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ТРИ ВЕКА РОССИЙСКОЙ МАТЕМАТИКИ

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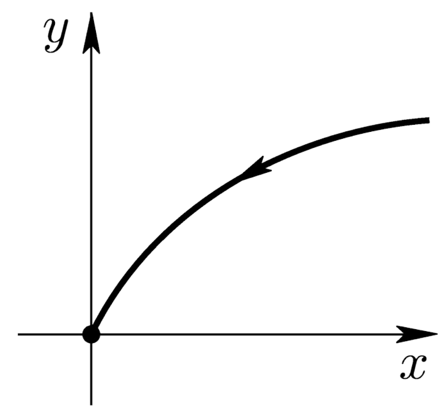
$$e^{ix} = \cos x + i \sin x$$
$$e^{i\pi} = -1$$
$$\log(-1) = i\pi + 2ki\pi \ (k = 0,\ \pm 1,\ \pm 2,\ \ldots)$$
$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right) \qquad \cos x = \prod_{n=1}^{\infty}\left(1 - \frac{4x^2}{(2k+1)^2\pi^2}\right)$$
$$1 + \frac{1}{2^{2k}} + \frac{1}{3^{2k}} + \cdots + \frac{1}{n^{2k}} + \ldots = \frac{(-1)^{k-1}2^{2k-1}B_{2k}}{(2k)!}\pi^{2k}$$
$$\frac{\pi}{\sin s\pi} = \frac{1}{s} + \sum_{n=1}^{\infty}(-1)^n\left(\frac{1}{n+s} - \frac{1}{n-s}\right)$$
$$\pi \operatorname{ctg} s\pi = \frac{1}{s} + \sum_{n=1}^{\infty}\left(\frac{1}{n+s} - \frac{1}{n-s}\right)$$
$$\frac{\pi}{3\sqrt{3}} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{5} + \frac{1}{7} - \frac{1}{8} + \frac{1}{10} - \ldots$$
$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \ldots$$
$$\frac{\pi}{3} = 1 + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} - \ldots$$
$$\frac{\pi^2}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \ldots$$
$$\frac{\pi^2}{6\sqrt{3}} = 1 - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} - \ldots$$
$$\sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{\text{простые числа } p}\left(1 - \frac{1}{p^s}\right)^{-1}$$

$$\frac{1 - 2^{m-1} + 3^{m-1} - \cdots}{1 - 2^{-m} + 3^{-m} - \cdots} = -\frac{1\cdot 2\cdot 3\cdot \ldots \cdot (m-1)\,(2^m - 1)}{(2^{m-1} - 1)\,\pi^m \cos\frac{m\pi}{2}}$$
$$\sum_{n>m>0}\frac{1}{n^2 m} = \sum_{n>0}\frac{1}{n^3}$$
$$\prod_{n=1}^{\infty}(1 - x^n) = 1 + \sum_{n=1}^{\infty}(-1)^n\left(x^{\frac{3n^2-n}{2}} + x^{\frac{3n^2+n}{2}}\right)$$
$$1 - 1!x + 2!x^2 - 3!x^3 + \ldots = \frac{1}{1+}\,\frac{x}{1+}\,\frac{x}{1+}\,\frac{2x}{1+}\,\frac{2x}{1+}\,\frac{3x}{1+}\,\frac{3x}{1+}\ldots$$
$$1 - 1! + 2! - 3! + \ldots = 0{,}596347362123\ldots$$
$$\sum_{k=0}^{m} f(k) = \int_0^m f(x)\,dx + \frac{1}{2}\left(f(0) + f(m)\right) +$$
$$+ \sum_{k\geqslant 1}\frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(m) - f^{(2k-1)}(0)\right)$$
$$\frac{\partial}{\partial y}F(x,\ y,\ y') = \frac{d}{dx}\left(\frac{\partial}{\partial y'}F(x,\ y,\ y')\right)$$

$$1 - 1! + 2! - 3! + 4! - \cdots = 0.5963\ldots \tag{E}$$



$$\dot{x} = x - y, \qquad \dot{y} = -y^2$$

$$x(t) \sim \sum_{k=1}^{\infty} (-1)^{k+1} \frac{(k-1)!}{t^k}, \qquad y(t) = \frac{1}{t}$$

A solution

$$x(t) = e^t \int_t^{\infty} \frac{e^{-u}}{u}\, du, \qquad y(t) = \frac{1}{t}$$

$$t = 1: \qquad e \int_1^{\infty} \frac{e^{-u}}{u}\, du = 0.5963\ldots$$

A generalization: $\dot{x} = v(x)$, $v(0) = 0$; $x \in \mathbb{R}^n$, $v \in C^\infty$.

THEOREM (A. N. Kuznetsov, 1972). *Let*

$$\sum_{j=1}^{\infty} \frac{x^{(j)}}{(t^\mu)^j}, \qquad x^{(j)} \in \mathbb{R}^n, \quad \mu = \text{const} > 0.$$

be a formal solution. Then there exists a true solution $t \mapsto x(t)$ *such that*

1) $x(t) \to 0$ *as* $t \to \infty$,

2) $x(t) - \sum_{j=1}^{N} \frac{x^{(j)}}{(t^\mu)^j} = O\left(\frac{1}{t^{(N+1)\mu}}\right), \qquad t \to \infty.$

An application: an energy criterion for stability

Lagrange's equations

$$\left(\frac{\partial T}{\partial \dot{x}}\right)^{\cdot} - \frac{\partial T}{\partial x} = -\frac{\partial V}{\partial x}, \qquad x \in \mathbb{R}^n,$$

$$T = \frac{1}{2}\sum g_{ij}(x)\,\dot{x}_i\,\dot{x}_j \quad \text{is the kinetic energy}$$

$$V : \mathbb{R}^n \to \mathbb{R} \quad \text{is the potential energy}$$

$$g_{ij}, V \in C^\infty(\mathbb{R}^n)$$

Let $dV(0) = 0$. Then there is an equilibrium at $x = 0$

$$V = V(0) + V_2 + V_m + V_{m+1} + \cdots \quad \text{is the Maclaurin series, } m \geqslant 3$$

$$V_k(\lambda x) = \lambda^k V_k(x)$$

Lyapunov's theorem: *if* V_2 *has no minimum at* $x = 0$, *then this is an unstable equilibruim.*

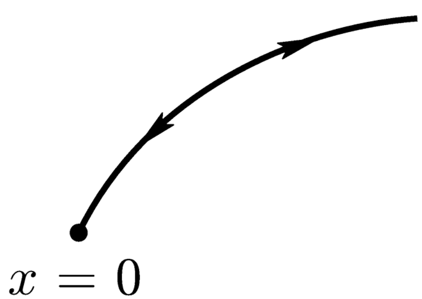
Assume that $V_2 \geq 0$. If V_2 is positive definite, then the equilibrium is stable (the Lagrange–Dirichlet theorem).

$\Pi = \{x \in \mathbb{R}^n \colon V_2(x) = 0\}$ is a linear space; assume that $\dim \Pi \geqslant 1$ and let W_m be the restriction of V_m to Π.

Theorem (1986). *If* W_m *has no minimum at* $x = 0$, *then the equilibrium at* $x = 0$ *is unstable.*

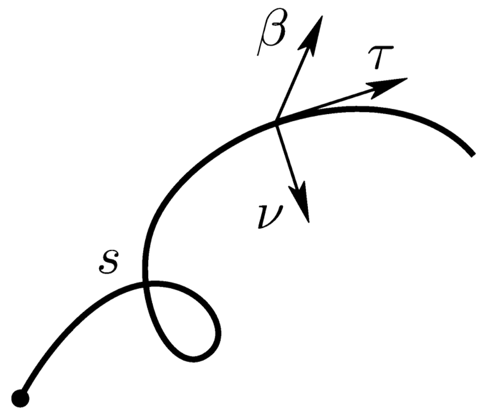
$\mu = \dfrac{2}{m-2}$; for even $m \geqslant 4$ the coefficients $x^{(j)}$ are polynomials of $\ln t$.

If $t \mapsto x(t)$ is a solution, then $t \mapsto x(-t)$ is also a solution.



Euler and mechanics

A. *Natural* equations of motion



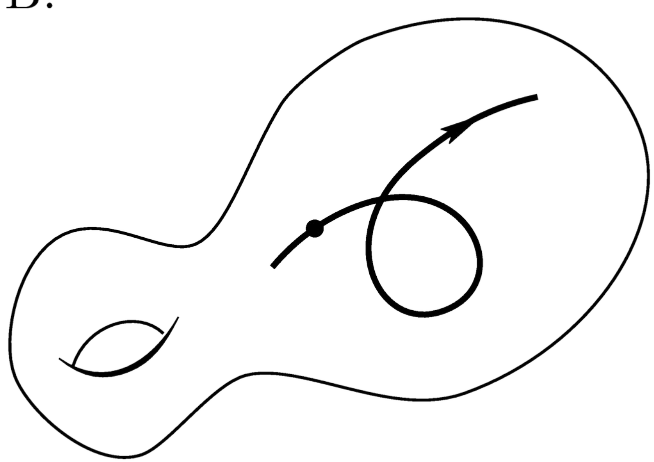
$$m\bar{a} = \overline{F}, \quad \overline{F} = F_\tau \bar{\tau} + F_\nu \bar{\nu} + F_\beta \bar{\beta}$$

$$m\dot{v} = F_\tau, \quad \frac{mv^2}{\rho} = F_\nu, \quad F_\beta = 0$$

$\dot{s} = v$ is the velocity

ρ is the curvature radius of the trajectory

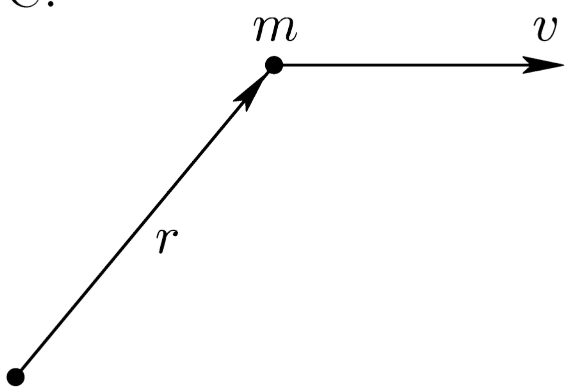
B.



$F = 0$: inertial motion

Trajectories $\equiv$ geodesics

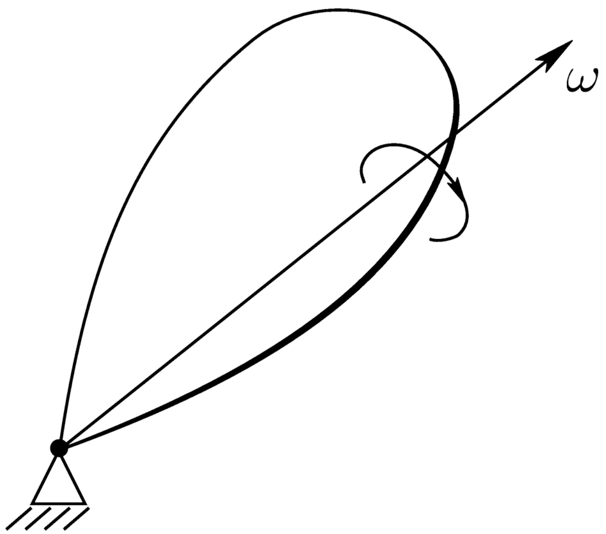
C.



$K = r \times (mv) = mr \times v$ is the angular momentum

$\dot{K} = M, \ \ M = r \times F$ is the torque

D.



$I\dot{\omega} + \omega \times I\omega = M$

ω is the angular velocity of the body in moving space

I is the inertia operator

M is the total torque

E.

$$\delta \int_{t_1}^{t_2} T\,dt = 0, \qquad T + V = h = \text{const}$$

$$T = \frac{1}{2} \sum g_{ij}\, \dot{x}_i\, \dot{x}_j, \quad ds^2 = (h - V) \sum g_{ij}\, dx_i\, dx_j \ \text{ is the Jacobi metric}$$

"MAUPERTUIS PRINCIPLE": the trajectories of the motion with total energy $h = T + V$ are geodesics of the metric ds.

F. Equations of motion of an ideal fluid:

$$\rho \left(\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\, v \right) = -\frac{\partial p}{\partial x} - \rho\, \frac{\partial V}{\partial x}, \qquad \frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0$$

Euler's equation — the equation of continuity

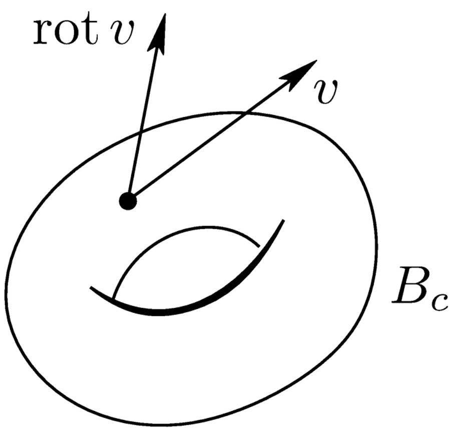
$$\frac{\partial v}{\partial t} + (\operatorname{rot} v) \times v = -\frac{\partial h}{\partial x} \quad \text{Lamb's equation,}$$

$$h = \frac{v^2}{2} + \mathscr{P} + V \quad \text{is the Bernoulli function,}$$

$\mathscr{P}$ is the pressure function: $\rho^{-1}\, dp = d\mathscr{P}$

Stationary flows in a bounded domain

$B_c = \{x \colon h(x) = c\}$ are the Bernoulli surfaces

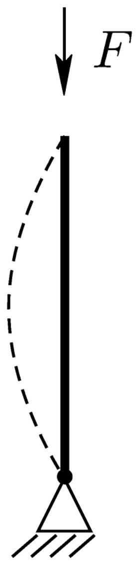


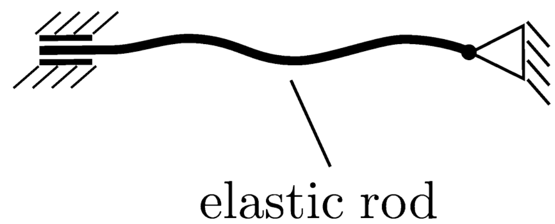
v and $\operatorname{rot} v$ are tangent to B_c,

furthermore, v and $\dfrac{\operatorname{rot} v}{\rho}$ commute

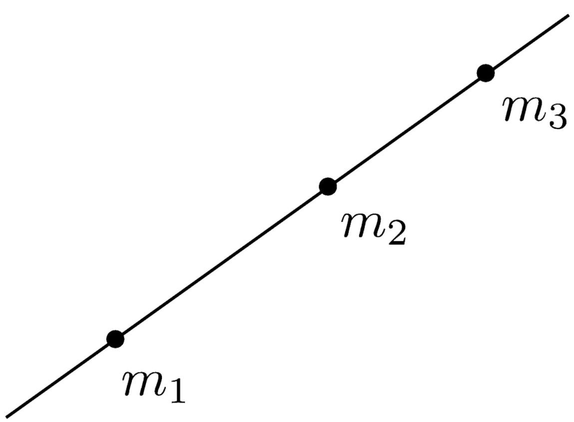
B_c are 2-tori foliated by almost periodic trajectories

G. Elasticity theory

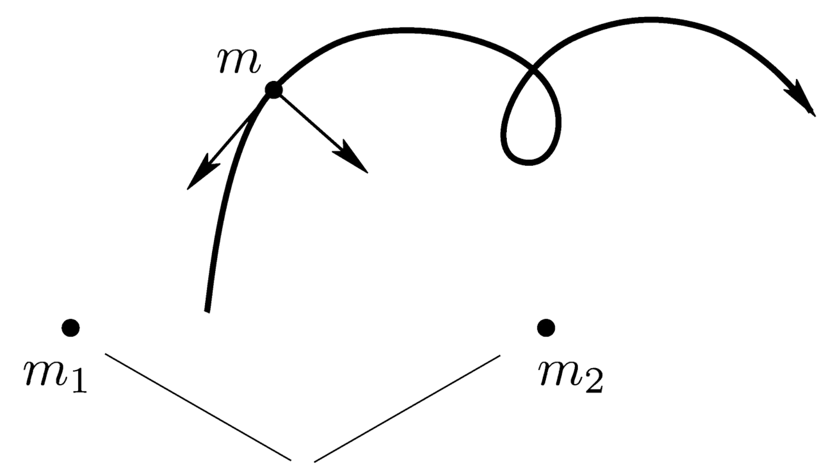




H. Celestial mechanics



Collinear solutions
of the three body problem



Two-center problem

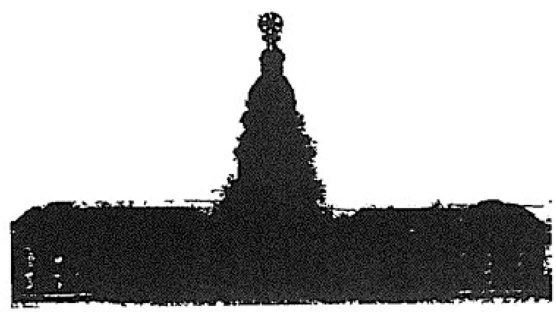
РУКОВОДСТВО
КЪ
АРИѲМЕТИКѢ
ДЛЯ УПОТРЕБЛЕНІЯ
ГИМНАЗІИ

при

ИМПЕРАТОРСКОЙ

АКАДЕМІИ НАУКЪ

переведено съ Нѣмецкаго языка
чрезъ Василья Адодурова
Академіи Наукъ Адъюнкта.



ВЪ САНКТПЕТЕРБУРГѢ

1740


АКАДЕМИЯ НАУК СССР

В. И. СМИРНОВ и Е. С. КУЛЯБКО

МИХАИЛ СОФРОНОВ
РУССКИЙ МАТЕМАТИК
СЕРЕДИНЫ XVIII ВЕКА

ИЗДАТЕЛЬСТВО
АКАДЕМИИ НАУК СССР
1954

$x^2 - 2x\cos\varphi + 1 = 0$; корни $e^{i\varphi}$ и $e^{-i\varphi}$

Формула Эйлера: $e^{i\varphi} = 2\cos\varphi - \frac{1}{e^{i\varphi}}$

$$e^{i\varphi} = 2\cos\varphi - \cfrac{1}{2\cos\varphi - \cfrac{1}{2\cos\varphi\cdots}}$$

Расходится при всех $x \neq \pi k$ (k — целое)

Подходящая дробь с номером n равна

$$\frac{\sin(n+1)\varphi}{\sin n\varphi} = \cos\varphi + \frac{\cos n\varphi}{\sin n\varphi}\sin\varphi$$

сдвиг $\varphi \mapsto \varphi + i\varepsilon$, $\varepsilon \neq 0$ $\mathrm{ctg}\varphi \mapsto f_\varepsilon(\varphi) = \frac{g_\varepsilon'}{g_\varepsilon}$, $g_\varepsilon(\varphi) = e^{i\varphi}e^{-\varepsilon} - e^{-i\varphi}e^{\varepsilon}$.

При $\varepsilon \neq 0$ функция g_ε не имеет особенностей на $\mathbb{R} = \{\varphi\}$.

Теорема Г. Вейля: если φ несоизмеримо с π, то

$$f_\varepsilon(n\varphi) \to \frac{1}{2\pi}\int_0^{2\pi} f_\varepsilon(x)dx \quad (C).$$

Интеграл равен $-2\pi i$ $(2\pi i)$, если $\varepsilon > 0$ $(\varepsilon < 0)$.

$\varepsilon \to 0 \Rightarrow n$-ая подходящая дробь сходится в среднем (после сколь угодно малого смещания ε в $\mathbb{C}$) как раз к $\cos\varphi \mp i\sin\varphi = e^{\mp i\varphi}$.

Обобщение: пусть $z \mapsto f(z)$ — мероморфная функция на $\mathbb{C}$, причем

1) $f(z+2\pi) = f(z)$ для всех $z \in \mathbb{C}$

2) f принимает вещественные значения при $z \in \mathbb{R}$

Положим $f_\varepsilon(\varphi) = f(\varphi + i\varepsilon)$, $\varepsilon \in \mathbb{R}$.

Теорема. Пусть α/π иррационально. Тогда при малых $\varepsilon \neq 0$

$$\lim_{n\to\infty} \frac{f_\varepsilon(\alpha) + f_\varepsilon(2\alpha) + \ldots f_\varepsilon(n\alpha)}{n} = a \pm ib \quad (\star)$$

причем

1) a и b не зависят от ε,

2) знак в правой части ($\star$) зависит только от знака ε,

3) $2b$ равно сумме вычетов функции f в полюсах на отрезке $\{0 \leqslant \varphi < 2\pi\} \subset \mathbb{R}$.

Пример: если $f(z) = \mathrm{ctg} z$, то $a = 0$ и на отрезке $\varphi \mod 2\pi$ она имеет два полюса, вычеты в которых равны 1. Следовательно, $b = 1$.

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