Trapping wave fields in an expulsive potential by means of linear coupling

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(1) Introduction

The **nonlinear Schrödinger** (**NLS**) equation including a trapping (**harmonic-oscillator**) potential is a commonly known model with many physical realizations, such as waveguides for photonic and matter waves (**BEC**):

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}-\frac{1}{2}x^2u+\sigma|u|^2u=0,$$

with ***σ*** **= +1** (***self-focusing***), **-1** (***defocusing***), or **0** (the **linear Schrödinger** equation). The equation is written in the notation adjusted to ***optics in the spatial domain***, ***z*** being the propagation distance. In terms of BEC, it is the **Gross-Pitaevskii** equation, with ***z*** replaced by time, ***t***.

A straightforward generalization is a ***symmetric*** system of two ***linearly-coupled*** **NLS** equations for wave fields ***u*** and ***v***, which models a set of two ***parallel waveguides*** **("*cores*")** ***coupled by tunneling*** of photons (in optics) or atoms (in **BEC**):

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\lambda v-\frac{1}{2}x^2 u+\sigma|u|^2 u=0,$$

$$\mathrm{i}\frac{\partial v}{\partial z}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+\lambda u-\frac{1}{2}x^2 v+\sigma|v|^2 v=0,$$

where real $\lambda>0$ is the coupling constant.

In terms of **BEC** (but **not in optics**), it is also relevant to consider a two-dimensional (**2D**) version of the system, with the evolution variable ***z*** replaced by time ***t***, and the **2D isotropic trapping potential**. The
2D system for a set of **parallel BEC layers** coupled by tunneling of atoms is written in **polar coordinates** (***r***,***θ***):

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)u+\lambda v-\frac{1}{2}r^2u+\sigma|u|^2u=0,$$

$$\mathrm{i}\frac{\partial v}{\partial t}+\frac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)v+\lambda u-\frac{1}{2}r^2v+\sigma|v|^2v=0.$$

The **2D** system admits ***vortex solutions*** (which carry the **angular momentum**), in the form of

$$\{u(r,\theta,t)\} = \exp\left(-i\mu t + iS\theta\right)\{U(r), V(r)\},$$

where real μ is the chemical potential (-μ is the propagation constant, alias wavenumber, in terms of the optics model),

integer S is the **vorticity** (winding number),

and real functions $U(r)$ and $V(r)$ satisfy the radial equations:

$$\mu U + \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{S^2}{r^2}\right)U + \lambda V - \frac{1}{2}r^2 U + \sigma U^3 = 0,$$

$$\mu V + \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{S^2}{r^2}\right)V + \lambda U - \frac{1}{2}r^2 V + \sigma V^3 = 0.$$

In the case of the **self-attractive nonlinearity** ($\sigma = +1$), the ***interplay*** between the ***intra-core self-attraction*** and ***inter-core linear coupling*** gives rise to ***spontaneous symmetry breaking***, in the **1D** and **2D** systems alike:

PHYSICAL REVIEW A **96**, 033621 (2017)

Spontaneous symmetry breaking of fundamental states, vortices, and dipoles in two- and one-dimensional linearly coupled traps with cubic self-attraction

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A dynamical effect: ***Josephson oscillations*** between the linearly coupled cores, if the input is loaded into one core of the **1D** system (**Symmetry** **13**, 372 (2021)):

symmetry

MDPI

Article

Nonlinear Dynamics of Wave Packets in Tunnel-Coupled Harmonic-Oscillator Traps

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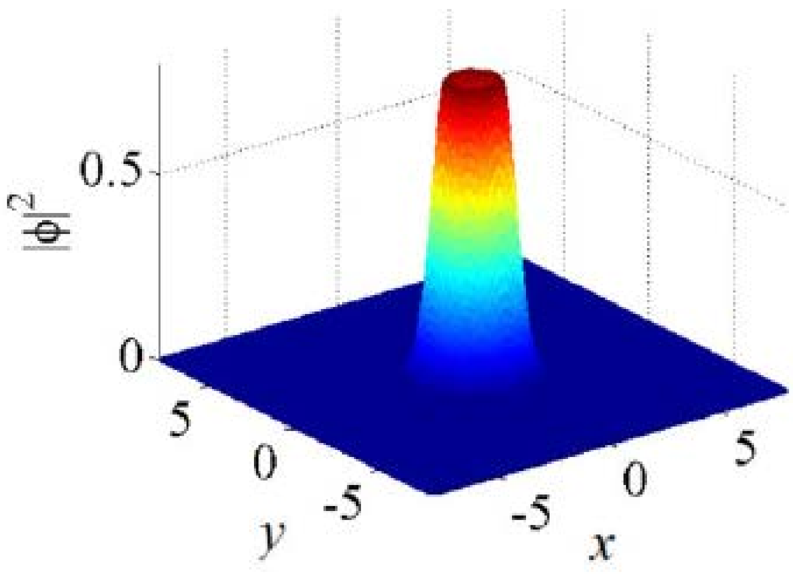
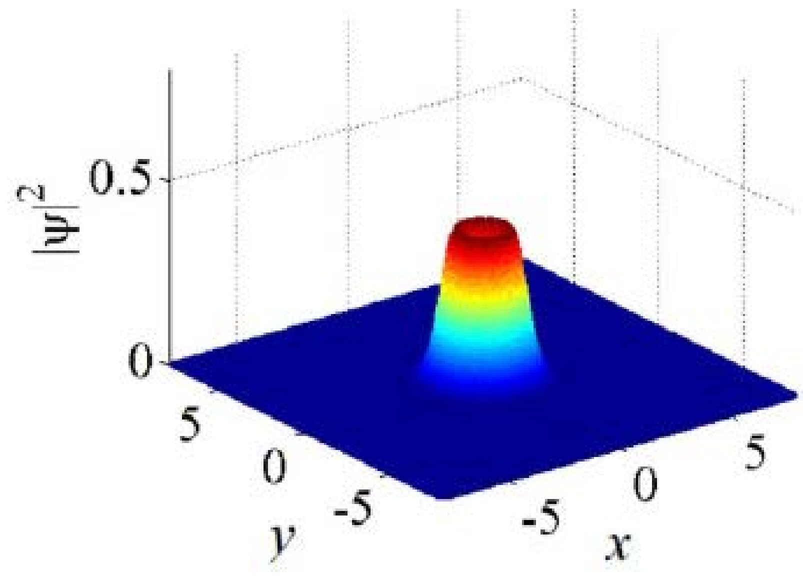
An example of a state with ***broken symmetry***: an ***asymmetric stable*** **vortex mode** with ***S* = 1** and ***different amplitudes*** of the two components, produced by the **2D system** with the **inter-core coupling constant *λ* = 0.4.** The total norm of the state is

$$N = N_u + N_v \equiv 2\pi \int_0^\infty \left[U^2(r) + V^2(r) \right] r dr \approx 8.8.$$

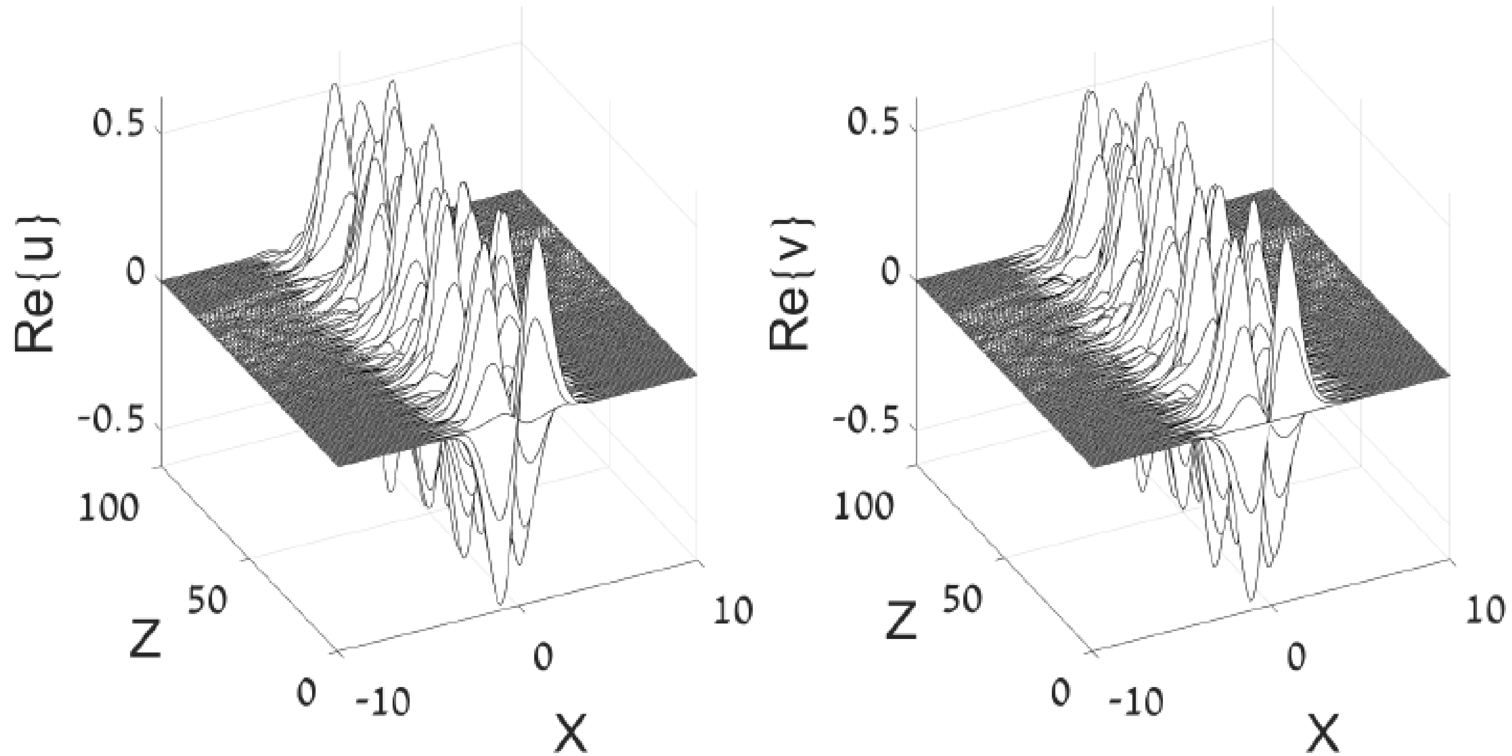
The **asymmetric vortices** with $S = 1$ exist (i.e., the **symmetry breaking takes place**) at

$$N > N_{\mathrm{cr}} \approx 0.57 + 19.06\lambda.$$

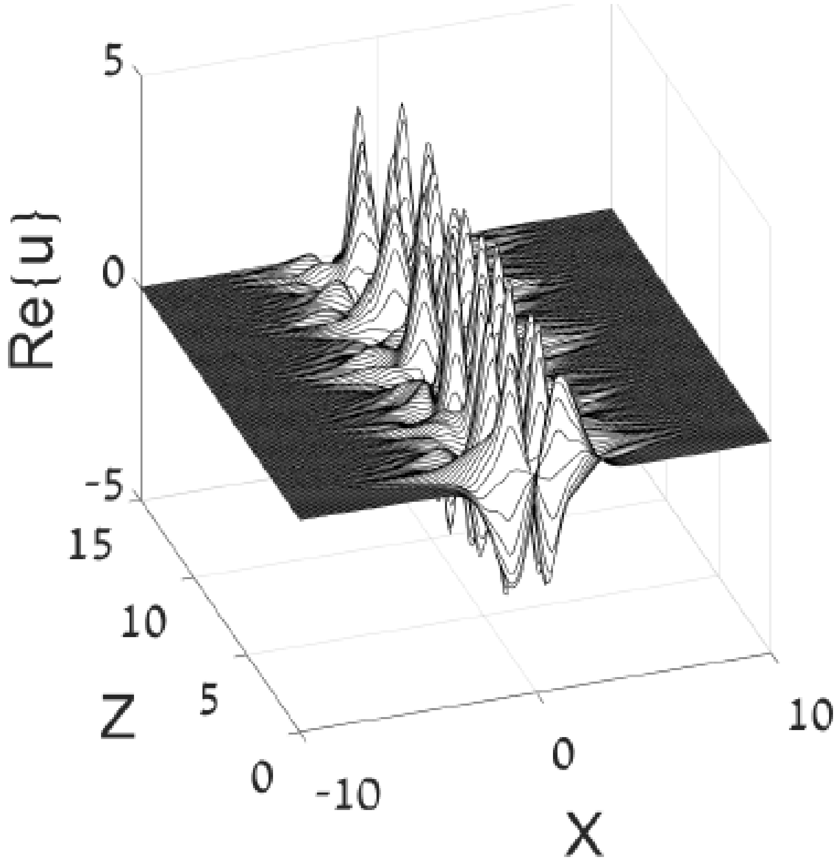


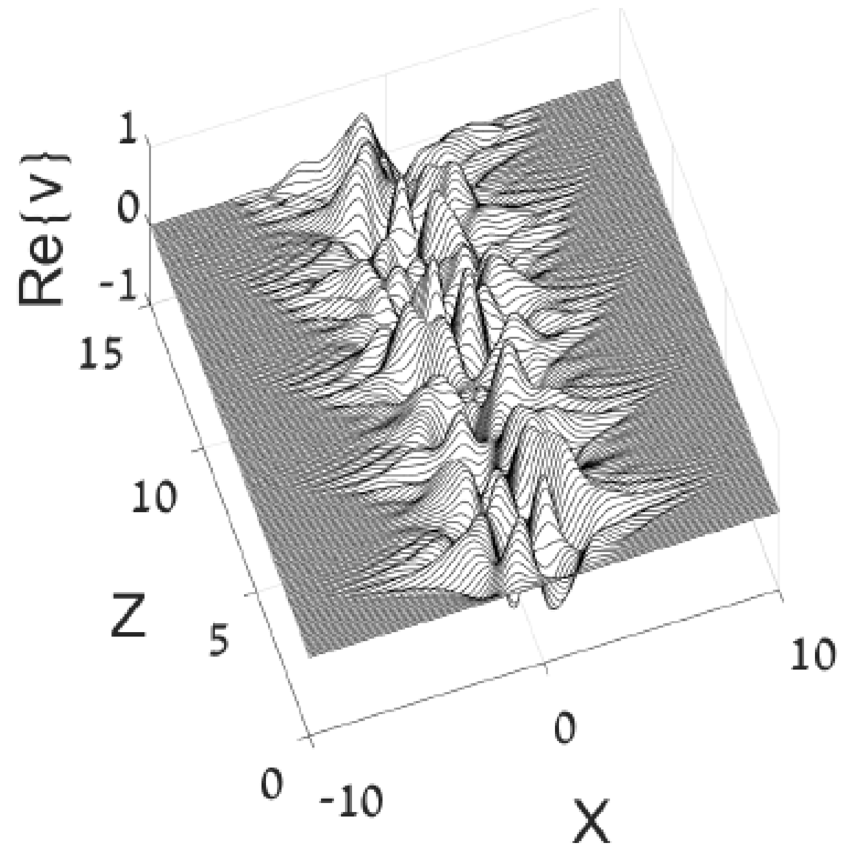


Below a critical value of the norm, the system performs ***regular*** (***non-chaotic***) Josephson oscillations, maintaining ***dynamical symmetry*** between the two components (cores):



Above the critical norm, the system performs ***chaotic*** oscillations, which ***spontaneously break*** the dynamical symmetry between the component:





(2) A new model: an *asymmetric linearly coupled system* with the *trapping* (harmonic-oscillator, HO) potential in one core, and an *expulsive* (*inverted HO*) potential in the other

The results have been reported in:

PHYSICAL REVIEW E **105**, 034213 (2022)

Trapping wave fields in an expulsive potential by means of linear coupling

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Expulsive potentials (in the **1D NLS** equation) were considered in optics, as they create ***anti-waveguiding structures***, that may find various applications in design of ***all-optical signal-processing systems***:

B. V. Gisin and A. A. Hardy, Stationary solutions of plane nonlinear-optical antiwaveguides, Opt. Quant. Electron **27**, 565 (1995).

B. V. Gisin, A. Kaplan, and B. A. Malomed, Spontaneous symmetry breaking and switching in planar nonlinear optical antiwaveguides, Phys. Rev. E **62**, 2804 (2000).

D. Bortman-Arbiv, A. D. Wilson-Gordon, and H. Friedmann, Strong parametric amplification by spatial soliton-induced cloning of transverse beam profiles in an all-optical antiwaveguide, Phys. Rev. A **63**, 031801(R) (2001).

O. N. Verma and T. N. Dey, Steering, splitting, and cloning of an optical beam in a coherently driven Raman gain system, Phys. Rev. A **91**, 013820 (2015).

A. Kaplan, B. V. Gisin, and B. A. Malomed, Stable propagation and all-optical switching in planar waveguide-antiwaveguide periodic structures, J. Opt. Soc. Am. B **19**, 522 (2002).

Expulsive potentials were also studied in various contexts in similar **BEC** models:

L. D. Carr and Y. Castin, Dynamics of a matter-wave bright soliton in an expulsive potential, Phys. Rev. A **66**, 063602 (2002).

L. Salasnich, Dynamics of a Bose-Einstein-condensate bright soliton in an expulsive potential, Phys. Rev. A **70**, 053617 (2004).

Z. X. Liang, Z. D. Zhang, and W. M. Liu, Dynamics of a Bright Soliton in Bose-Einstein Condensates with Time-Dependent Atomic Scattering Length in an Expulsive Parabolic Potential, Phys. Rev. Lett. **94**, 050402 (2005).

The system of the coupled equations with ***trapping*** and ***anti-trapping*** potentials:

$$i\frac{\partial u}{\partial z}+\frac{1}{2}\frac{\partial^2 u}{\partial x^2}+\lambda v-\frac{1}{2}x^2u+\sigma|u|^2u=-\omega u,$$

$$i\frac{\partial v}{\partial z}+\frac{1}{2}\frac{\partial^2 v}{\partial x^2}+\lambda u+\frac{\kappa}{2}x^2v+\sigma|v|^2v=0,$$

where ω is a possible *mismatch* between the coupled components, and $\kappa>0$ is the strength of the **expulsive** potential acting in the *v*-component.

The main question: can the linear coupling maintain **stable two-component bound (localized) states**, in spite of the obvious **delocalization effect** produced by the expulsive potential, in such **1D** and **2D** coupled systems?

In optics, the physical realization of such a **1D** system is obvious: a planar ***dual-core coupler***, with the ***waveguiding*** and ***antiwaveguiding*** structures induced by the corresponding patterns of the transversely modulated refractive index in the coupled cores.

The **2D** variant of the system cannot be realized in optics, but ***it is possible*** (as well as the **1D case**) in **BEC**: ***trapping*** and ***antitrapping*** optical potentials may be induced, respectively, by ***red***- and ***blue***-detuned laser beams focused on two **tunnel-coupled** parallel layers of **BEC**. The separation between the layers is expected to be **a few microns**, which is also **sufficient** to separate (resolve) the two optical trapping patterns.

Stationary solutions for the linearly-coupled **1D** system, with propagation constant ***-μ*** (in terms of BEC, with ***z*** replaced by ***t***, ***μ*** is the chemical potential), are looked for as:

$$\{u(x,z),v(x,z)\}=\{U(x),V(x)\}\exp\left(-i\mu z\right),$$

with real functions $U(x)$ and $V(x)$ satisfying equations

$$\left(\mu+\omega\right)U+\frac{1}{2}\frac{d^2U}{dx^2}+\lambda V-\frac{1}{2}x^2U+\sigma U^3=0,$$

$$\mu V+\frac{1}{2}\frac{d^2V}{dx^2}+\lambda U+\frac{\kappa}{2}x^2V+\sigma V^3=0,$$

In terms of this system of equations, a mathematical problem is: does the system give rise to solutions which are localized at $|x|\ \rightarrow\infty$ **in both components**, while the potential term $\sim\kappa$ **tends to expel** the *v* component?

The plan of the subsequent presentation:

(3) Exact **non-generic** solutions for the bound states in the **1D linear system**.

(4) The variational (***Rayleigh-Ritz***) approximation and numerical results for **generic** bound states in the **linear system**.

(5) Coexistence of the **discrete 1D** bound states with the **continuum** of **delocalized** (unbound) states (the realization of "***bound states in the continuum***").

(6) Nonlinear effects in the **1D** system.

(7) **2D** systems (linear and nonlinear): vorticity, stability, etc.

(8) Conclusion.

(3) Exceptional (*codimension-1*) exact solutions of the 1D linearized system

First of all, to confirm the existence of the bound states in the system, it is possible to find an ***exact*** spatially-symmetric (***even***) solution of the ***linearized*** (**$\sigma = 0$**) coupled system, which is valid **under a special condition** imposed on $\boldsymbol{\omega}$ and $\boldsymbol{\lambda}$ (while $\boldsymbol{V_0}$ is an **arbitrary amplitude**):

$$U(x) = \left(U_0 + U_2 x^2\right) \exp\left(-\frac{x^2}{2}\right),$$

$$V(x) = V_0 \exp\left(-\frac{x^2}{2}\right),$$

$$U_0 = \frac{1 - 2\lambda^2 + \kappa}{4\lambda} V_0,$$

$$U_2 = -\frac{1 + \kappa}{2\lambda} V_0,$$

$$\mu_{\text{even}} = \frac{1}{2}\left(\lambda^2 + \frac{1}{2}\right) - \frac{\kappa}{4},$$

This solution exists under the ***restriction*** imposed on the parameters (note that the ***restriction*** may hold for ***arbitrarily large*** values of strength ***κ*** of the expulsive potential):

$$\omega_{\text{even}} = \frac{9}{4} - \frac{\lambda^2}{2} + \frac{\kappa}{4}.$$

Therefore it is categorized as a **codimension-1** exact solution.

The ratio of norms of the **trapped** and **anti-trapped** components in the exact solution:

$$\frac{N_u}{N_v} = \frac{\lambda^2}{4} + \frac{(1+\kappa)^2}{8\lambda^2}.$$

The trapped component (u) is a **dominant** one, with $N_u > N_v$,

if the strength of the expulsive potential is large enough,

$\kappa > 2\sqrt{2} - 1 \approx 1.83$. Otherwise, $N_u < N_v$ is possible.

It is also possible to find an ***exact solution*** of the linearized system for a **spatially-odd** (antisymmetric, alias ***dipole***) mode, with an arbitrary amplitude $\boldsymbol{V_1}$:

$$U(x) = \left(U_1 x + U_3 x^3\right)\exp\left(-x^2/2\right),$$

$$V(x) = V_1 x \exp\left(-x^2/2\right),$$

$$U_1 = (4\lambda)^{-1}\left(3 - 2\lambda^2 + 3\kappa\right)V_1,$$

$$U_3 = -(2\lambda)^{-1}(1+\kappa)V_1,$$

$$\mu_{\text{odd}} = \frac{1}{2}\left(\lambda^2 + \frac{3}{2}\right) - \frac{3}{4}\kappa.$$

This exact solution too exists under the ***restriction*** imposed on the parameters (and again, the ***restriction*** may hold for ***arbitrarily large*** values of strength $\boldsymbol{\kappa}$ of the expulsive potential):

$$\omega_{\text{odd}} = \frac{11}{4} - \frac{\lambda^2}{2} + \frac{3\kappa}{4}.$$

(4) To construct *generic* bound eigenstates of the **linear system**, one can use the **variational (**alias **Rayleigh-Ritz) approximation** (**VA**). It is based on the integral expression for $\boldsymbol{\mu}$ following from the stationary equations:

$$\mu = -\int_{-\infty}^{+\infty} dx \left[U \left(\omega U + \frac{1}{2}\frac{d^2U}{dx^2} - \frac{x^2}{2}U \right) + V \left(\frac{1}{2}\frac{d^2V}{dx^2} + \frac{\kappa x^2}{2}V \right) + 2\pi UV \right]$$

(assuming that the total norm of the wave function is $N \equiv N_u + N_v = 1$).

The variational *ansatz* for the **spatially even** eigenstates with free parameter η is adopted as

$$\{U_{\mathrm{VA}}(x), V_{\mathrm{VA}}(x)\} = \pi^{-1/4}\{\cos\eta, \sin\eta\}\exp\left(-x^2/2\right).$$

The substitution of the *ansatz* in the expession for μ yields

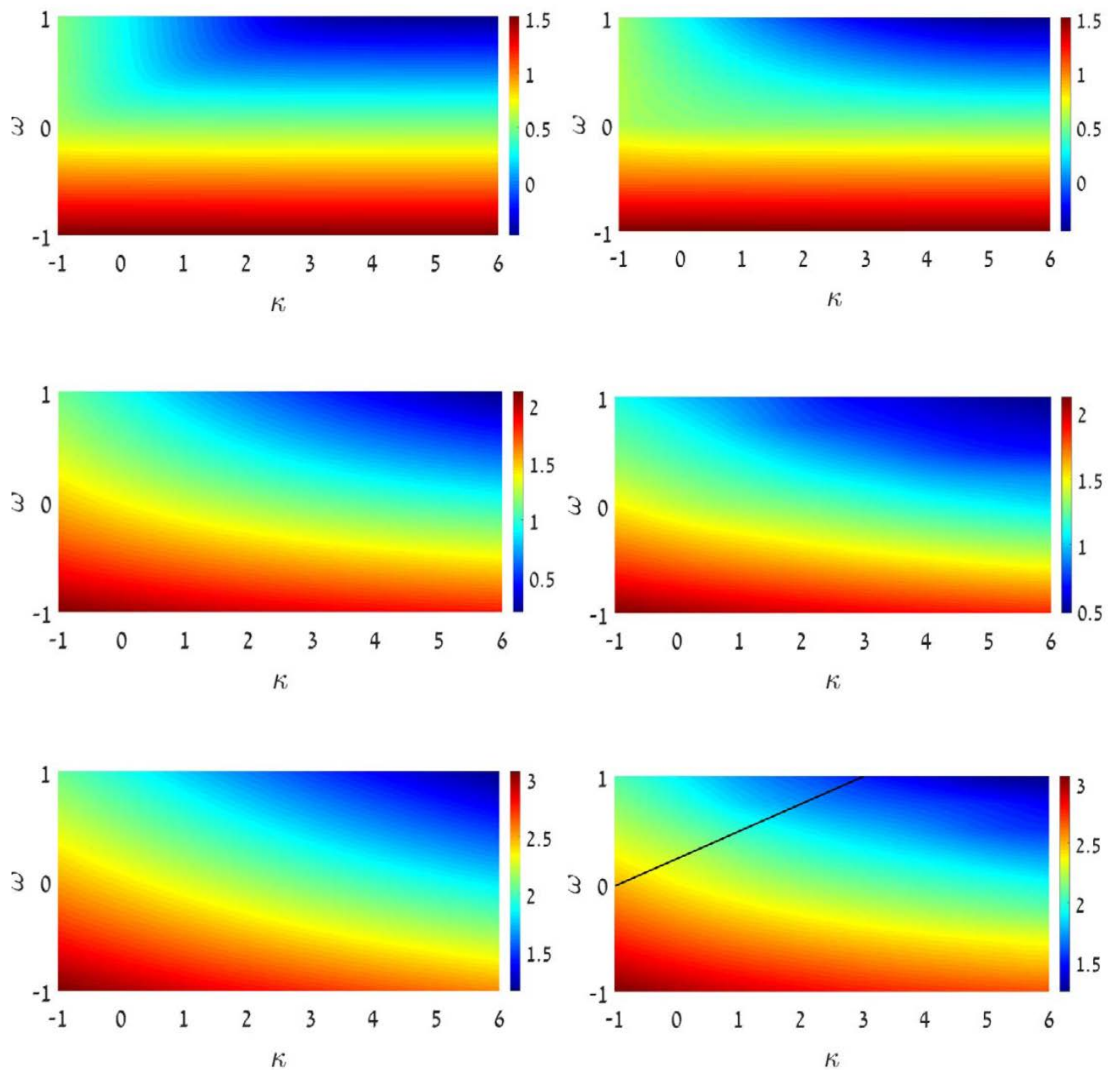
$$\mu_{\mathrm{VA}} = \left(1/2 - \omega\right)\cos^2\eta + (1/4)(1-\kappa)\sin^2\eta - \lambda\sin\left(2\eta\right).$$

The **variational equation** is $d\mu_{\mathrm{VA}}/d\eta = 0$. It yields

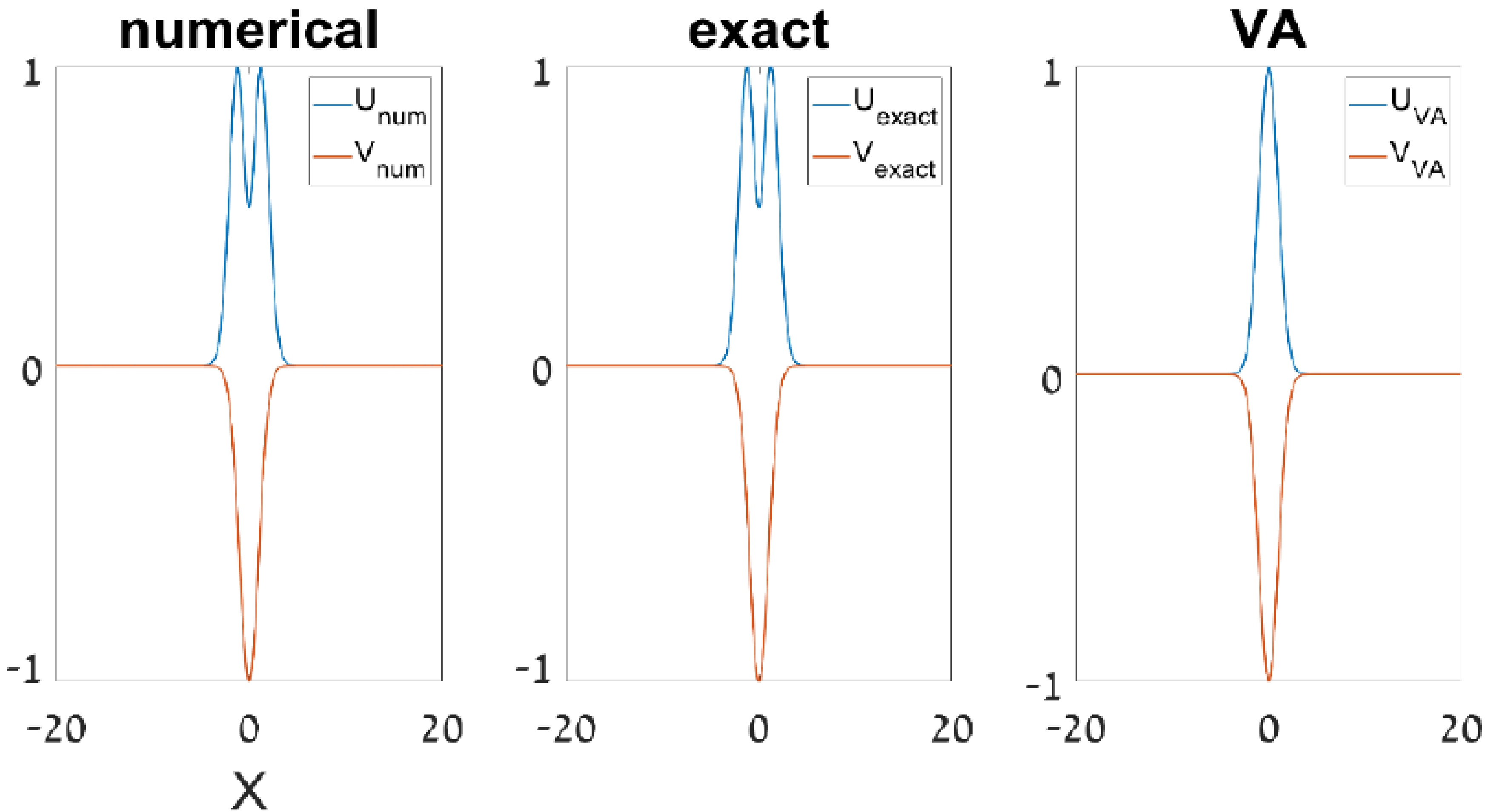
$$\mu_{\mathrm{VA}} = \left(1/2 - \omega - q\right) + \sqrt{q^2+\lambda^2},\ q \equiv (1/2)\left[(1/4)(1+\kappa) - \omega\right].$$

In particular, in the limit of $\kappa \to \infty$ (**very strong expulsive potential**) the bound state **persists**, with $\mu_{\mathrm{VA}} \approx 1/2 - \omega + 4\lambda^2/\kappa$, while the amplitude of the v component, which is subject to the action of the expulsive potential, **is vanishing**: $\eta \approx -4\lambda/\kappa$.

Heatmaps for the **VA-predicted** (**left**) and **numerically found** (**right**) eigenvalues ***μ*** in the (***κ***,***ω***) plane for ***λ* = 0.1, 1, 2** (top to bottom). The **black line** designates the **exact solution** of **codimension 1**.



Comparison of the variational, numerically found, and exact (**codimension-1**) shapes of the **spatially even** wave functions for **λ = 2, ω = 1, κ = 3**. All the three solutions have **μ = 1.5.** The **VA** predicts **μ** **accurately**, in spite of the discrepancy in the shape of the wave function.



The **VA** can be also developed for eigenvalues of the **spatially odd** (dipole) eigenstates, using the ansatz

$$\left\{U_{\mathrm{DM}}^{(\mathrm{VA})}(x), V_{\mathrm{DM}}^{(\mathrm{VA})}(x)\right\} = \sqrt{2}\pi^{-1/4}\{\cos\eta, \sin\eta\}x\exp\left(-\frac{x^2}{2}\right), \tag{53}$$

cf. Eq. (42), which is also subject to normalization Eq. (43). Substituting this in Eq. (44) yields

$$\mu_{\mathrm{DM}} = \left(\frac{3}{2} - \omega\right)\cos^2\eta + \frac{3}{4}(1-\kappa)\sin^2\eta - \lambda\sin(2\eta), \tag{54}$$

cf. Eq. (45). Then, the variational Eq. (46), applied to Eq. (54), produces the result

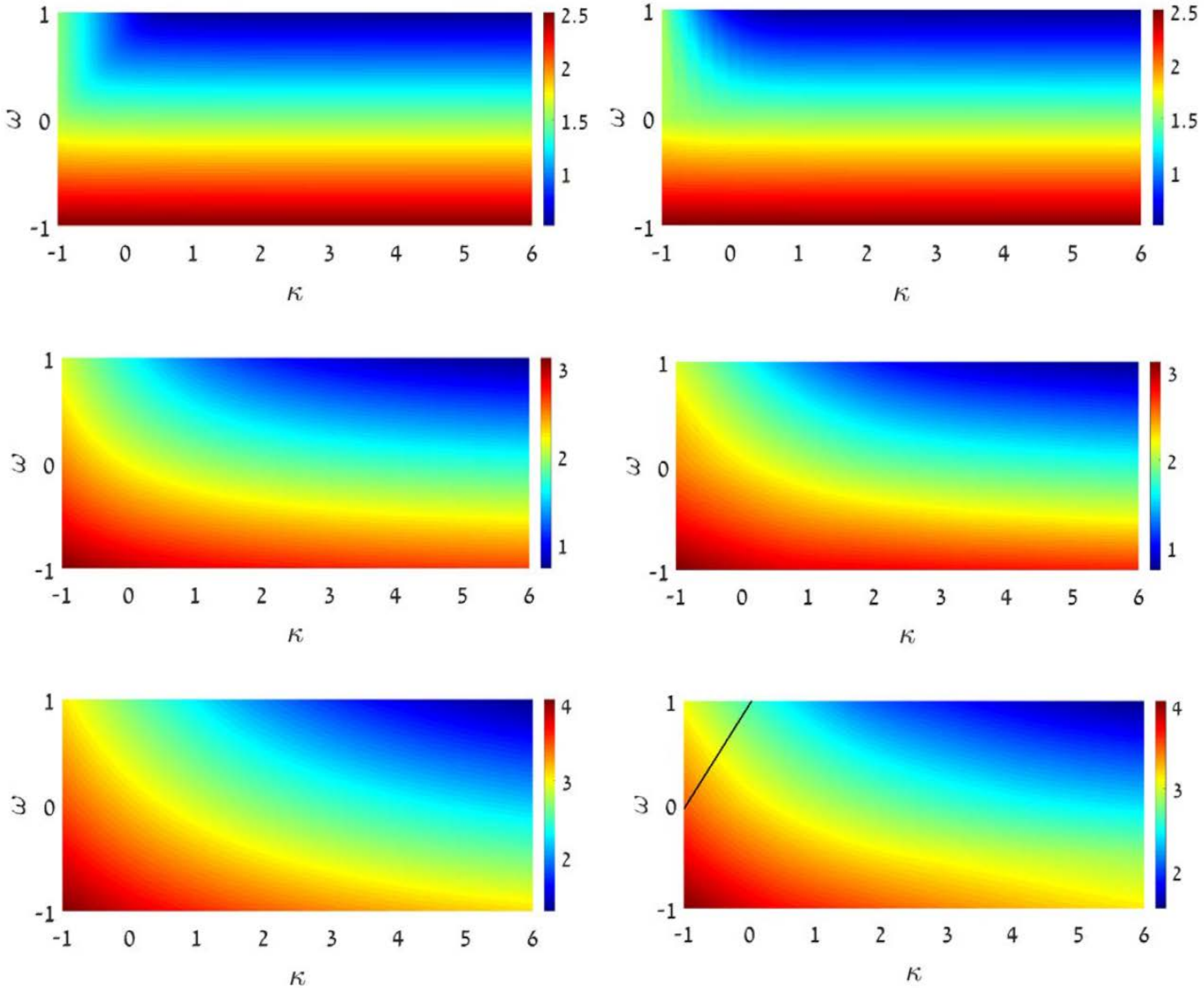
$$\tan(2\eta) = -\lambda/q_{\mathrm{DM}}, \tag{55}$$

$$q_{\mathrm{DM}} \equiv \frac{1}{2}\left[\frac{3}{4}(1+\kappa) - \omega\right], \tag{56}$$

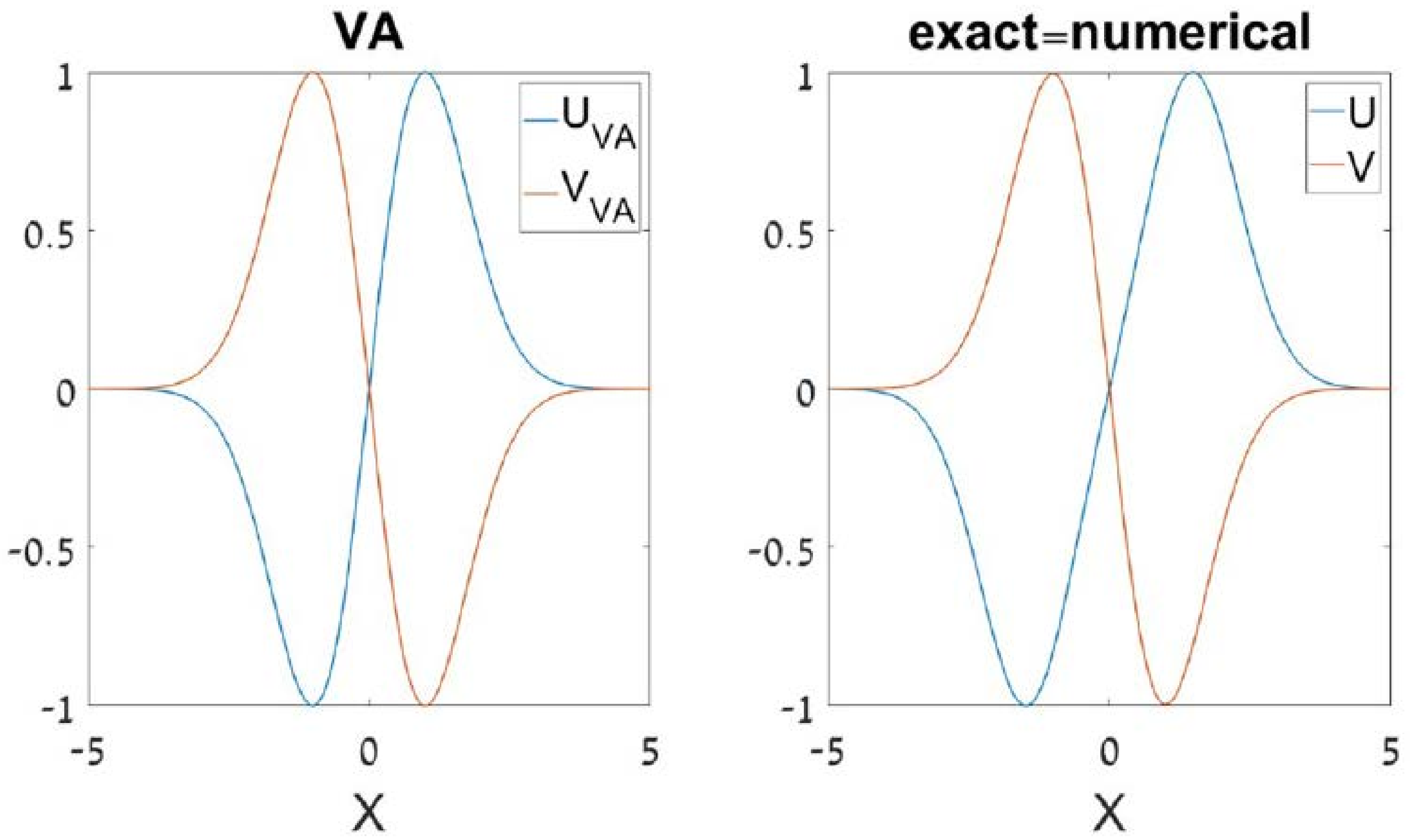
cf. Eqs. (47) and (48). The substitution of this in Eq. (54) leads to the following eigenvalue:

$$\mu_{\mathrm{DM}}^{(\mathrm{VA})} = \frac{3}{2} - \omega - q_{\mathrm{DM}} + \sqrt{q_{\mathrm{DM}}^2 + \lambda^2}, \tag{57}$$

The comparison of the variational (**left**) and numerical (**right**) results for **eigenvalues *μ*** of the **dipole** (spatially odd) **eigenstate**, for ***λ* = 0.1, 1, 2** (top to bottom), with the **black line** denoting the **exact solution of codimension 1** :



Comparison of the variational, numerically found, and exact shapes of the **spatially odd** eigenstates for $\boldsymbol{\lambda = 2, \omega = 1, \kappa = 1/3}$. All the three solutions have $\boldsymbol{\mu = 2.5}$.



(5) The same system admits a continuum of *delocalized states*, at all values of *μ*. Therefore, the localized eigenstates, existing at discrete values of ***μ***, may be categorized as ***bound states in the continuum*** (**BIC**), alias ***embedded states***, **cf**.

Stillinger, F.H.; Herrick, D.R. *Bound states in continuum*. Phys. Rev. A **11**, 446 (1975);

Kodigala, A.; Lepetit, T.; Gu, Q.; Bahari, B.; Fainman, Y.; Kante, B. *Lasing action from photonic bound states in continuum*. Nature **54**, 196 (2017).

Champneys, A.R.; Malomed, B.A.; Yang, J.; Kaup, D.J. *"Embedded solitons": solitary waves in resonance with the linear spectrum*. Physica D **152**, 340 (2001).

The analytical asymptotic form of the ***delocalized solutions***, which exist at ***all values*** of $\boldsymbol{\mu}$, and thus indeed form a ***continuum***:

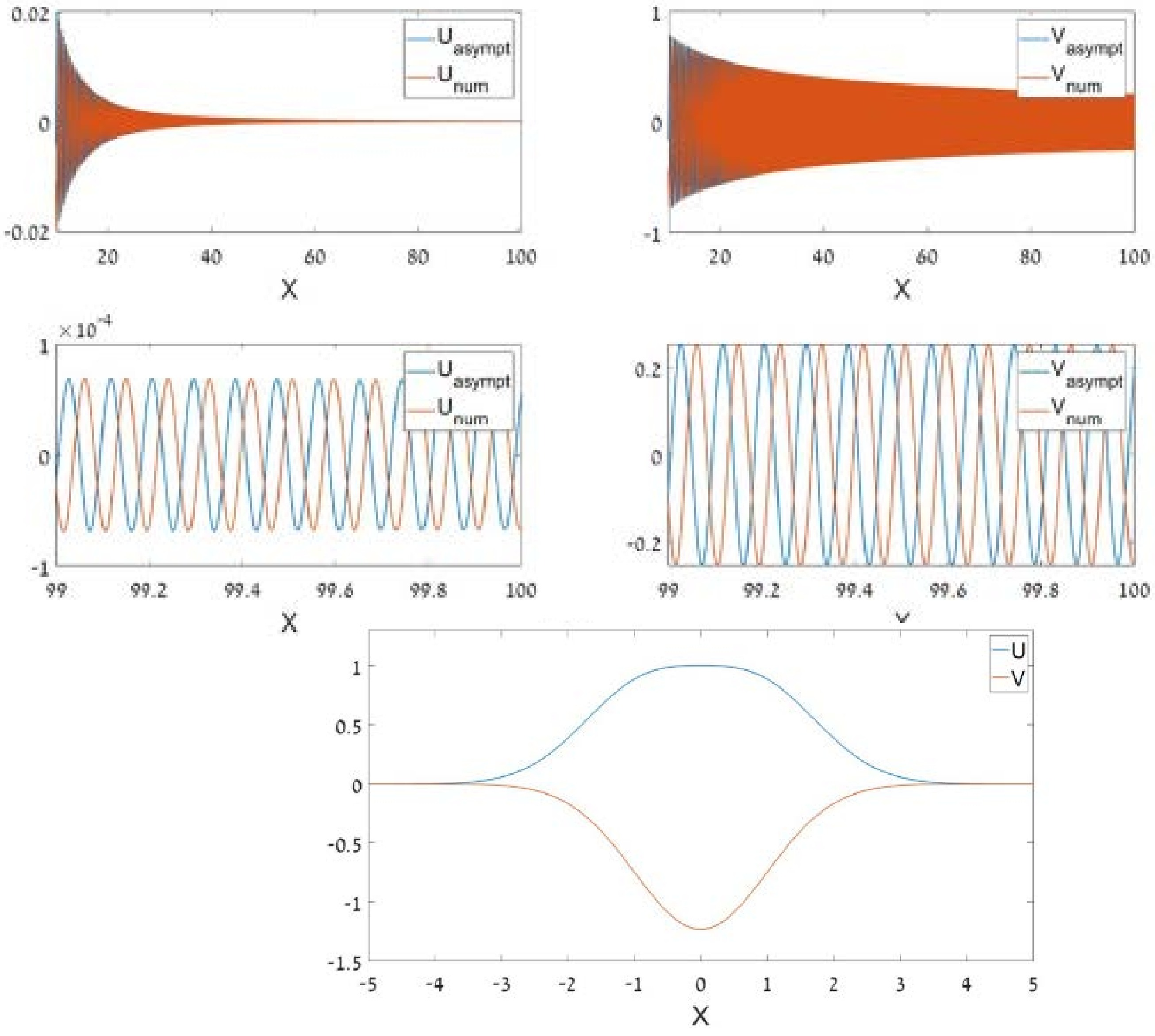
$$V_{\mathrm{deloc}}(x) \underset{|x|\to\infty}{\approx} V_0 |x|^{-1/2} \cos\left(\frac{\sqrt{\kappa}}{2}x^2 + \frac{\mu}{\sqrt{\kappa}}\ln\left(|x|\right)\right), \quad (60)$$

$$U_{\mathrm{deloc}}(x) \underset{|x|\to\infty}{\approx} V_0 \frac{2\lambda}{1+\kappa}|x|^{-5/2} \cos\left(\frac{\sqrt{\kappa}}{2}x^2 + \frac{\mu}{\sqrt{\kappa}}\ln\left(|x|\right)\right), \quad (61)$$

Note that the ***quadratic term*** is the ***leading one*** in the phase of these expressions.

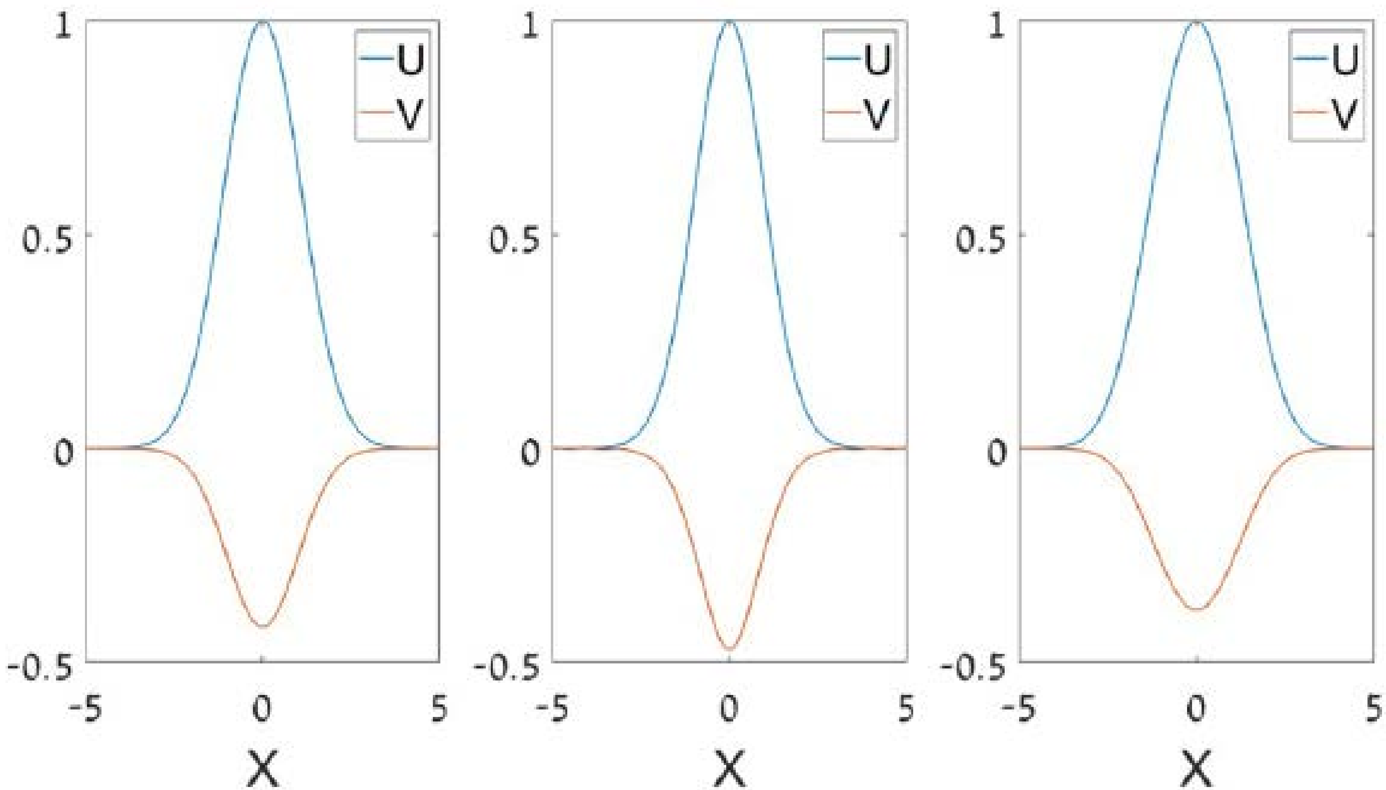
The ***v*** component, subject to the action of the expulsive potential, obviously **dominates** in this asymptotic solution. These eigenstates may be considered as delocalized ones because their norm ***diverges*** (slowly) as $\int$ **d*x*/|*x*|** at **|*x*|**→**∞**.

An example of the delocalized state, and the spatially even ***exact*** bound eigenstate existing ***at the same values of parameters***:
$\boldsymbol{\kappa}$ **= 0.5,** $\boldsymbol{\lambda}$ **= 2,** $\boldsymbol{\omega}$ **=0.375**, and with ***equal eigenvalues***, $\boldsymbol{\mu}$ **= 2.125**:

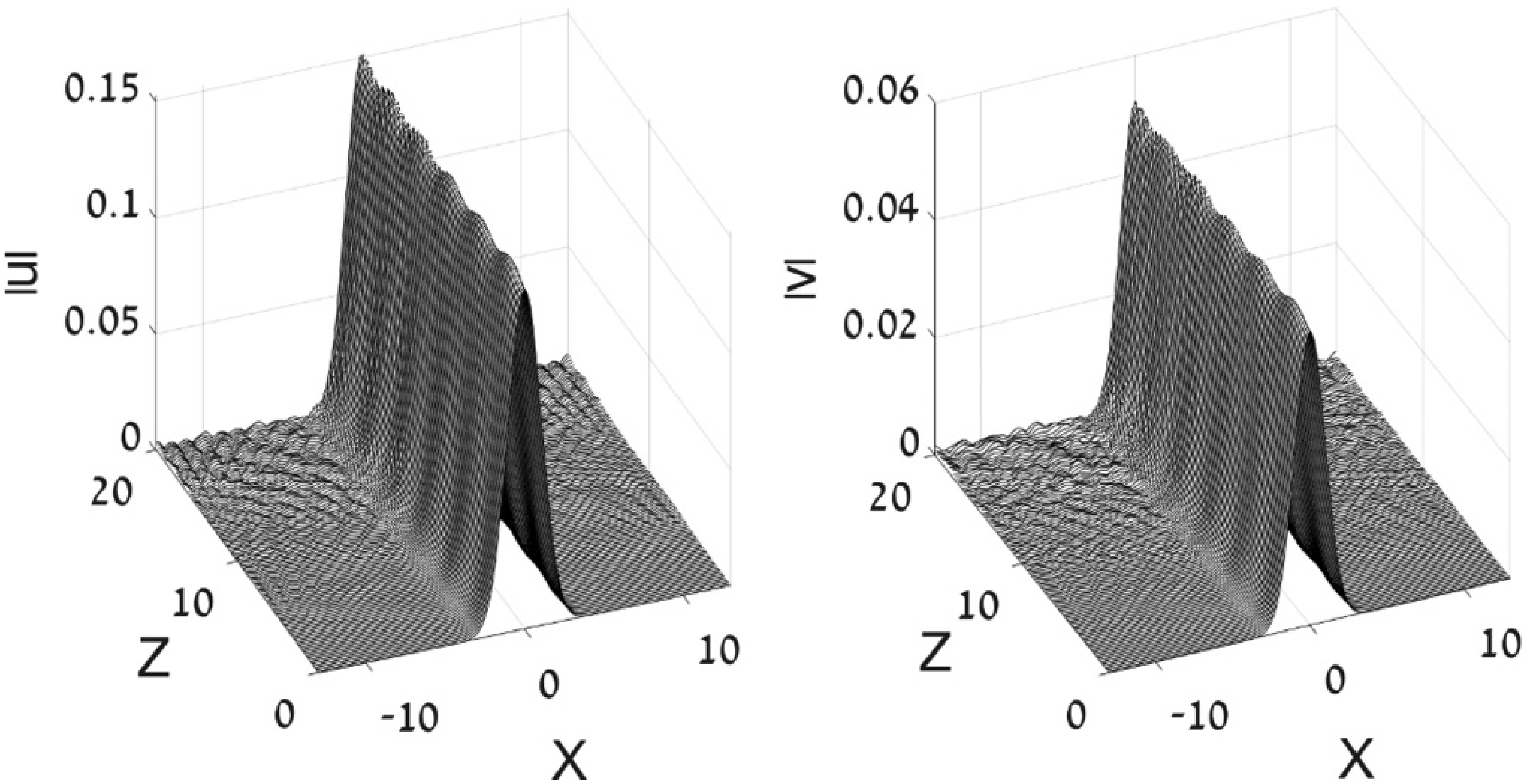


(6) Effects of the cubic self-focusing ($\sigma = +1$) and defocusing ($\sigma = -1$) nonlinearities

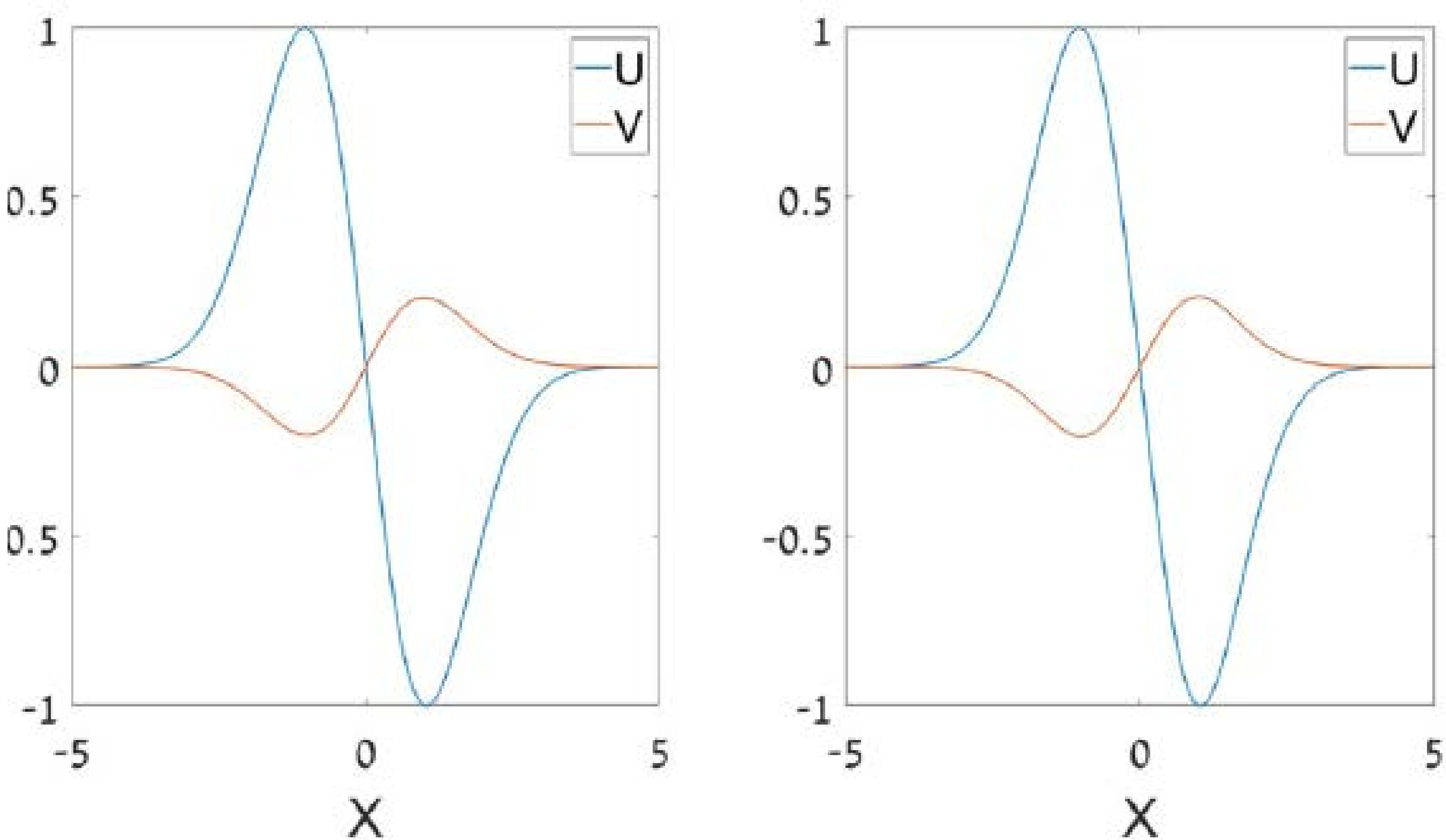
Comparison of the ***exact*** **spatially-even** bound state at $\sigma = 0$, $\kappa = 1$, $\lambda = 5$, $\omega = -10$, $\mu = 12.5$ (the **left panel**) and its ***numerically found counterparts*** with $\sigma = +1$, $\mu = 11.97$ (**center**) and $\sigma = -1$, $\mu = 13.19$ (**right**). Bound states remain ***stable*** in the nonlinear system:



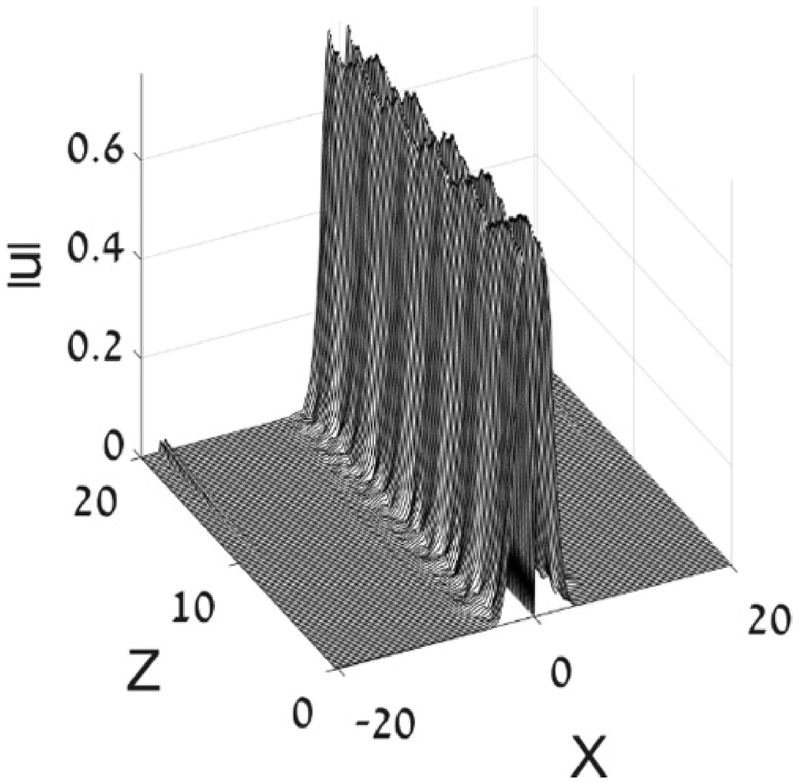
A ***stability test***: take the exact **spatially even** bound-state solution of the ***linearized system*** (here, with $\boldsymbol{\kappa = 1, \lambda = 6,}$ $\boldsymbol{\omega = -15.5}$, and amplitude $\boldsymbol{U_{max} = 0.146}$), and use it as the ***input*** for simulations of the ***nonlinear system*** with $\boldsymbol{\sigma = +1}$ (**self-focusing**). The result is a ***robust breather***, which emits a small amount of “radiation”:

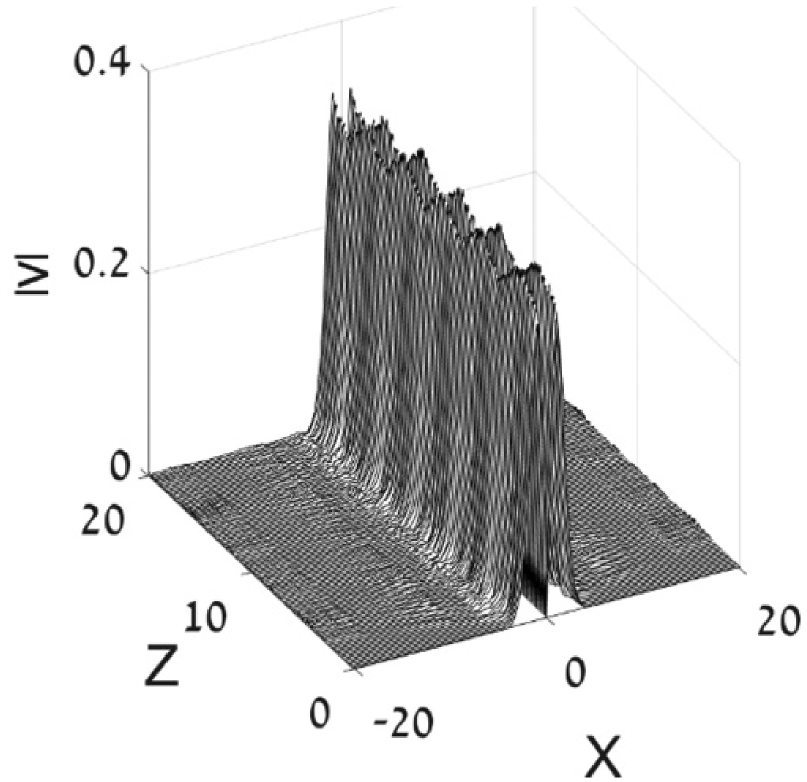


Similar comparison for **spatially-odd** solutions at **$\sigma = 0$**,**$\kappa = 2.5$, $\lambda = 10$, $\omega = -45.375$** (left: the **exact solution** with **$\mu = 48.875$)** and its ***numerically found counterpart*** with **$\sigma = +1$**, **$\mu = 48.363$** (right). The odd bound states are **stable** in the nonlinear system.

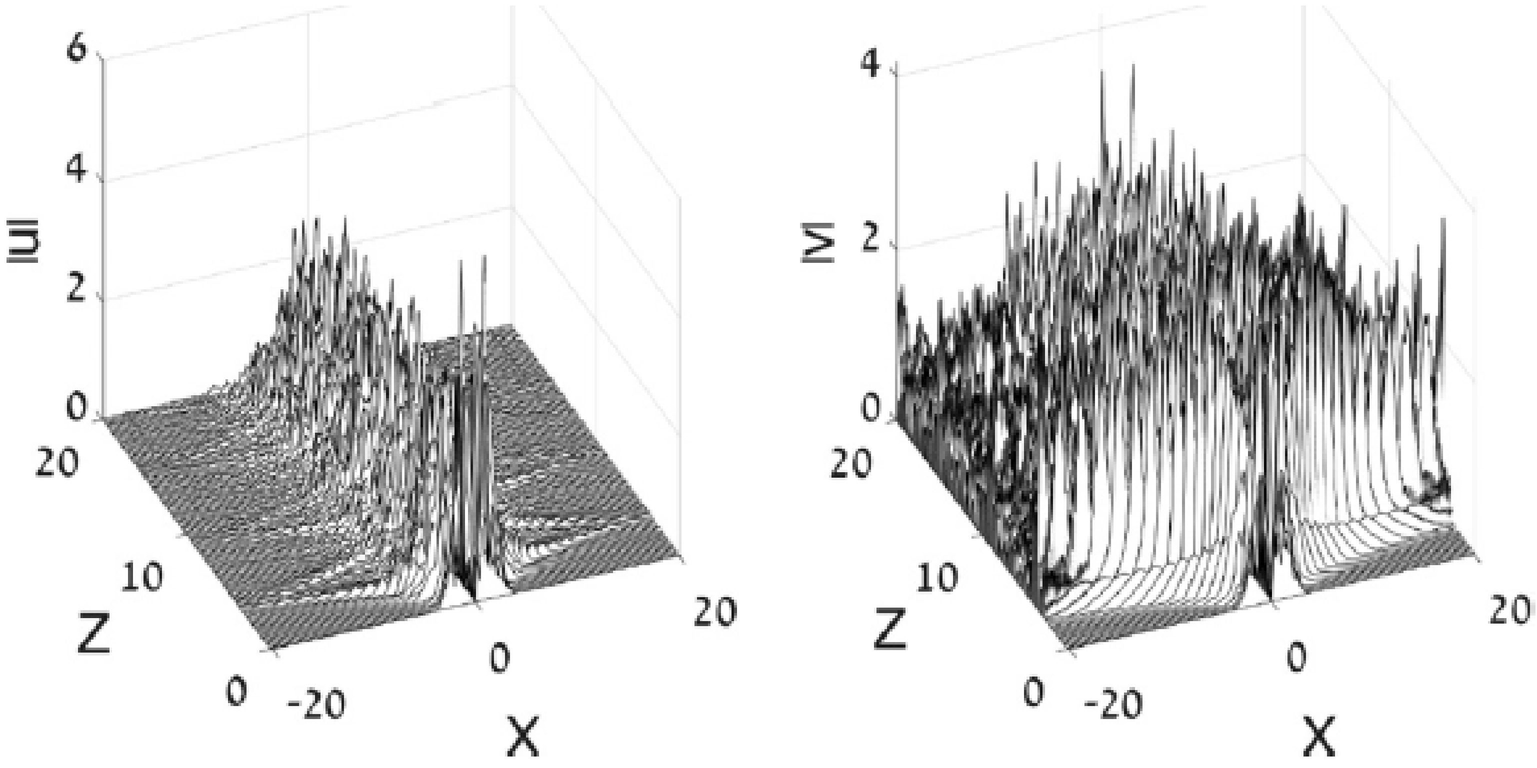


The ***stability test*** with the ***input*** taken as per the exact **spatially odd** solution of the ***linearized system*** with $\boldsymbol{\kappa = 0.5}$, $\boldsymbol{\lambda = 5}$, $\boldsymbol{\omega = -9.375}$, and amplitude $\boldsymbol{U_{max} = 0.716}$. A very "clean" breather is produced by the simulations, with virtually no emission of "radiation".





On the other hand, the simulations with a **much larger amplitude** of the input demonstrate ***chaotization*** of the ensuing dynamics. It ***suppresses*** the **effective confinement** imposed by the **linear coupling** onto the ***v*** component, which is subject to the action of the **expulsive potential**. This leads to the ***loss of the localization***. An example: the simulation initiated by the same **spatially odd input** as before, but with the amplitude **×5:**



(7) Fundamental and vortex eigenstates of the two-dimensional system

A straightforward **2D** extension of the linearly-coupled system (written in the polar coordinates):

$$iu_z + \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)u + \lambda v - \frac{1}{2}r^2 u + \sigma |u|^2 u$$
$$= -\omega u, \tag{6}$$

$$iv_z + \frac{1}{2}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)v + \lambda u + \frac{1}{2}\kappa r^2 v + \sigma |v|^2 v$$
$$= 0. \tag{7}$$

Stationary **2D** solutions for bound states with propagation constant ***–μ*** and embedded vorticity ***S* = 0,1,2,3**,… are looked for as

$$\{u, v\} = \exp(-i\mu z + iS\theta)\{U(r), V(r)\}, \tag{8}$$

where real functions U and V satisfy radial equations

$$(\mu+\omega)U + \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{S^2}{r^2}\right)U + \lambda V - \frac{1}{2}r^2 U + \sigma U^3 = 0, \tag{9}$$

$$\mu V + \frac{1}{2}\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{S^2}{r^2}\right)V + \lambda U + \frac{1}{2}\kappa r^2 V + \sigma V^3 = 0. \tag{10}$$

The **linearized version** of these equations admits an **exact codimension-1 solution** too, with **any integer vorticity *S***:

$$U(r) = \left(U_0^{(2D)} + U_2^{(2D)} r^2\right) r^S \exp\left(-\frac{r^2}{2}\right),$$

$$V(r) = V_0^{(2D)} r^S \exp\left(-\frac{r^2}{2}\right),$$

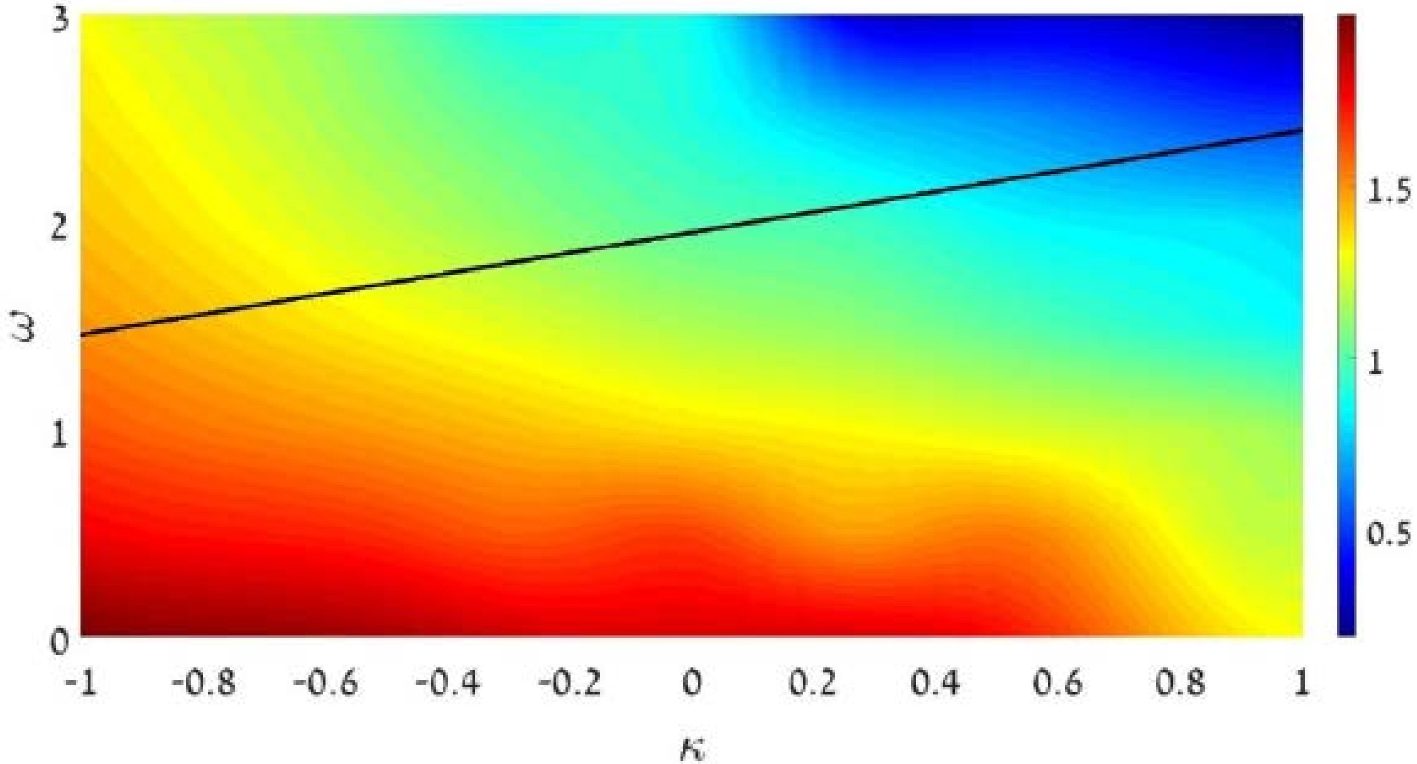
$$U_0^{(2D)} = \frac{(S+1)(1+\kappa) - \lambda^2}{2\lambda} V_0^{(2D)},$$

$$U_2^{(2D)} = -\frac{1+\kappa}{2\lambda} V_0^{(2D)},$$

$$\mu_{2D} = \frac{1}{2}[\lambda^2 + (S+1)(1-\kappa)],$$

This solution is valid under the following constraint imposed on parameters of the system: $\omega_{2D} = (1/2)\left[5 + S - \lambda^2 + (S+1)\kappa\right]$.

The heatmap of **numerically found** eigenvalues of the **2D** bound states in the linear system with ***σ* = 0, *S* = 0** and ***λ*** = 1 (the **exact codimension-1** solutions exist along the black line):



Similarly to the **1D** setting, ***all bound states*** of the linearized **2D** system may be considered as eigenmodes ***embedded into the continuum of delocalized states***.

The asymptotic form of the **2D** delocalized states at ***r*** **→∞** is

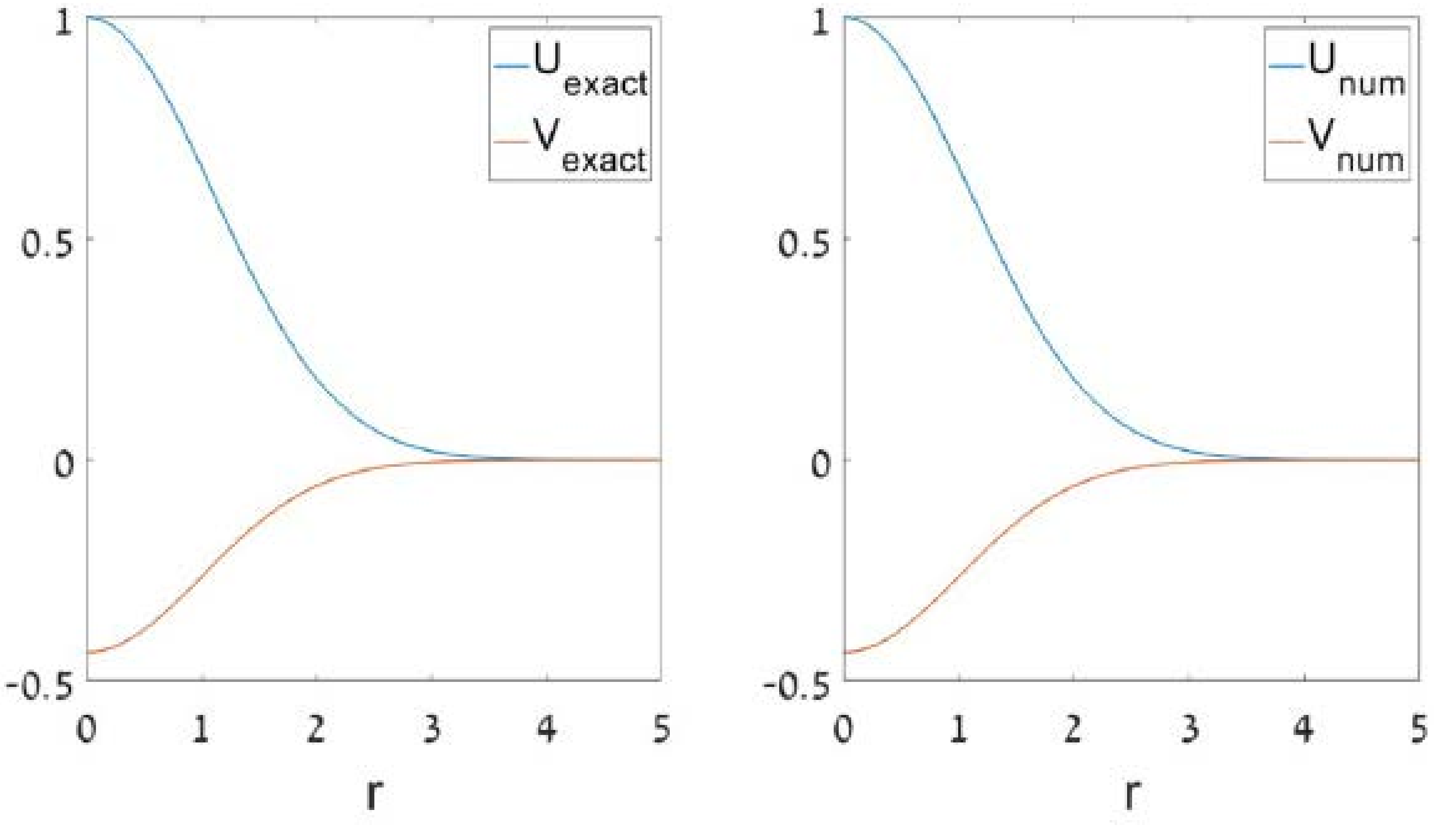
$$V_{\text{deloc}}^{(2\text{D})}(r) \underset{r\to\infty}{\approx} V_0 r^{-1} \cos\left(\frac{\sqrt{\kappa}}{2} r^2 + \frac{\mu}{\sqrt{\kappa}} \ln r\right),$$

$$U_{\text{deloc}}^{(2\text{D})}(r) \underset{r\to\infty}{\approx} V_0 \frac{2\lambda}{1+\kappa} r^{-3} \cos\left(\frac{\sqrt{\kappa}}{2} r^2 + \frac{\mu}{\sqrt{\kappa}} \ln r\right).$$

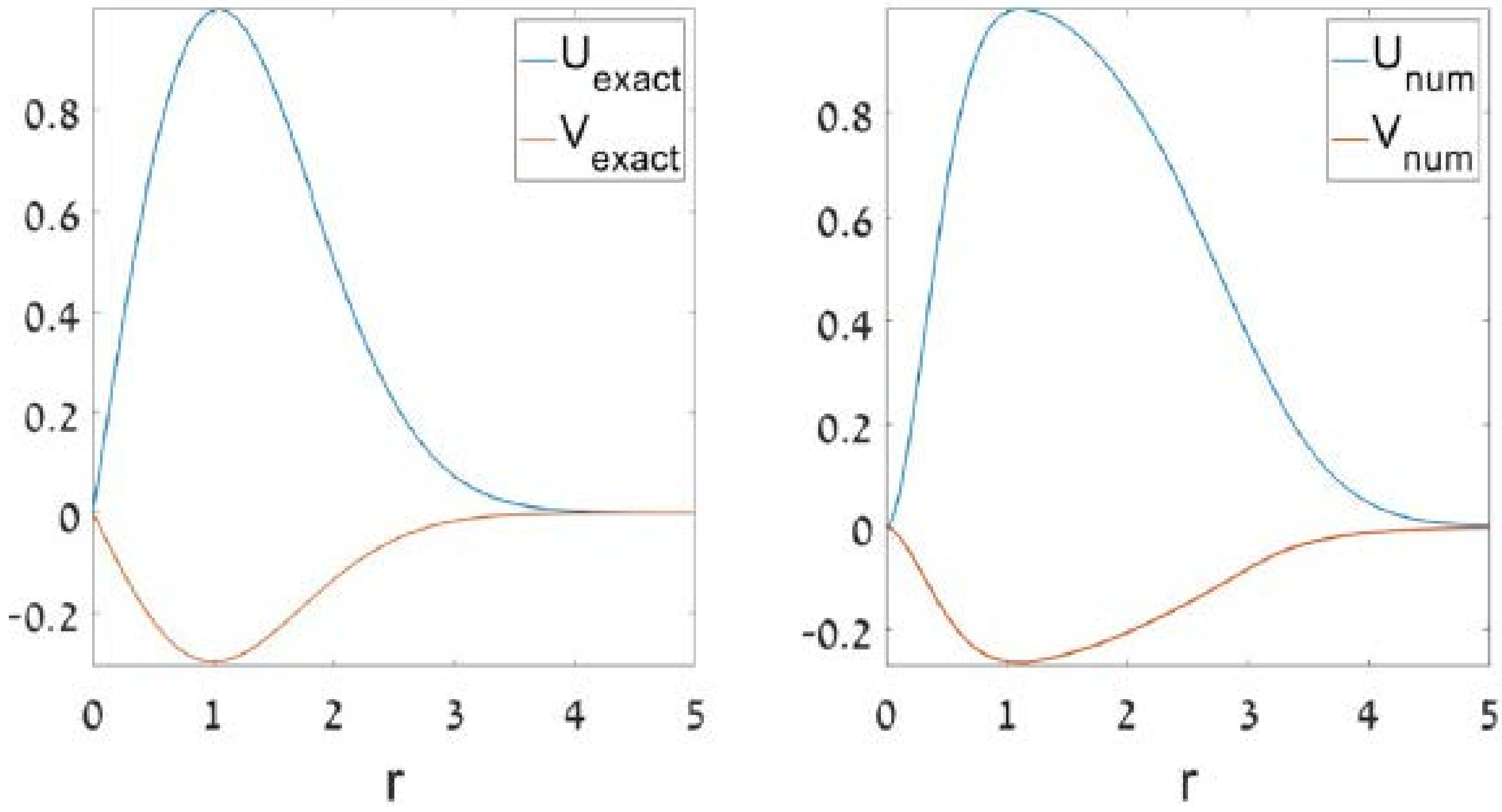
Note the ***quadratic term ~ r²*** in the phase of these expressions. At ***r*** **→∞**, their total norm slowly diverges as $\int dr/r$.

Also similar to the **1D** case, the ***v*** component, which is subject to the action of the expulsive potential, **dominates** in this solution.

The (weak) **effect of the self-focusing nonlinearity** on the **2D** state with ***S* = 0** and ***κ* = 1, *λ* = 5, *ω* =-10**: the ***exact solution*** with ***σ* = 0, *μ* = 12.5** (**left**), and its ***numerically found counterpart*** with ***σ* = +1, *μ* = 12.10** (**right**). The **2D** bound states with ***S* = 0** remain ***stable*** in the nonlinear system **with either sign of *σ*** .



The **effect of the** **self-defocusing** **nonlinearity** on the **2D** bound state with vorticity ***S* =1** and ***κ* = 0.5, *λ* = 10, *ω* =-46.5**: the ***exact solution*** with ***σ* = 0**, ***μ* = 24.5** (left), and its ***numerically found counterpart*** with ***σ* = -1**, ***μ* = 27.01** (right).

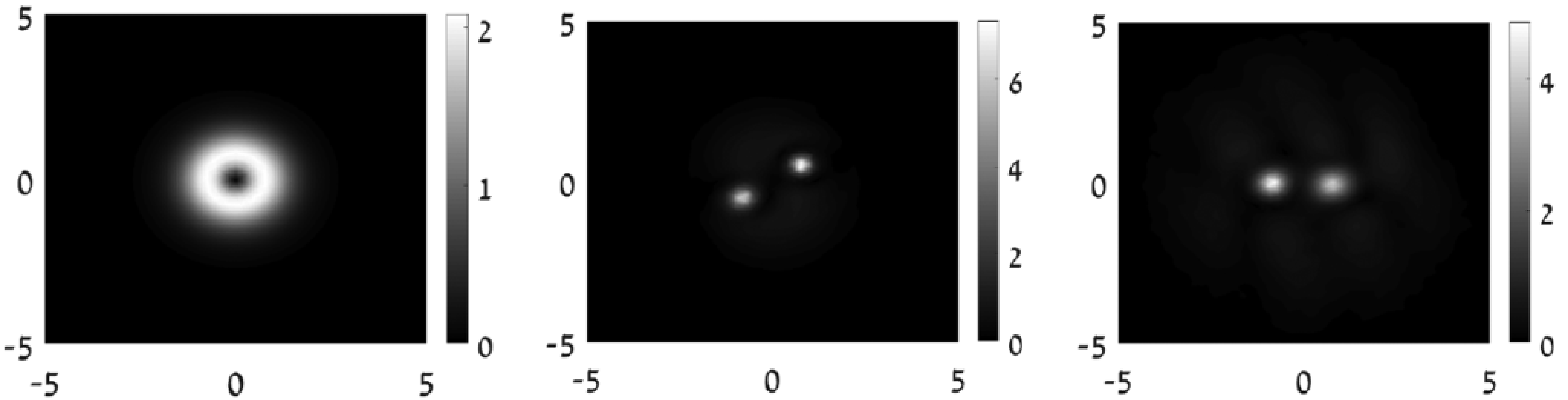


The situation concerning the **stability** of the **2D** bound states with **embedded vorticity** in the system with the **self-focusing** is ***different***, in comparison with the case of ***S* = 0**. As in other models [T. J. Alexander and L. Bergé, Ground states and vortices of matter-wave condensates and optical guided waves, Phys. Rev. E **65**, 026611 (2002);

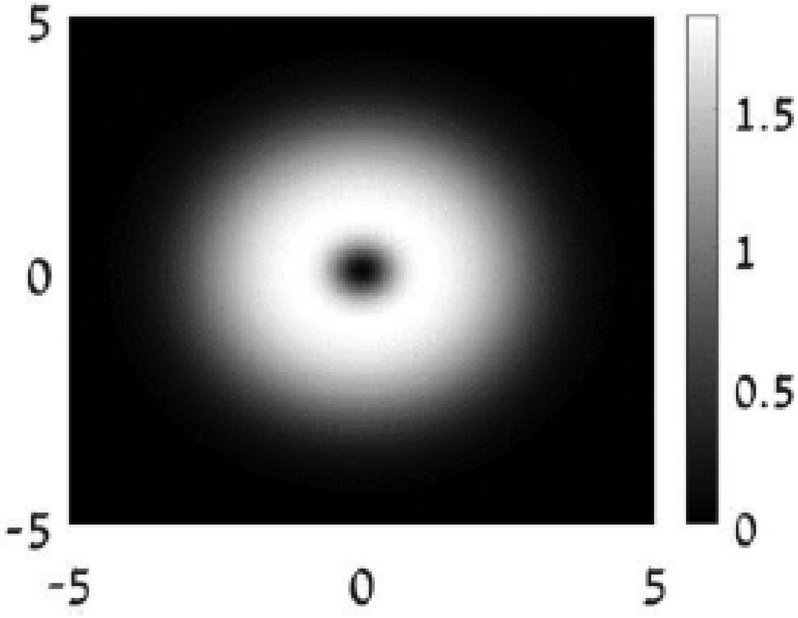
B. A. Malomed, Vortex solitons: Old results and new perspectives, Physica D **399**, 108 (2019)],

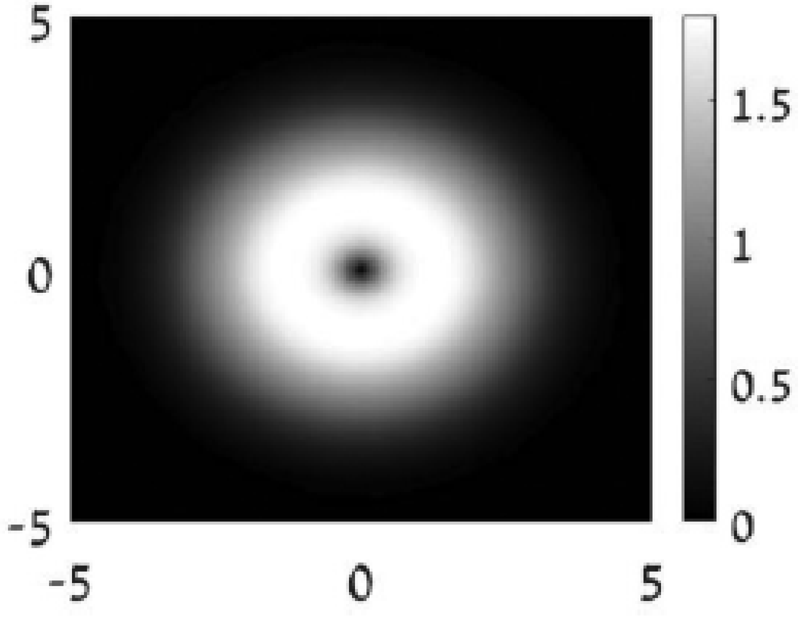
the vortex is ***unstable*** against ***spontaneous splitting*** under the action of the ***self-focusing nonlinearity*** (**σ = +1**).

An example: the unstable evolution of the vortex state with **S = 1** and **κ = 0.5, λ = 10, ω =-46.5**, **U_{max} = 1**. Shown are the shapes at **z = 0, 3.8**, and **6.2**.



On the other hand, the **vortex eigenstates** remain ***stable*** in the ***nonlinear system*** with ***self-defocusing*** (**σ = -1**). An example: the ***stable evolution*** of the **eigenstate** with **S = 1** in the system with **κ = 1, λ = 7, ω =-20.5**, and **U_{max} = 1**. Shown are the shapes at **z = 0** and **z = 10**.





(8) Conclusions

(i) It is demonstrated that, quite **counter-intuitively**, **bound states** of the wave function subject to the action of the **1D** and **2D** **expulsive** parabolic potential may be supported by the linear coupling of the wave function to a mate one, which is confined by the **trapping potential**.

(ii) This finding is precisely corroborated by the **exact analytical solutions** of **codimension 1**, for both the even and odd eigenmodes in the **1D** system, and for eigenstates with all values of vorticity ***S*** in **2D**.

(iii) Generic spatially even and odd **1D** eigenstates are found by means of the variational (***Rayleigh-Ritz***) approximation. Along with the systematically reported numerical findings, these results corroborate the **existence of the bound states for all values of strength κ of the expulsive potential**. In the limit of $\kappa \to \infty$, this is explained by the fact that the amplitude of the component of the bound state which is subject to the action of the expulsive potential becomes **vanishingly small**, $\sim 1/\kappa$.

(iv) All the **bound eigenstates** coexist with the **delocalized states** which form the **continuous spectrum**, therefore the bound eigenstates may be categorized as ***localized modes embedded into the continuum***.

(v) Both the self-focusing and defocusing nonlinearity produces a weak deformation of the bound states, and **does not break their stability** in the **1D** system.

(vi) In the **2D** system, bound eigenstates with ***embedded vorticity*** are ***unstable against spontaneous splitting*** under the action of the ***self-focusing***, but remain ***stable*** in the case of ***defocusing***.

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Thank you for your interest!