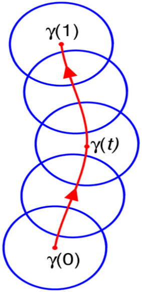
$$(**) \rightsquigarrow (***) \quad \vec{f}(z) = \vec{f_0} + \vec{f_1}\,(z - z_0) + \vec{f_2}\,(z - z_0)^2 + \cdots \quad \vec{f_i} \in \mathbb{C}^n$$

Given $\vec{f_0}$, there exists a unique $\vec{f}(z) \in \mathcal{O}\,(D_\varepsilon)^n$ s.t. $\vec{f}\,(z_0) = \vec{f_0}$.

Hilbert's 21st problem

- For a path $\gamma : [0,1] \to U$, analytic continuation gives rise to a linear transformation

$$\rho(\gamma) : \mathbb{C}^n \to \mathbb{C}^n, \quad \vec{f}(\gamma(0)) \mapsto \vec{f}(\gamma(1))$$



Fact: $\rho(\gamma)$ depends only on the homotopy class of γ.

- This induces a representation

$$\rho : \pi_1(U, z_0) \to \mathrm{GL}_n(\mathbb{C}), \qquad [\gamma] \mapsto \rho(\gamma),$$

called the monodromy representation.

E.g. $zf'(z) = af(z), \quad U = \mathbb{C} - \{0\}$

$$f(z) = z^a \text{ (not well-defined if } a \notin \mathbb{Z}) = \exp(a \log z)$$

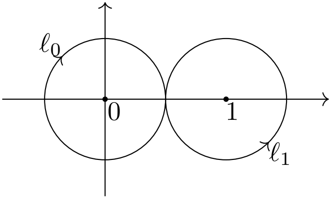
$$\rightsquigarrow \rho : \pi_1(\mathbb{C} - \{0\}) \cong \mathbb{Z} \longrightarrow \mathbb{C}^\times, \qquad 1 \longmapsto e^{2\pi i a}$$

Hilbert's 21st problem

- E.g. hypergeometric equation: $(a, b, c) = \left(\frac{1}{2}, \frac{1}{2}, 1\right)$

$$\rho : \pi_1(\mathbb{C} - \{0, 1\}) \cong \langle l_0, l_1 \rangle \to \mathrm{GL}_2(\mathbb{C})$$

$$\rho\,(l_0) = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \qquad \rho\,(l_\infty) = \begin{pmatrix} -1 & 0 \\ 2 & -1 \end{pmatrix}, \quad l_0 l_1 l_\infty = 1$$



- Q: which representations of π_1 come from (how many) differential equations?

  E.g. both $f' = 0$ $(f = c)$ and $f' = f$ $(f = ce^z)$ give rise to trivial monodromy representations.

Hilbert's 21st problem

- $U = \mathbb{C} - \{z_1, \ldots, z_r\} = \mathbb{P}^1 - \{z_1, \ldots, z_r, z_{r+1}(= \infty)\}$, $A(z)$ is meromorphic on $\mathbb{P}^1$, holomorphic on U.
- z_j is called a regular singular point if over any sector

$$\{z = z_j + re^{i\theta} | a < \theta < b < a + \pi\},$$

  for any solution $\vec{f}$ of $(**)$, $|\vec{f}(z)| < C|z - z_j|^{-N}$ for some $C, N > 0$ as $z \to z_j$.
- Fuchs criterion: for (*), z_j is regular if and only if $a_i(z)$ has a pole of order at most i at z_j.
- E.g. hypergeometric differential equations have regular singularities at $0, 1, \infty$; Bessel equation has regular singularity at 0, irregular singularity at ∞.

Hilbert's 21st problem

- Hilbert's 21st Problem: "In the theory of linear differential equations with one independent variable z, I wish to indicate an important problem one which very likely Riemann himself may have had in mind. This problem is as follows: To show that there always exists a linear differential equation of the Fuchsian class, with given singular points and monodromic group......"

- Q: Given $\rho : \pi_1(\mathbb{P}^1 - \{z_1, \ldots, z_r, \infty\}) \to \mathrm{GL}_n(\mathbb{C})$, does there exist a linear ODE with monodromy representation ρ and regular singularities at $z_1, \ldots, z_r, \infty$? If so, is it unique in a suitable sense?

  A: Yes, there is a unique one (in a suitable sense)!

Deligne's Riemann-Hilbert correspondence

- X : connected smooth complex algebraic variety
- $(\mathcal{E}, \nabla)$: vector bundle with a *flat connection* on X (fancy version of differential equations)

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes \Omega^1_X, \quad \nabla(fe) = f \cdot \nabla e + e \otimes df$$

$$\text{flatness} : \mathcal{E} \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^1_X \xrightarrow{\nabla} \mathcal{E} \otimes \Omega^2_X \to \cdots, \quad \nabla^2 = 0$$

Suppose $(x_1, \ldots, x_d)$ form a system of local coordinates,

$$\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{O}_X \cdot v_i, \quad e = v \cdot \vec{f}, \quad \nabla_{\frac{\partial}{\partial x_i}}(\vec{f}) = A_i \vec{f}, \quad A_i \in M_{n \times n}(\mathcal{O}_X).$$

- $(\mathcal{E}, \nabla)$ is called of *regular singularities* if for any algebraic map $\Sigma \to X$ from a Riemann surface Σ, $(\mathcal{E}, \nabla)|_\Sigma$ is of regular singularities.
- Existence and uniqueness of solutions $\rightsquigarrow$ monodromy representation $\rho : \pi_1(X^{\mathrm{top}}) \to \mathrm{GL}_n(\mathbb{C})$ of $(\mathcal{E}, \nabla)$.

Deligne's Riemann-Hilbert correspondence

Theorem (Deligne)

Let X be a connected smooth complex algebraic variety. There is an equivalence of categories

$$\left\{ \begin{array}{c} \textit{vector bundles with flat connections} \\ \textit{and regular singularities} \end{array} \right\} \simeq \left\{ \begin{array}{c} \textit{finite-dimensional complex} \\ \textit{representations of } \pi_1(X^{\mathrm{top}}) \end{array} \right\}$$

- The RHS is just the local systems of finite-dimensional $\mathbb{C}$-vector spaces on X^{top}.
- $(\mathcal{E}, \nabla) \mapsto$ monodromy representation ρ (given by $\mathcal{E}^{\nabla=0}$)
- $(\mathcal{O}_X, d) \mapsto$ trivial representation of $\pi_1(X^{\mathrm{top}}) \mapsto$ constant sheaf $\underline{\mathbb{C}}$ $\rightsquigarrow$ de Rham comparison theorem:

$$\begin{array}{ccc} \mathrm{Ext}^i((\mathcal{O}_X, d), (\mathcal{O}_X, d)) & \xrightarrow{\simeq} & \mathrm{Ext}^i(\underline{\mathbb{C}}, \underline{\mathbb{C}}) \\ \| & & \| \\ \mathrm{H}^i_{\mathrm{dR}}(X, \mathbb{C}) & \xrightarrow{\simeq} & \mathrm{H}^i_{\mathrm{Betti}}(X, \mathbb{C}) \end{array}$$

p-adic numbers

- Let p be a prime number. It gives rise to the p-adic valuation

$$v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}, \quad v_p(a/b) = m - n, \quad p^m \parallel a, \quad p^n \parallel b$$

  ($v_p(0) = \infty$); independent of the choice of the presentation a/b. It satisfies

$$v_p(xy) = v_p(x) + v_p(y), \quad v_p(x+y) \geq \min\{v_p(x), v_p(y)\}.$$

- Define the p-adic absolute value $|\cdot|_p = p^{-v_p(\cdot)}$, which satisfies

$$|xy|_p = |x|_p|y|_p$$

  and the strong triangle inequality

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}.$$

- $$|9|_3 = |\frac{9}{5}|_3 = \frac{1}{9}, \qquad |\frac{1}{4}|_2 = 4.$$

p-adic numbers

Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|\cdot|_p$. The following theorem was proved by Ostrowski in 1916.

Theorem

Every non-trivial absolute value on $\mathbb{Q}$ is equivalent to either the natural absolute value $|\cdot|_\infty$ or the p-adic absolute value $|\cdot|_p$ for some prime p.

- The key difference between $|\cdot|_\infty$ and $|\cdot|_p$:

$$|n|_\infty \to \infty, \qquad \text{(Archimedean)};$$

$$|n|_p \leq 1, \qquad \text{(Non-Archimedean)}.$$

- This results in some "pathologies" in p-adic geometry from the point view of real (complex) geometry. Easy example: inside a p-adic disk or ball, every point is a center.
- Product formula: $a \in \mathbb{Q} \rightsquigarrow \prod_{p \leq \infty} |a|_p = 1$.

A p-adic Riemann-Hilbert functor

Let k be a finite extension of $\mathbb{Q}_p$.

Theorem (Diao-Lan-L.-Zhu, 2023)

Let X be a smooth algebraic variety over k. Then there is a tensor functor $D_{\mathrm{dR}}^{\mathrm{alg}}$:

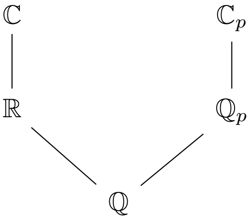
$$\left\{ \begin{array}{c} p\text{-adic de Rham} \\ \text{local systems on } X_{\mathrm{et}} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{vector bundles with filtered flat} \\ \text{connections and regular singularities} \end{array} \right\}$$

- The functor $D_{\mathrm{dR}}^{\mathrm{alg}}$ may be regarded as a p-adic analogue of Deligne's Riemann-Hilbert correspondence.
- In addition, there is a p-adic de Rham comparison isomorphism

$$H^i_{\mathrm{et}}(X_{\bar{k}}, \mathbb{L}) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}} \cong H^i_{\mathrm{dR}}(X, D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L})) \otimes_k B_{\mathrm{dR}},$$

which may be regarded as a culmination of the works of Fontaine, Faltings, Tsuji, Beilinson and Scholze et al. on de Rham comparison.

A p-adic Riemann-Hilbert functor

- 

  $\overline{\mathbb{Q}_p}$ is an infinite extension of $\mathbb{Q}_p$ and is not complete! $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ is complete and algebraically closed.
- The p-adic analogue of complex numbers is Fontaine's p-adic de Rham period ring B_{dR} ($\neq \mathbb{C}_p$!).
- The proof builds upon recent advances in p-adic geometry: pefectoid spaces, relative p-adic Hodge theory （Scholze, Kedlaya-L.） ......

Applications

- Fontaine-Mazur conjecture: global p-adic Galois representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\overline{\mathbb{Q}}_p),$$

  which are geometric in the sense of Fontaine-Mazur (p-adic Hodge theory), ought to come from p-adic étale cohomology of algebraic varieties over $\mathbb{Q}$.

- The Fontaine-Mazur conjecture is a vast generalization of the Taniyama-Shimura-Weil conjecture: if E is an elliptic curve over $\mathbb{Q}$, then there is an integer $N \geq 1$ and a weight-two cuspidal Hecke eigenform f of level N, such that $a_p(E) = a_p(f)$, for all primes p of good reduction for E.

- The Taniyama-Shimura-Weil conjecture is proved by Wiles and Taylor-Wiles (Fermat's Last Theorem), Breuil-Conrad-Diamond-Taylor.

Applications

- The Fontaine-Mazur conjecture is a far-reaching central problem in the Langlands program for number theory.

Theorem (rigidity, L.-Zhu, 2017)

Let X be a connected algebraic variety over a number field E and let $\mathbb{L}$ be a p-adic étale local system on X. If for some $x \in X$, the stalk $\mathbb{L}_{\bar{x}}$ is geometric in the sense of Fontaine-Mazur, then so is the stalk $\mathbb{L}_{\bar{y}}$ for any $y \in |X|$.

- Based on the above observation, we are dared to make the following stronger conjecture

Conjecture (relative Fontaine-Mazur, L.-Zhu, 2017)

Let $\mathbb{L}$ be a geometric p-adic étale local system over a geometrically connected smooth algebraic variety X over E. Let η be the generic point of X. Then there exists some algebraic variety Y over η such that $\mathbb{L}_{\bar{\eta}}$ appears as a subquotient of the étale cohomology of $Y_{\bar{\eta}}$.

Applications

- Let $\mathbb{L}$ be a geometric p-adic étale local system over a geometrically connected smooth algebraic variety X over E.

- Fix an isomorphism $\iota : \overline{\mathbb{Q}}_p \cong \mathbb{C}$ and a field homomorphism $\sigma : E \to \mathbb{C}$. Let v be the p-adic place of E determined by $\iota^{-1} \circ \sigma$. The following conjecture compares the classical and p-adic Riemann-Hilbert:

Conjecture (Diao-Lan-L.-Zhu, 2023)

The flat connections $D(\mathbb{L}|_{\sigma X} \otimes_\iota \mathbb{C})$ and $D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L}|_{X_{E_v}}) \otimes_{E_v,\iota} \mathbb{C}$ are canonically isomorphic. Moreover, $(D_{\mathrm{dR}}^{\mathrm{alg}}(\mathbb{L}|_{X_{E_v}}) \otimes_{E_v,\iota} \mathbb{C}, \mathrm{Fil}^\bullet)$ is a complex variation of Hodge structures.

- The above conjecture relates "periods" on different places (such as the product formula......).

Applications

Let (G, X) be a Shimura datum. For a neat open compact subgroup $K \subset G(\mathbb{A}_f)$, let $\mathrm{Sh}_K(G, X)$ be the (canonical model of) corresponding Shimura variety. Let $\mathbb{L}_{V,p}$ be the p-adic étale local system over $\mathrm{Sh}_K(G, X)$ induced by a $\mathbb{Q}$-rational representation V of G.

Theorem (L.-Zhu, 2017; Diao-Lan-L.-Zhu, 2023)

1. *The p-adic etale local system $\mathbb{L}_{V,p}$ is geometric.*
2. *The second conjecture holds for $\mathbb{L}_{V,p}$.*

- The cohomology of $\mathbb{L}_{V,p}$ are expected to realize the Langlands correspondence for the reductive group G.
- A crucial ingredient of (2) is Margulis' superrigidity theorem.
- In the recent work of Pila-Shankar-Tsimerman on Andre-Oort for general Shimura varieties, this theorem is applied to construct canonical heights on *arbitrary* Shimura varieties.

Heights function over projective spaces

- Height over $\mathbb{Q}$: for $(p,q)=1$, $H(\frac{p}{q})=\max\{|p|_\infty,|q|_\infty\}$.
- Height over $\mathbb{P}^n(\mathbb{Q})$: $H([x_0,x_1,\ldots,x_n])=\max\{|x_0|_\infty,|x_1|_\infty,\ldots,|x_n|_\infty\}$ with relatively coprime integers $x_0,x_1,\ldots,x_n$.
- A more canonical definition: for $[x_0,x_1,\ldots,x_n]\in\mathbb{P}^n(\mathbb{Q})$,

$$H([x_0,x_1,\ldots,x_n])=\prod_{p\leq\infty}\max\{|x_0|_p,|x_1|_p,\ldots,|x_n|_p\}.$$

The two definitions coincide due to the product formula!

Thank you for your attention!