Lindblad dynamics of multi-mode bosonic systems: Exceptional points and speed of evolution

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Remarks on quantum channels and master equations

Quantum channels

are completely positive trace preserving (CPTP) mappings describing temporal evolution of density operators.

Lindblad-type master equations

- As a consequence of the Markovian approximation, the evolution of the system is effectively local in time;
- A general Markovian superoperator is generated by a Liouvillian (Lindbladian) as described by the master equation;
- The form of a Lindblad-type master equation can be inferred using the Kraus representation of quantum channels.

Intermode couplings in photonic systems

- quantum polarization modes propagating in an anisotropic environment such as optical fibers;
- overlapping modes in open resonators;
- quantized fields propagating in (evanescently) coupled waveguides;
- quantum optical scattering by lossy bianisotropic metasurfaces.

General problem

Understanding effects of the environment induced and dynamical intermode couplings in quantum dynamics of open photonic systems.

Topics to be discussed

- Mathematical tools for modeling Lindblad dynamics of multimode systems;
- Quantum speed limits and speed of evolution of bimodal photonic systems.

1. **Lindblad master equation approach**
   - Algebraic approach
   - Method of characteristic functions

2. **Speed of evolution**
   - Gaussian states
   - Polarization qubit states

Outline

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Lindblad master equation

Master equation for a multimode bosonic system

$$\boxed{\frac{\partial \hat{\rho}}{\partial t} = \hat{\mathcal{L}}\hat{\rho} = -i \sum_{n,m=1}^{N} \Omega_{nm}[\hat{a}_n^\dagger \hat{a}_m, \hat{\rho}] + \sum_{n,m=1}^{N} \left(\Gamma_{nm}^{(-)} \mathcal{D}_{\hat{a}_m \hat{a}_n^\dagger}\hat{\rho} + \Gamma_{nm}^{(+)} \mathcal{D}_{\hat{a}_n^\dagger \hat{a}_m}\hat{\rho}\right),}$$

$$\mathcal{D}_{\hat{A}\hat{B}} : \hat{\rho} \mapsto \mathcal{D}_{\hat{A}\hat{B}}\hat{\rho} = 2\hat{A}\hat{\rho}\hat{B} - \hat{B}\hat{A}\hat{\rho} - \hat{\rho}\hat{B}\hat{A} = [\hat{A}, \hat{\rho}\hat{B}] + [\hat{A}\hat{\rho}, \hat{B}],$$

where $\mathbf{\Omega} = \mathbf{\Omega}^\dagger$ is the frequency matrix, and $\mathbf{\Gamma}_\pm$, $\mathbf{\Gamma} = \mathbf{\Gamma}_- - \mathbf{\Gamma}_+ \geq 0$ are the relaxation matrices.

Thermal bath limiting case

$$\boxed{\mathbf{\Gamma}_- = (n_T + 1)\mathbf{\Gamma}, \quad \mathbf{\Gamma}_+ = \mathrm{e}^{-z_T}\mathbf{\Gamma}_- = n_T \mathbf{\Gamma}, \quad z_T \equiv \frac{\hbar\Omega_0}{k_B T},}$$

where $n_T = (\mathrm{e}^{z_T} - 1)^{-1}$ is the mean number of thermal photons.

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General strategy [Gaidash et al. arXiv:2412.13890[quant-phys]]

Basic steps

1. Structure of Lindblad superoperator and extending Jordan-Schwinger approach to the case of left and right superoperators ($\overleftarrow{\hat{A}} : \hat{\rho} \mapsto \hat{A}\hat{\rho}$ and $\overrightarrow{\hat{A}} : \hat{\rho} \mapsto \hat{\rho}\hat{A}$ are the left-hand and right-hand action superoperators).
2. Commutation relations and algebraic identities of the algebra of bilinear superoperators.
3. Eliminating quantum jump terms of the form $\hat{A}\hat{\rho}\hat{B}$ using the similarity transformation:

$$\boxed{\hat{\mathcal{L}} \to \hat{\mathcal{T}}\hat{\mathcal{L}}\hat{\mathcal{T}}^{-1} = \hat{\mathcal{L}}_d = \overleftarrow{\hat{L}_{\mathrm{eff}}} + \overrightarrow{\hat{L}^{\dagger}_{\mathrm{eff}}},\ \hat{L}_{\mathrm{eff}} = -i\hat{H}_{\mathrm{eff}}.}$$

4. Solving spectral problems for $\hat{\mathcal{L}}_d$ and $\hat{\mathcal{L}}$ and deriving formulas for the superpropagator $e^{\hat{\mathcal{L}}t}$.

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Algebraic structure of Liouvillian superoperator

$$\boxed{\hat{\mathcal{L}} = -i\hat{\mathcal{N}}_{\Omega}^{(-)} + 2(\hat{\mathcal{K}}_{\Gamma_0}^{(0)} + \hat{\mathcal{K}}_{\Gamma_+}^{(+)} + \hat{\mathcal{K}}_{\Gamma_-}^{(-)}) + \mathrm{Tr}\,\Gamma,}$$

$$\hat{\mathcal{L}}^{\sharp} = i\hat{\mathcal{N}}_{\Omega}^{(-)} + 2(\hat{\mathcal{K}}_{\Gamma_0}^{(0)} + \hat{\mathcal{K}}_{\Gamma_-}^{(+)} + \hat{\mathcal{K}}_{\Gamma_+}^{(-)}) + \mathrm{Tr}\,\Gamma,$$

$$\Gamma_0 = -(\Gamma_+ + \Gamma_-),$$

$$\hat{\mathcal{N}}_{\Omega}^{(-)} = \sum_{n,m=1}^{N} \Omega_{nm}\hat{\mathcal{N}}_{nm}^{(-)}, \quad \hat{\mathcal{K}}_{\Gamma_\nu}^{(\nu)} = \sum_{n,m=1}^{N} \Gamma_{nm}^{(\nu)}\hat{\mathcal{K}}_{nm}^{(\nu)}$$

$$\hat{\mathcal{N}}_{nm}^{(-)} = \overleftarrow{\hat{a}_n^{\dagger}\hat{a}_m} - \overrightarrow{\hat{a}_n^{\dagger}\hat{a}_m}, \quad \hat{\mathcal{K}}_{nm}^{(0)} = \frac{1}{2}\left(\overleftarrow{\hat{a}_n^{\dagger}\hat{a}_m} + \overrightarrow{\hat{a}_m\hat{a}_n^{\dagger}}\right).$$

$$\hat{\mathcal{K}}_{nm}^{(+)} = \overleftarrow{\hat{a}_n^{\dagger}}\overrightarrow{\hat{a}_m}, \quad \hat{\mathcal{K}}_{nm}^{(-)} = \overleftarrow{\hat{a}_m}\overrightarrow{\hat{a}_n^{\dagger}},$$

where $\hat{\mathcal{L}}^{\sharp}$ is the adjoint of $\hat{\mathcal{L}}$: $\mathrm{Tr}\{\hat{A}\hat{\mathcal{L}}(\hat{\rho})\} = \mathrm{Tr}\{\hat{\mathcal{L}}^{\sharp}(\hat{A})\hat{\rho}\}$
$(\hat{\mathcal{L}} \to \hat{\mathcal{L}}^{\sharp} \Rightarrow \Omega \to -\Omega,\ \Gamma_{\pm} \to \Gamma_{\mp})$

Jordan-Schwinger map

Quadratic bosonic operators associated with matrices

$$A \mapsto \hat{J}_A = \sum_{n,m=1}^{N} A_{nm} \hat{a}_n^\dagger \hat{a}_m$$

and their commutation relations

$$[\hat{J}_A, \hat{J}_B] = \hat{J}_{[A,B]}.$$

$$e^{\hat{J}_V} \hat{J}_A e^{-\hat{J}_V} = \hat{J}_{A'}, \quad A' = e^V A e^{-V},$$

$$e^{\hat{J}_V} \hat{a}_i^\dagger e^{-\hat{J}_V} = \sum_{j=1}^{N} U_{ji} \hat{a}_j^\dagger, \quad U = e^V,$$

$$\hat{\mathcal{L}} e^{-\hat{J}_V} = 0 \Rightarrow \Gamma_+ - \frac{i}{2}[\Omega, e^{-V}] + \frac{1}{2}\{\Gamma_0, e^{-V}\} + e^{-V}\Gamma_- e^{-V} = 0.$$

Algebraic relations

Identities for exponentiated superoperators acting by similarity

$$e^{\hat{\mathcal{K}}_B^{(\nu)}}\hat{\mathcal{N}}_A^{(-)}e^{-\hat{\mathcal{K}}_B^{(\nu)}} = \hat{\mathcal{N}}_A^{(-)} - \hat{\mathcal{K}}_{[A,B]}^{(\nu)}, \quad \nu \in \{+,-\},$$

$$e^{\hat{\mathcal{K}}_B^{(\nu)}}\hat{\mathcal{K}}_A^{(0)}e^{-\hat{\mathcal{K}}_B^{(\nu)}} = \hat{\mathcal{K}}_A^{(0)} - \nu\hat{\mathcal{K}}_{\frac{1}{2}\{A,B\}}^{(\nu)},$$

$$e^{\hat{\mathcal{K}}_B^{(\nu)}}\hat{\mathcal{K}}_A^{(-\nu)}e^{-\hat{\mathcal{K}}_B^{(\nu)}} = \hat{\mathcal{K}}_A^{(-\nu)} - \nu\hat{\mathcal{K}}_{\{A,B\}}^{(0)} + \hat{\mathcal{N}}_{\frac{1}{2}[A,B]}^{(-)} + \hat{\mathcal{K}}_{BAB}^{(\nu)}.$$

Transforming Liouvillian superoperator

$$\boxed{\hat{\mathcal{L}} \mapsto e^{\hat{\mathcal{K}}_B^{(\nu)}}\hat{\mathcal{L}}e^{-\hat{\mathcal{K}}_B^{(\nu)}} = \hat{\mathcal{L}}' = -i\hat{\mathcal{N}}_{\Omega'}^{(-)} + 2\sum_{\alpha\in\{0,\pm\}}\hat{\mathcal{K}}_{\Gamma'_\alpha}^{(\alpha)} + \operatorname{Tr}\Gamma}$$

$$\Omega' = \Omega + i[\Gamma_{-\nu}, B], \quad \Gamma'_0 = \Gamma_0 - \nu\{\Gamma_{-\nu}, B\},$$

$$\Gamma'_{-\nu} = \Gamma_{-\nu}, \quad \Gamma'_\nu = \Gamma_\nu + \frac{i}{2}[\Omega, B] - \frac{\nu}{2}\{\Gamma_0, B\} + B\Gamma_{-\nu}B.$$

Eliminating jumps using $\hat{\mathcal{T}}_{+-}(A_+, A_-)$

Effective Hamiltonian

$$\hat{\mathcal{T}}_{+-} \equiv \hat{\mathcal{T}}_{+-}(A_+, A_-) = e^{\hat{\mathcal{K}}^{(+)}_{A_+}} e^{\hat{\mathcal{K}}^{(-)}_{A_-}} : \boxed{\hat{\mathcal{T}}_{+-}\hat{\mathcal{L}}\hat{\mathcal{T}}^{-1}_{+-} = \hat{\mathcal{L}}_d}$$

$$\hat{\mathcal{L}}_d = \overleftarrow{\hat{L}}_{\mathrm{eff}} + \overrightarrow{\hat{L}^{\dagger}_{\mathrm{eff}}}, \quad \hat{L}_{\mathrm{eff}} = -i\hat{H}_{\mathrm{eff}} = \hat{J}_L, \quad \mathbf{L} = -i\mathbf{\Omega}' + \mathbf{\Gamma}'_0,$$

$$\Omega' = \Omega + i[\Gamma_+, A_-], \quad \Gamma'_0 = \Gamma_0 + \{\Gamma_+, A_-\} \le 0.$$

Equations for the matrices $A_{\pm}$

$$\Gamma_- + \frac{i}{2}[\Omega, A_-] + \frac{1}{2}\{\Gamma_0, A_-\} + A_-\Gamma_+A_- = 0,$$

$$\Gamma_+ + \frac{i}{2}[\Omega', A_+] - \frac{1}{2}\{\Gamma'_0, A_+\} = 0 \Rightarrow \boxed{2\Gamma_+ = LA_+ + A_+L^{\dagger}}$$

Solutions and structure of steady state

Steady state characteristic function and matrix

$$\boxed{\chi_N(\boldsymbol{\alpha}, t) \xrightarrow{t\to\infty} \chi_{\mathrm{eq}}(\boldsymbol{\alpha}) = \mathrm{e}^{-(\boldsymbol{\alpha}^*, \mathbf{W}\boldsymbol{\alpha})},}$$

$$W = 2\int_0^\infty P(t)\Gamma_+ P^\dagger(t)\mathrm{d}t, \quad P(t) = \mathrm{e}^{Lt},$$

$$W \to W_1 = (W^{-1} + I)^{-1} \Rightarrow \quad 2\Gamma_+ + LW + WL^\dagger = 0 \to$$

$$\to \Gamma_+ - \frac{i}{2}[\Omega, W_1] + \frac{1}{2}\{\Gamma_0, W_1\} + W_1\Gamma_- W_1 = 0$$

$$\Rightarrow \boxed{\hat{\rho}_{\mathrm{eq}} \propto e^{-\hat{J}_V}, \quad e^{-V} = W_1 = (W^{-1} + I)^{-1}}$$

$A_\pm$ and L

$$\boxed{A_- = I} \Rightarrow \boxed{L = -i\Omega - \Gamma} \Rightarrow \boxed{A_+ = -W.}$$

Eliminating jumps using $\hat{\mathcal{T}}_{-+}(B_-, B_+)$

$$\hat{\mathcal{T}}_{-+} \equiv \hat{\mathcal{T}}_{-+}(B_-, B_+) = e^{\hat{\mathcal{K}}^{(-)}_{B_-}} e^{\hat{\mathcal{K}}^{(+)}_{B_+}}$$

$$\Gamma_+ + \frac{i}{2}[\Omega, B_+] - \frac{1}{2}\{\Gamma_0, B_+\} + B_+\Gamma_- B_+ = 0,$$

$$L^\dagger B_- + B_- L + 2\Gamma_- = 0$$

$$B_+ = -W_1, \quad B_- = 2\int_0^\infty P^\dagger(t)\Gamma_- P(t)dt.$$

Thermal bath limit

$$A_\pm = \alpha_\pm I, \quad \alpha_+ = -n_T, \quad \alpha_- = 1,$$

$$B_\pm = \beta_\pm I, \quad \beta_+ = -\mathrm{e}^{-z_T}, \quad \beta_- = n_T + 1 \equiv Z,$$

$$\hat{\mathcal{T}}_{+-} = \hat{\mathcal{T}}_{+-}(-n_T, 1), \quad \hat{\mathcal{T}}_{-+} = \hat{\mathcal{T}}_{-+}(Z, -e^{-z_T})$$

Spectral problem

$$\hat{\mathcal{L}}\hat{\rho}_\lambda = \lambda\hat{\rho}_\lambda, \quad \hat{\mathcal{L}}^\sharp\hat{\sigma}_\lambda = \lambda\hat{\sigma}_\lambda, \quad \hat{\mathcal{L}}_d\,\hat{\rho}_\lambda^{(d)} = \lambda\hat{\rho}_\lambda^{(d)}, \quad \hat{\mathcal{L}}_d^\sharp\,\hat{\sigma}_\lambda^{(d)} = \lambda\hat{\sigma}_\lambda^{(d)},$$

$$\boxed{\hat{\rho}_\lambda = \hat{\mathcal{T}}_{-+}^{-1}\hat{\rho}_\lambda^{(d)} = q_\lambda\hat{\mathcal{T}}_{+-}^{-1}\hat{\rho}_\lambda^{(d)},} \quad \boxed{\hat{\sigma}_\lambda = \hat{\mathcal{T}}_{+-}^\sharp\hat{\sigma}_\lambda^{(d)}} \propto \hat{\mathcal{T}}_{-+}^\sharp\hat{\sigma}_\lambda^{(d)},$$

$$\mathrm{Tr}\{\hat{\sigma}_{\lambda'}^{(d)}\hat{\rho}_\lambda^{(d)}\} = \delta_{\lambda'\lambda} \Rightarrow \mathrm{Tr}\{\hat{\sigma}_{\lambda'}\hat{\rho}_\lambda\} = q_\lambda\delta_{\lambda'\lambda}$$

Single-mode effective Hamiltonian

$$\hat{L}_{\mathrm{eff}} = -(i\Omega + \Gamma)\hat{a}^\dagger\hat{a},$$

$$\hat{\rho}_\lambda^{(d)} = \hat{\rho}_{mn}^{(d)} = |m\rangle\langle n| \equiv \hat{\Pi}_{mn} = [\hat{\sigma}_\lambda^{(d)}]^\dagger,$$

$$\hat{\sigma}_\lambda^{(d)} = [\hat{\rho}_\lambda^{(d)}]^\dagger = \hat{\sigma}_{nm}^{(d)} = \hat{\Pi}_{nm},$$

$$\lambda \equiv \lambda_{mn} = -i\Omega(m-n) - \Gamma(m+n).$$

Single-mode eigenoperators and biorthogonality relations

Eigenoperators

$$\hat{\rho}_{mn} = \hat{\mathcal{T}}_{+-}(e^{-z_T}, -Z)\hat{\Pi}_{mn} = \sum_{k=0}^{\min(m,n)} \frac{(-Z)^k\sqrt{m!n!}}{k!(m-k)!(n-k)!}(\hat{a}^\dagger)^{m-k}\hat{\rho}_0\hat{a}^{n-k},$$

$$\hat{\rho}_0 = e^{e^{-z_T}\hat{\mathcal{K}}^{(+)}}|0\rangle\langle 0| = e^{-z_T\hat{a}^\dagger\hat{a}},$$

$$\hat{\sigma}_{nm} = \hat{\mathcal{T}}_{+-}(1, -n_T)\hat{\Pi}_{nm} = \sum_{k=0}^{\min(m,n)} \frac{(-n_T)^k\sqrt{m!n!}}{k!(m-k)!(n-k)!}(\hat{a}^\dagger)^{n-k}\hat{a}^{m-k}.$$

Biorthogonality relations

$$\mathrm{Tr}\{\hat{\sigma}_{n'm'}\hat{\rho}_{mn}\} = Z^{m+n+1}\delta_{n'n}\delta_{m'm}.$$

Temporal evolution in terms of $P(t)$

$$\boxed{\hat{\rho}(t) = e^{\hat{\mathcal{L}}t}\hat{\rho}(0) = \sum_{\mathbf{m},\mathbf{n},\mathbf{m}',\mathbf{n}'} \frac{U_{\mathbf{m}'\mathbf{m}}(t)U^*_{\mathbf{n}'\mathbf{n}}(t)}{q_{\mathbf{m}'\mathbf{n}'}}\hat{\rho}_{\mathbf{m}'\mathbf{n}'}\mathrm{Tr}\{\hat{\sigma}_{\mathbf{nm}}\hat{\rho}(0)\},}$$

$$\hat{\mathcal{L}} = \hat{\mathcal{T}}_{+-}^{-1}\hat{\mathcal{L}}_d\hat{\mathcal{T}}_{+-}, \quad \hat{\sigma}_{\mathbf{nm}} = \hat{\mathcal{T}}_{+-}^{\sharp}\hat{\Pi}_{\mathbf{nm}}, \quad \hat{\rho}_{\mathbf{nm}} = \hat{\mathcal{T}}_{-+}^{-1}\hat{\Pi}_{\mathbf{nm}} = q_{\mathbf{nm}}\hat{\mathcal{T}}_{-+}^{-1}\hat{\Pi}_{\mathbf{nm}},$$

$$\hat{\mathcal{T}}_{+-}\hat{\rho}(0) = \sum_{\mathbf{m},\mathbf{n}}\hat{\Pi}_{\mathbf{mn}}\langle\mathbf{m}|\hat{\mathcal{T}}_{+-}\hat{\rho}(0)|\mathbf{n}\rangle = \sum_{\mathbf{m},\mathbf{n}}\hat{\Pi}_{\mathbf{mn}}\mathrm{Tr}\{\hat{\sigma}_{\mathbf{nm}}\hat{\rho}(0)\},$$

$$e^{\hat{\mathcal{L}}_d t}\hat{\Pi}_{\mathbf{mn}} = \sum_{\mathbf{m}',\mathbf{n}'} U_{\mathbf{m}'\mathbf{m}}(t)U^*_{\mathbf{n}'\mathbf{n}}(t)\hat{\Pi}_{\mathbf{m}'\mathbf{n}'}, \quad U_{\mathbf{m}'\mathbf{m}}(t) = \langle\mathbf{m}'|e^{\hat{L}_{\mathrm{eff}}t}|\mathbf{m}\rangle,$$

$$e^{\hat{L}_{\mathrm{eff}}t}|\mathbf{m}\rangle = \prod_{i=1}^{N}\frac{(\hat{b}_i^\dagger)^{m_i}}{\sqrt{m_i!}}|\mathbf{0}\rangle\langle\mathbf{0}|, \quad \hat{b}_i^\dagger = e^{\hat{L}_{\mathrm{eff}}t}\hat{a}_i^\dagger e^{-\hat{L}_{\mathrm{eff}}t} = \sum_{j=1}^{N}P_{ji}(t)\hat{a}_j^\dagger,$$

$$q_{\mathbf{mn}} = \prod_{i=1}^{N}q_{m_in_i}, \quad q_{m_in_i} = Z^{n_i+m_i+1}.$$

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Characteristic function [Kiselev et al. Entropy 23, 1409 (2021), Symmetry 13, 2309 (2021)]

Normally ordered characteristic function

$$\chi_N(\boldsymbol{\alpha}) = \langle \mathrm{e}^{(\boldsymbol{\alpha},\hat{\mathbf{a}}^\dagger)} \mathrm{e}^{-(\boldsymbol{\alpha}^*,\hat{\mathbf{a}})} \rangle,$$

$$(\boldsymbol{\alpha}^*, \hat{\mathbf{a}}) \equiv \sum_{i=1}^{N} \alpha_i^* \hat{a}_i, \quad (\boldsymbol{\alpha}, \hat{\mathbf{a}}^\dagger) \equiv \sum_{i=1}^{N} \alpha_i \hat{a}_i^\dagger.$$

$$\boxed{\frac{\partial \chi_N(\boldsymbol{\alpha}, t)}{\partial t} = \hat{L} \chi_N(\boldsymbol{\alpha}, t)}$$

$$\hat{L} = i \sum_{n,m=1}^{N} \Omega_{nm} \hat{D}_{mn}^{(-)} - \sum_{n,m=1}^{N} (\Gamma_{nm} \hat{D}_{mn}^{(+)} + 2\Gamma_{nm}^{(+)} \alpha_m \alpha_n^*),$$

where

$$\hat{D}_{mn}^{(\pm)} = \alpha_m \frac{\partial}{\partial \alpha_n} \pm \alpha_n^* \frac{\partial}{\partial \alpha_m^*}. \tag{1}$$

Method of characteristics

Solution of characteristic equations

$$\boldsymbol{\alpha} = \mathbf{A}(-t)\boldsymbol{\alpha}_0, \quad \mathbf{A}(t) = P^\dagger(t) = \mathrm{e}^{(i\boldsymbol{\Omega}-\boldsymbol{\Gamma})t},$$

where $\boldsymbol{\alpha}_0 = \boldsymbol{\alpha}(0)$.
Solution along the characteristic curves

$$\chi_N(\boldsymbol{\alpha}_0, t) = \exp\left[-2\int_0^t (\boldsymbol{\alpha}_0^*, \mathbf{A}^\dagger(-\tau)\boldsymbol{\Gamma}_+\mathbf{A}(-\tau)\boldsymbol{\alpha}_0)\mathrm{d}\tau\right]\chi_{\mathrm{ini}}(\boldsymbol{\alpha}_0).$$

Expression for the characteristic function

$$\chi_N(\boldsymbol{\alpha}, t) = \mathrm{e}^{-(\boldsymbol{\alpha}^*, \mathbf{B}(t)\boldsymbol{\alpha})}\chi_{\mathrm{ini}}(\mathbf{A}(t)\boldsymbol{\alpha})$$

$$\boxed{\mathbf{B}(t) = 2\int_0^t \mathbf{A}^\dagger(\tau)\boldsymbol{\Gamma}_+\mathbf{A}(\tau)\mathrm{d}\tau = n_T(\mathbf{I}_N - \mathbf{A}^\dagger(t)\mathbf{A}(t)), \quad W = \mathbf{B}(\infty).}$$

where $\mathbf{I}_N$ is the identity $N \times N$ matrix.

Second order moments

Exact relations for the second order moments

$$\boxed{\langle \hat{a}_i^\dagger \hat{a}_j\rangle(t) = -\left.\frac{\partial^2 \chi_N(\boldsymbol{\alpha}, t)}{\partial \alpha_i \partial \alpha_j^*}\right|_{\boldsymbol{\alpha}=0} = B_{ji}(t) + \sum_{n,m} A_{ni}(t) A_{mj}^*(t) \langle \hat{a}_n^\dagger \hat{a}_m\rangle(0),}$$

$$\boxed{\langle \hat{a}_i^\dagger \hat{a}_j^\dagger\rangle(t) = \left.\frac{\partial^2 \chi_N(\boldsymbol{\alpha}, t)}{\partial \alpha_i \partial \alpha_j}\right|_{\boldsymbol{\alpha}=0} = \sum_{n,m} A_{ni}(t) A_{mj}(t) \langle \hat{a}_n^\dagger \hat{a}_m^\dagger\rangle(0).}$$

can be easily generalized to the case of higher order moments.

Phase-space representation of superpropagator

Glauber-Sudarshan representation

$$\hat{\rho}(t) = \int \mathrm{d}\mu(\boldsymbol{\alpha}) P(\alpha, t) |\boldsymbol{\alpha}\rangle\langle\boldsymbol{\alpha}|, \quad \mathrm{d}\mu(\boldsymbol{\alpha}) = \prod_{i=1}^{N} \mathrm{d}\mu(\alpha_i), \quad \mathrm{d}\mu(\alpha_i) = \frac{\mathrm{d}^2\alpha_i}{\pi}.$$

Temporal evolution of P function

$$\boxed{P(\boldsymbol{\alpha}, t) = \int \mathrm{d}\mu(\boldsymbol{\beta}) G(\boldsymbol{\alpha}, \boldsymbol{\beta}, t) P(\boldsymbol{\beta}, 0).}$$

Green function (kernel of the superpropagator)

$$G(\boldsymbol{\alpha}, \boldsymbol{\beta}, t) = \tfrac{1}{\det \mathbf{B}(t)} \mathrm{e}^{-(\mathbf{q}^*(t), \mathbf{B}^{-1}(t)\mathbf{q}(t))}, \quad \mathbf{q}(t) = \mathbf{P}(t)\boldsymbol{\beta} - \boldsymbol{\alpha}.$$

Two-mode system

Intermode coupling vectors

$$i\boldsymbol{\Omega} + \boldsymbol{\Gamma} = \sum_{k=0}^{3} (i\omega_k + \gamma_k)\boldsymbol{\sigma}_k = (i\omega_0 + \gamma_0)\boldsymbol{\sigma}_0 + (i\boldsymbol{\omega} + \boldsymbol{\gamma}, \boldsymbol{\sigma}),$$

where $\boldsymbol{\gamma} = \gamma\mathbf{n}_\Gamma = \gamma(\sin\theta_\Gamma\cos\phi_\Gamma, \sin\theta_\Gamma\sin\phi_\Gamma, \cos\theta_\Gamma)$ is the relaxation rate vector and $\boldsymbol{\omega} = \omega\mathbf{n}_\Omega = \omega(\sin\theta_\Omega\cos\phi_\Omega, \sin\theta_\Omega\sin\phi_\Omega, \cos\theta_\Omega)$ is the frequency vector.

$$\boxed{\mathbf{P}(t) = \mathrm{e}^{-(i\omega_0+\gamma_0)t}\left\{\cosh(qt)\boldsymbol{\sigma}_0 - \frac{\sinh(qt)}{q}(i\boldsymbol{\omega} + \boldsymbol{\gamma}, \boldsymbol{\sigma})\right\},}$$

$$q = \sqrt{(\boldsymbol{\gamma} + i\boldsymbol{\omega}, \boldsymbol{\gamma} + i\boldsymbol{\omega})} = \sqrt{\gamma^2 - \omega^2 + 2i\gamma\omega(\mathbf{n}_\Omega, \mathbf{n}_\Gamma)}.$$

Comparison between the methods

Algebraic approach

- Gives transformations that explicitly relate quadratic Lindbladians and effective non-Hermitian Hamiltonians;
- Demonstrates efficiency in solving single-mode spectral problem (expressions for the eigenoperators and the biorthogonality relations are relatively easy to derive);
- Applicable for developing of algebraic perturbative techniques and has potential as a tool for improving numerical algorithms.

Method of characteristic function

- Solving equation for the normally ordered characteristic function avoids dealing with matrix Riccati equations;
- Higher order moments can be easily computed;
- Gives the Green function;
- Provides the steady state matrix W used in the algebraic analysis.

Exceptional points (EP)

EP of a parameter dependent matrix

The point in the parameter space where two or more eigenvalues and the corresponding eigenvectors coalesce (reduced dimensionality of eigenspaces and normal Jordarn block structure).

EP of $\hat{L}_{\mathrm{eff}} = \hat{J}_L$ is the EP of the matrix $L = -i\Omega - \Gamma$

EP condition at $N = 2$

$$q = 0 \Rightarrow \omega = \gamma \text{ and } \mathbf{n}_\Gamma \perp \mathbf{n}_\Omega \Rightarrow$$

$$\Rightarrow P(t)|_{q=0} = \mathrm{e}^{-(i\omega_0+\gamma_0)t}\Big\{I - \gamma t(i\boldsymbol{\omega} + \boldsymbol{\gamma}, \boldsymbol{\sigma})\Big\}.$$

Outline

Quantum speed limits for pure states and unitary evolution

Mandelstam-Tamm uncertainty relation

$$\partial_t \hat{S} = \frac{i}{\hbar}[\hat{H}, \hat{S}] \Rightarrow \Delta H \Delta S \geq \frac{\hbar}{2}|\langle \partial_t \hat{S} \rangle| \xrightarrow{\hat{S}=|\psi(t)\rangle\langle\psi(t)|}$$

$$t \geq \frac{\hbar}{\Delta H} \arccos |\langle \psi(0)|\psi(t)\rangle| \leq t_{QSL} = \frac{\pi}{2}\frac{\hbar}{\Delta H},$$

$$\boxed{t \geq \frac{\pi}{2}\frac{\hbar}{\Delta H},} \quad \Delta H = \sqrt{\langle \hat{H}^2 \rangle - \langle \hat{H} \rangle^2}.$$

Unified (Mandelstam-Tamm and Margolus-Levitin) QSL bound

$$t \geq t_{QSL} = \frac{\pi}{2} \max\left\{ \frac{\hbar}{\Delta H}, \frac{\hbar}{\langle \hat{H} \rangle - E_g} \right\}.$$

QSL speeds and times for mixed states

Distinguishability measures

$\|\hat{\rho}(t) - \hat{\rho}(0)\|_p, \quad \|\hat{A}\|_p = (\mathrm{Tr}\,|\hat{A}|^p)^{1/p}$ is the Schatten-p-norm,

$\mathcal{F}_U(\rho(\hat{0}), \hat{\rho}(t)) = \|\sqrt{\hat{\rho}(0)}\sqrt{\hat{\rho}(t)}\|_1^2$ is the Uhlmann fidelity $\xrightarrow{\hat{\rho}(0)=|\psi(0)\rangle\langle\psi(0)|}$

$\mathcal{F}(0,t) \equiv \mathrm{Tr}[\hat{\rho}(0)\hat{\rho}(t)]$, $\mathcal{F}(t_1, t_2) \equiv \mathrm{Tr}[\hat{\rho}(t_1)\hat{\rho}(t_2)]$ is the fidelity of evolution.

Speed of evolution as an upper bound of the rate of change

$$\boxed{|\dot{\mathcal{F}}(0,t)| \le v(t) = \left\|\frac{\partial\hat{\rho}(t)}{\partial t}\right\|_2,} \qquad \boxed{\left|\frac{\partial}{\partial t}\|\hat{\rho}(t) - \hat{\rho}(0)\|_2\right| \le v(t).}$$

Time-dependent quantum speed limit (QSL) times

$$t_{HS}(t) = \frac{\|\hat{\rho}(t) - \hat{\rho}(0)\|_2}{\langle v\rangle_t}, \quad t_F(t) = \frac{1 - \mathcal{F}(t)}{\langle v\rangle_t}, \quad \langle v\rangle_t \equiv \frac{1}{t}\int_0^t v(\tau)\mathrm{d}\tau,$$

$$t \ge t_{HS} \ge t_F.$$

Outline

1. **Lindblad master equation approach**
   - Algebraic approach
   - Method of characteristic functions
2. **Speed of evolution**
   - Gaussian states
   - Polarization qubit states

Speed of evolution [Kiselev et al. Entropy 24, 1774 (2022)]

Initial and evolved states

$$\boxed{|\psi(0)\rangle = \exp[\eta \hat{K}_+ - \eta^* \hat{K}_-]|0\rangle}, \quad \eta = r e^{i\theta_0}, \quad \hat{K}_+ = \hat{K}_-^\dagger = \hat{a}_1^\dagger \hat{a}_2^\dagger,$$

$$\chi_{\rm ini}(\boldsymbol{\alpha}) = \exp\Big\{-\frac{1}{2}(\boldsymbol{\alpha}^*, \mathbf{Q}_0\boldsymbol{\alpha}) + \frac{1}{4}[(\boldsymbol{\alpha}, \mathbf{P}_0\boldsymbol{\alpha}) + (\boldsymbol{\alpha}, \mathbf{P}_0\boldsymbol{\alpha})^*]\Big\}.$$

$$\mathbf{Q}_0 = \cosh(2r)\boldsymbol{\sigma}_0, \quad \mathbf{P}_0 = \sinh(2r)e^{-i\theta_0}\boldsymbol{\sigma}_1,$$

$$\chi(\mathbf{s}, t) = \exp\left\{-\frac{1}{4}(\mathbf{s}, \mathbf{G}(t)\mathbf{s})\right\}, \mathbf{s} = \sqrt{2}(\mathrm{Re}\,\alpha_1, \dots, \mathrm{Im}\,\alpha_1, \dots, \mathrm{Im}\,\alpha_N).$$

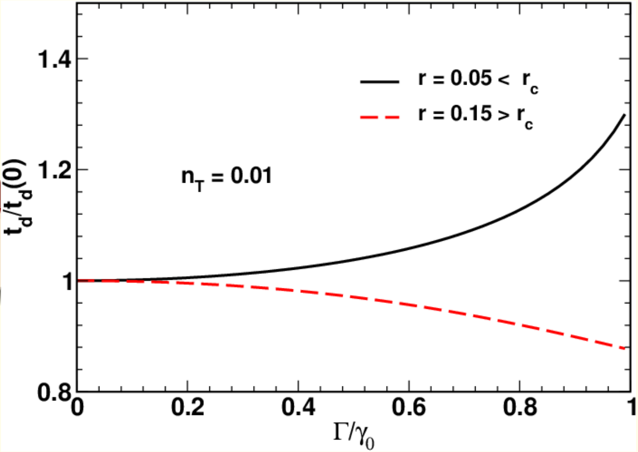
Fidelity and speed of evolution

$$\mathcal{F}(t_1, t_2) = 2^N[\det(\mathbf{G}(t_1) + \mathbf{G}(t_2))]^{-1/2},$$

$$v^2(t) = \frac{1}{16(2\pi)^N}\int (\mathbf{s}, \dot{\mathbf{G}}(t)\mathbf{s})^2 \exp\left\{-\frac{1}{2}(\mathbf{s}, \mathbf{G}(t)\mathbf{s})\right\} \mathrm{d}^{2N}\mathbf{s}$$

$$= \frac{1}{16\sqrt{\det \mathbf{G}}}\{[\mathrm{Tr}(\dot{\mathbf{G}}\mathbf{G}^{-1})]^2 + 2\,\mathrm{Tr}([\dot{\mathbf{G}}\mathbf{G}^{-1}]^2)\}.$$

Time of disentanglement

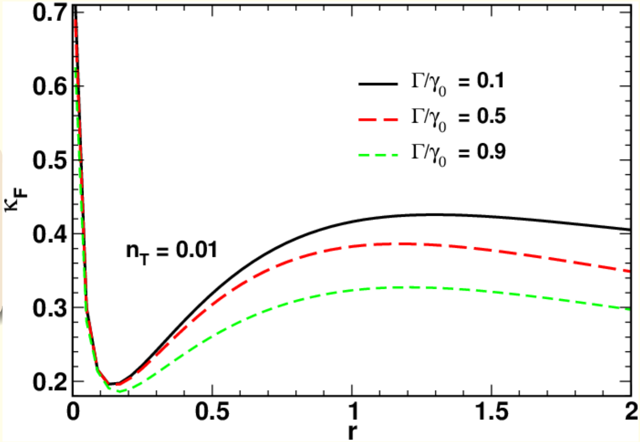


Parameters

$\gamma_0 t_{\mathrm{d}}(0) =$
$\ln(1 + (1 - \mathrm{e}^{-2r})/2n_T)$

Long-time asymptotic ratio for fidelity based QSL time at different temperatures

$$t_F/t \xrightarrow{t\to\infty} \kappa_F = (1 - \mathcal{F}(\infty))/s_\infty,$$
$$s_\infty = \int_0^\infty v(t)\mathrm{d}t$$



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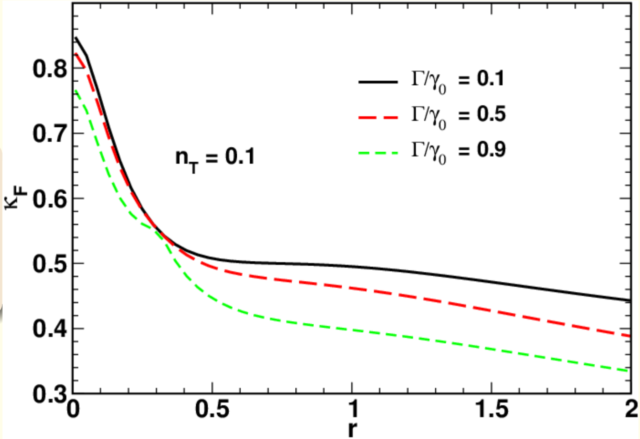

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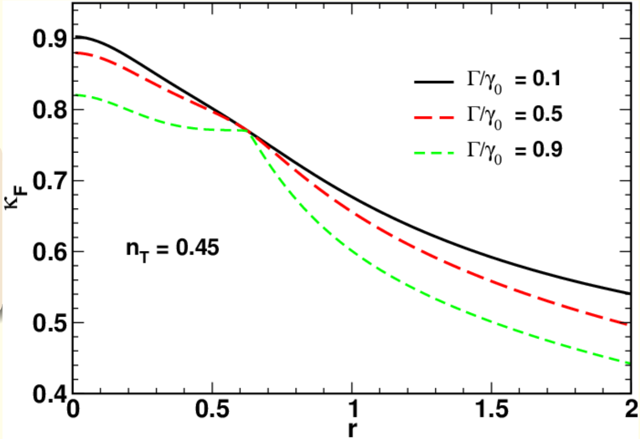
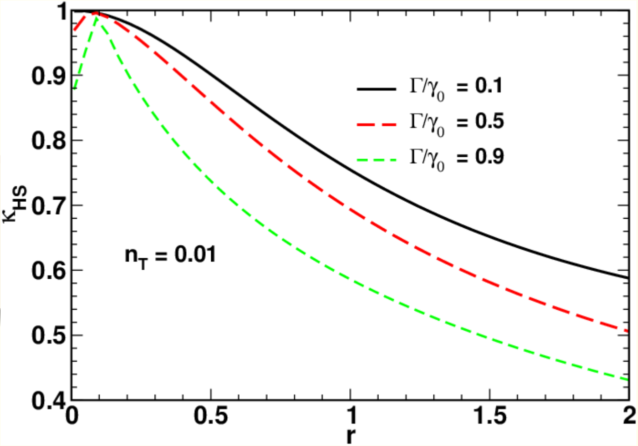


Long-time asymptotic ratio for Hilbert-Schmidt distance based QSL time at different temperatures

$$t_{HS}/t \xrightarrow{t\to\infty} \kappa_{HS} = \|\hat{\rho}_{\text{eq}} - \hat{\rho}(0)\|_2 / s_\infty,$$
$$s_\infty = \int_0^\infty v(t)\mathrm{d}t$$



Long-time asymptotic ratio for Hilbert-Schmidt distance based QSL time at different temperatures

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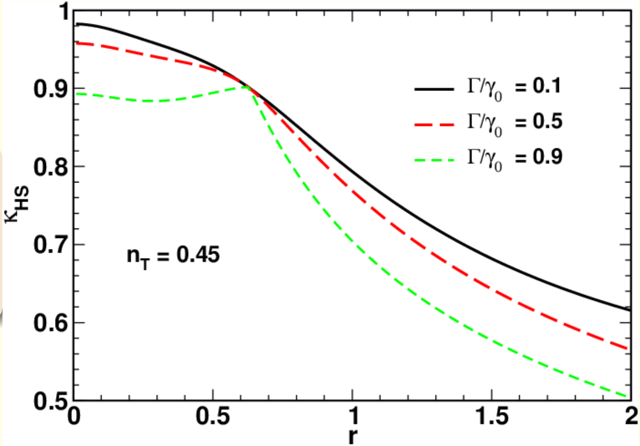

Long-time asymptotic ratio for Hilbert-Schmidt distance based QSL time at different temperatures

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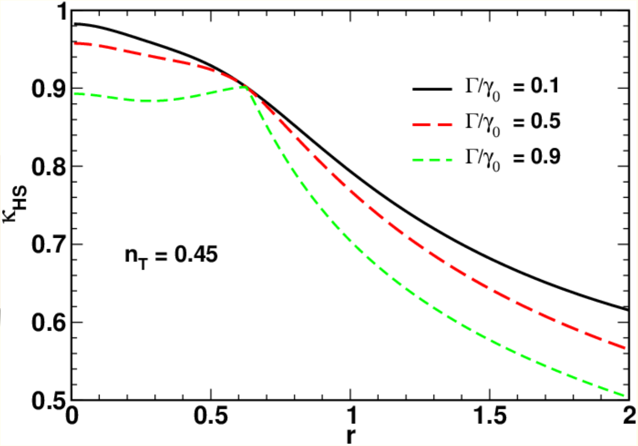


Outline

Low-temperature approximation

Initial polarization qubit state

$$|\psi(0)\rangle = \cos\frac{\theta}{2}|1_H, 0_V\rangle + e^{i\phi}\sin\frac{\theta}{2}|0_H, 1_V\rangle,$$

$$\hat{\rho}(0) = \hat{\mathcal{K}}^{(+)}_{R_0}|\mathbf{0}\rangle\langle\mathbf{0}|, \quad R_0 = \frac{1}{2}\left\{\sigma_0 + (\mathbf{n}, \boldsymbol{\sigma})\right\},$$

Linear approximation

$$\hat{\rho}(t) \approx \hat{\rho}_0(t) + n_T\hat{\rho}_1(t),$$

Zero-temperature and first-order terms

$$\hat{\rho}_0(t) = (\hat{\mathcal{K}}^{(+)}_{R} + r)|\mathbf{0}\rangle\langle\mathbf{0}|, \quad R = P(t)R_0P^\dagger(t),\ r = 1 - \operatorname{Tr} R,$$

$$\hat{\rho}_1(t) = (\hat{\mathcal{K}}^{(+)}_{Q}\hat{\mathcal{K}}^{(+)}_{R} + \hat{\mathcal{K}}^{(+)}_{V} + v)|\mathbf{0}\rangle\langle\mathbf{0}|, \quad Q = 1 - P(t)P^\dagger(t),$$

$$V = Q - Q\operatorname{Tr} R - R\operatorname{Tr} Q - \{R, Q\}, \quad v = \operatorname{Tr}(RQ) + \operatorname{Tr} R\operatorname{Tr} Q - \operatorname{Tr} Q.$$

Exact result

$$v^2(t) = \det G\Big\{F(\dot{R},\dot{R}) - 2n_T\left[F(\dot{P},\dot{R}) - F(\dot{P},\dot{R},R)\right] + n_T^2\left[F(\dot{P},\dot{P})\right.$$
$$\left. - 2F(\dot{P},\dot{P},R) + F(\dot{P},\dot{P},R,R)\right]\Big\}, \quad G = (I + 2n_T(I - PP^\dagger))^{-1},$$

where

$$F(A_1,A_2) = \sum_{p\in S_2}\sum_{i_1,i_2}(\tilde{A}_1)_{i_1 i_{p(1)}}(\tilde{A}_2)_{i_2 i_{p(2)}} = \mathrm{Tr}(\tilde{A}_1)\,\mathrm{Tr}(\tilde{A}_2) + \mathrm{Tr}(\tilde{A}_1\tilde{A}_2),$$

$$F(A_1,A_2,A_3) = \sum_{p\in S_3}\sum_{i_1,i_2,i_3}(\tilde{A}_1)_{i_1 i_{p(1)}}(\tilde{A}_2)_{i_2 i_{p(2)}}(\tilde{A}_3)_{i_3 i_{p(3)}}$$
$$= \mathrm{Tr}(\tilde{A}_1)\,\mathrm{Tr}(\tilde{A}_2)\,\mathrm{Tr}(\tilde{A}_3) + \mathrm{Tr}(\tilde{A}_1)\,\mathrm{Tr}(\tilde{A}_2\tilde{A}_3) + \mathrm{Tr}(\tilde{A}_2)\,\mathrm{Tr}(\tilde{A}_1\tilde{A}_3)$$
$$+ \mathrm{Tr}(\tilde{A}_3)\,\mathrm{Tr}(\tilde{A}_1\tilde{A}_2) + \mathrm{Tr}(\tilde{A}_1\tilde{A}_2\tilde{A}_3) + \mathrm{Tr}(\tilde{A}_1\tilde{A}_3\tilde{A}_2),$$

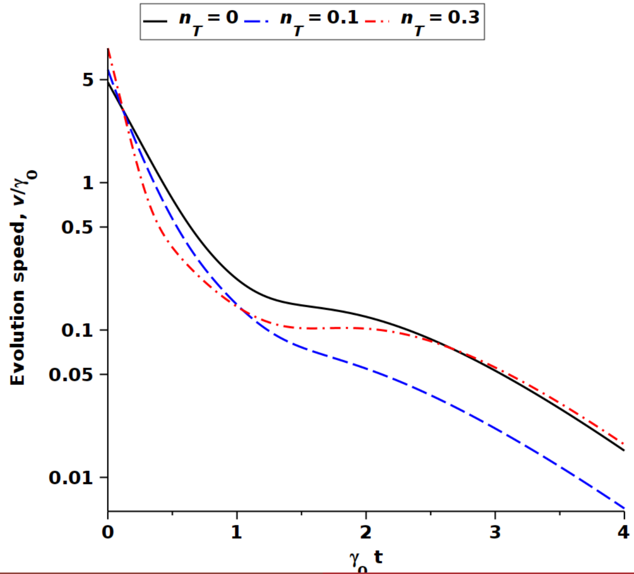
$$F(A_1,A_2,A_3,A_4) = \sum_{p\in S_4}\sum_{i_1,i_2,i_3,i_4}(\tilde{A}_1)_{i_1 i_{p(1)}}(\tilde{A}_2)_{i_2 i_{p(2)}}(\tilde{A}_3)_{i_3 i_{p(3)}}(\tilde{A}_4)_{i_4 i_{p(4)}} = \ldots,$$

$$\tilde{A}_i = A_i G.$$

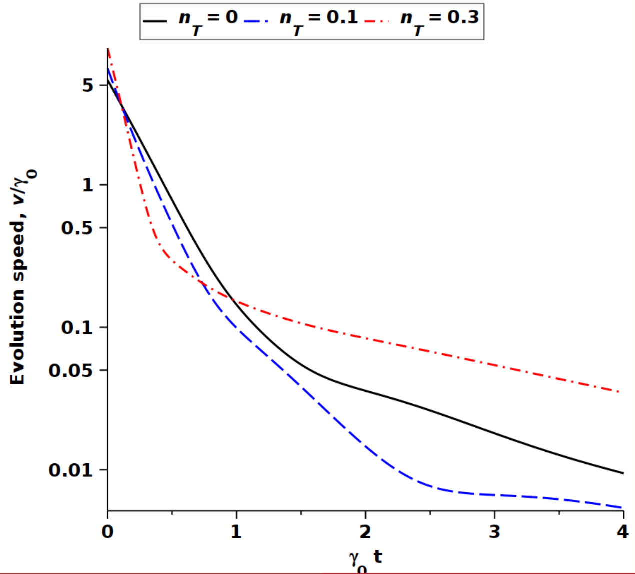
Speed of evolution as a function of time at $\omega = \gamma$ and different θ_Γ

Parameters

$\theta = \pi/4$, $\phi = 0$, $\boldsymbol{\omega} = 0.9\gamma_0(0, 0, 1)$, $\boldsymbol{\gamma} = 0.9\gamma_0(\sin\theta_\Gamma, 0, \cos\theta_\Gamma)$, $\theta_\Gamma = \pi/2$.



Speed of evolution as a function of time at $\omega = \gamma$ and different θ_Γ



Parameters

$\theta = \pi/4$, $\phi = 0$, $\boldsymbol{\omega} = 0.9\gamma_0(0, 0, 1)$, $\boldsymbol{\gamma} = 0.9\gamma_0(\sin\theta_\Gamma, 0, \cos\theta_\Gamma)$, $\theta_\Gamma = \pi/4$.

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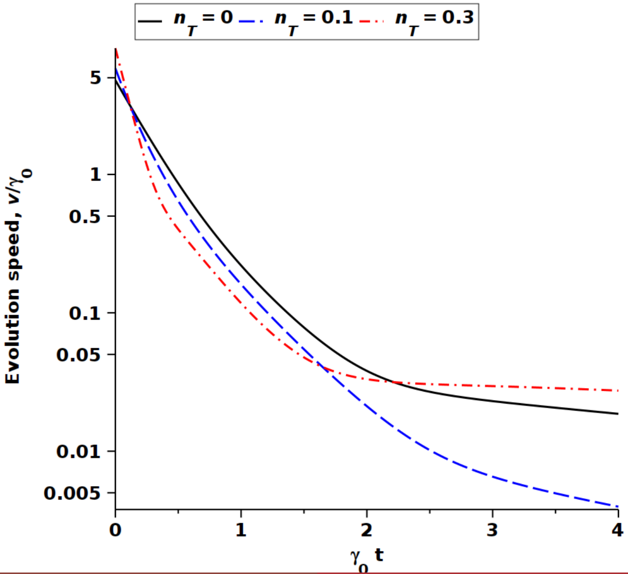


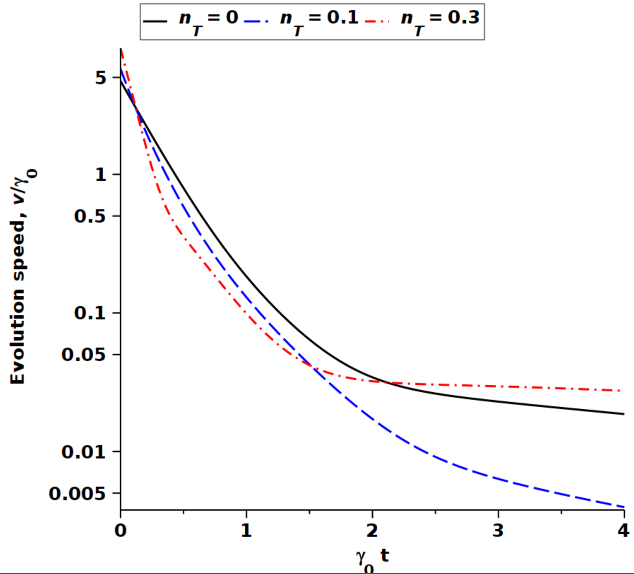
Parameters

$\theta = \pi/4$, $\phi = 0$,
$\boldsymbol{\omega} = 0.9\gamma_0(0, 0, 1)$, $\boldsymbol{\gamma} = 0.9\gamma_0(\sin\theta_\Gamma, 0, \cos\theta_\Gamma)$,
$\theta_\Gamma = 0$.

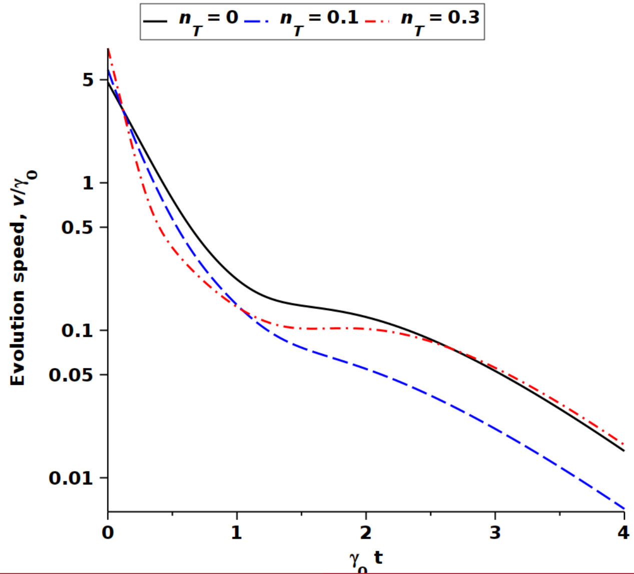
Speed of evolution as a function of time at $\theta_\Gamma = \pi/2$ and different ω

Parameters

$\theta = \pi/4$, $\phi = 0$,
$\boldsymbol{\omega} = \omega(0, 0, 1)$,
$\boldsymbol{\gamma} = 0.9\gamma_0(1, 0, 0)$,
$\omega = 0$.



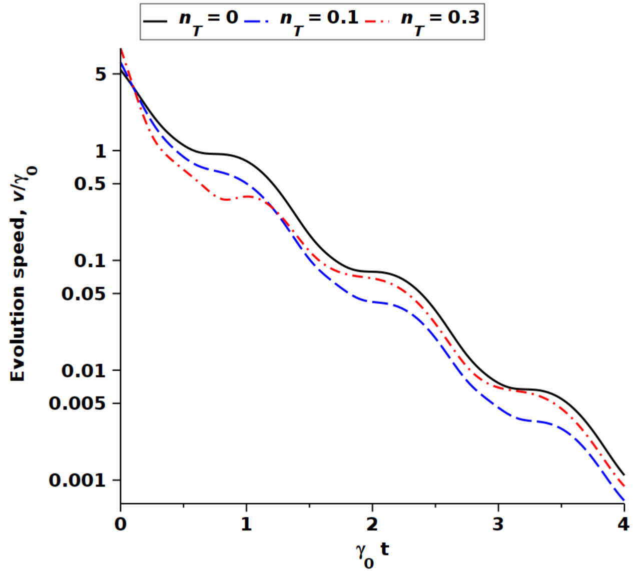
Speed of evolution as a function of time at $\theta_\Gamma = \pi/2$ and different ω



Parameters

$\theta = \pi/4$, $\phi = 0$,
$\boldsymbol{\omega} = \omega(0, 0, 1)$,
$\boldsymbol{\gamma} = 0.9\gamma_0(1, 0, 0)$,
$\omega = \gamma$.

Speed of evolution as a function of time at $\theta_\Gamma = \pi/2$ and different ω



Parameters

$\theta = \pi/4$, $\phi = 0$,
$\boldsymbol{\omega} = \omega(0, 0, 1)$,
$\boldsymbol{\gamma} = 0.9\gamma_0(1, 0, 0)$,
$\omega = 3\gamma$.

Conclusions

- Jump-eliminating transformations of the algebraic method are determined by the steady state density matrix.
- In two-mode bosonic system, geometry of coherent and incoherent interactions between modes and EP conditions can be described in terms of the frequency and the relaxation rate vectors.
- Intermode coupling have a pronounced effect on QSL times and speed of evolution.

Thank you for attention!