On Some Surfaces and Space Curves in Ancient Greek Mathematics*

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GREECE AND ITS COLONIES IN 550 B.C.

Geometers of Ancient Greece worked from Southern Italy to Asia Minor (modern Turkey). From the 4th century BC, Athens became the scientific center. From the 3rd century BC, Alexandria (Egypt) played a major role.

The founder of the Milesian school is **Thales of Miletus**, who was a contemporary of Solon of Athens. Miletus is located on the western coast of Asia Minor, southeast of Samos. Thales died at the age of 78 in the 58th Olympiad, 548–545 BC. However, all theorems attributed to Thales are related to plane geometry, not stereometry.

The school at Croton, the Achaean capital of Southern Italy, was founded by **Pythagoras of Samos** (ca. 570–490 BC). Famous Pythagoreans included Hippasus of Metapontum, **Philolaus** of Croton or Tarentum (born ca. 474 BC), Theodore of Cyrene, and **Archytas of Tarentum**, who was a contemporary of Plato. Archytas was a strategos (an army commander). He is considered the creator of the theory of mechanisms and machines. (W. K. C. Guthrie, 1962; Iamblichus.)

An elaborate history of Greek geometry before Euclid was written by Eudemus, the pupil of Aristotle, who lived about 330 BC.

Philolaus of Croton (ca. 474–385 BC) is a scholar in the Pythagorean tradition (Carl A. Huffman, 1993). He proposed a model of the world in which the Earth, Moon, Sun, and planets revolve around Hestia, i. e., the central fire. Another model of the world was proposed by Heraclides of Pontus (ca. 388–310 BC).

Diogenes Laertius (3rd century AD) (E. V. Afonasin, 2019)

> Heraclides, the son of Euthyphron, was a citizen of Heraclea in Pontus. He was very rich. At Athens he first followed Speusippus, but he also attended the lectures of the Pythagoreans and became an ardent follower of Plato. Afterwards he also heard Aristotle, as Sotion relates in his *Successions*. He wore soft clothes and was so plump that the Atticans called him not Pontic (Ποντικον), but Fat-Belly (Πομπικον). However, he seemed kind and majestic. Beautiful and magnificent writings are attributed to him.

In the 3rd century BC, the heliocentric model was created by Aristarchus of Samos. Hipparchus of Nicaea (ca. 190–120 BC) and Geminus worked in Rhodes.

Known surfaces:

sphere, hemisphere, cylinder, semi-cylinder, cone, truncated cone, spheroid, rectangular conoid, i.e., paraboloid of revolution, obtuse-angled conoid, i.e., one cavity of a two-sheeted hyperboloid of revolution, bicylinder, i.e., the Steinmetz solid surface (J. P. Hogendijk, 2002), vault, i.e., half of a bicylinder, torus, and polyhedral surfaces.

When did the torus (σπειρα) become known?

Latin *torus* means the base of a column.

The torus (**σπειρα**) is explicitly mentioned only by Perseus (2nd century BC) and Geminus of Rhodes (1st century BC) in the exposition of Heron of Alexandria (1st century AD) and Proclus Diadochus (5th century AD). (Chasles, 1837; Gow, 1884).

The surface of revolution can be defined in the same way that Euclid defined the sphere (σφαιρα) in the *Elements*.

ιζ΄. Διάμετρος δὲ τῆς σφαίρας ἐστὶν εὐθεῖά τις διὰ τοῦ κέντρου ἠγμένη καὶ περατουμένη ἐφ᾽ ἐκάτερα τὰ μέρη ὑπὸ τῆς ἐπιφανείας τῆς σφαίρας.

14. A sphere is the figure enclosed when, the diameter of a semicircle remaining (fixed), the semicircle is carried around, and again established at the same (position) from which it began to be moved.
(Richard Fitzpatrick, 2008, p. 424.)

Known space curves

The Archytas curve (torus–cylinder intersection). The work of Archytas of Tarentum is known from the commentaries written by Eutocius of Ascalon (ca. 480–540 AD). Eutocius' text does not contain a definition of the torus. It was only a matter of turning a semicircle about the end of its diameter. But one can see the torus using reconstruction. A similar description (without reference to Archytas) is given by Muhammad, al-Hasan, and Ahmad Banu Musa, who worked in Baghdad (9th century AD).

The hippopede of Eudoxus (spherical curve). The work of Eudoxus of Cnidus (4th century BC) is known from Aristotle's *Metaphysics* and the commentaries of Simplicius of Cilicia (ca. 490–560 AD). (A. A. Rossius, 2005) Several reconstructions are known (I. N. Veselovskii, 1974, 2010; I. Yavetz, 1998, 2001).

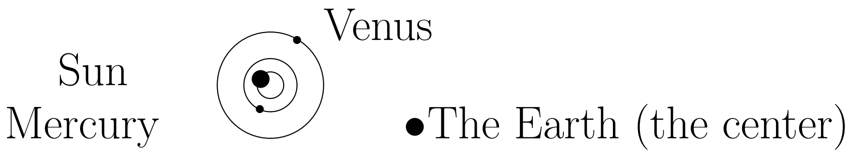
Helical lines. Proclus Diadochus (412–485 BC) in his commentary on Euclid's *Elements* mentioned Apollonius's treatise *On the Cylindrical Helix*. Apollonius of Perga (262–190 BC) could also have studied curves on the torus.

The model of the world in which both Venus and Mercury revolve around the mean position of the Sun, and they together revolve around the Earth, is attributed to Heraclides of Pontus (387–312 BC) (E. V. Afonasin, 2019).

Later, a similar model was used by Tycho Brahe (1546–1601 AD).

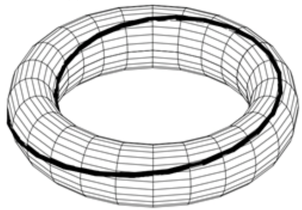
Calcidius (4th century AD), Commentary on Plato's *Timaeus*:

> Such an interpretation seems necessary to them for the reason that in the circle of the Sun, as well as in the circles of each of these moving stars, there is only one center and one point:



> So, Heraclides of Pontus, having drawn the circle of Venus and the circle of the Sun so that their orbits were constructed from one center and one middle, showed that sometimes [on this drawing] Venus appears higher, and sometimes lower than the Sun.

The details of Heraclides' model can only be guessed at, but its reconstruction also suggests that the torus was already being considered in Ancient Greece in the 4th century BC. Since Mercury (Hermes) and Venus (Aphrodite) deviate from the ecliptic plane, these planets move along a curve on a toroidal surface. There is a surface of revolution. The section of the surface by a plane passing through the axis of rotation is a pair of ellipses. But this surface can also be obtained by rotating a circle around an axis inclined to the plane of the circle at some angle.



The same curve on a toroidal surface also appears in the heliocentric system developed by Aristarchus of Samos when describing the motion of the Moon.

Mars: `https://www.astronet.ru/db/msg/1268817` (Tezel, 2012)

The angle of inclination of the orbit to the ecliptic plane equals 5.14° for the Moon, 7.0049863889° for Mercury, 3.3946619444° for Venus, 1.8497263889° for Mars, 1.3032697222° for Jupiter, and 2.4888780556° for Saturn (J2000).
`https://www.sai.msu.ru/neb/rw/natsat/plaorbw.htm`

The angle of inclination was known with sufficient accuracy by **Cleomedes**, who quotes Posidonius of Apamea (ca. 135–50 BC) (A. I. Schetnikov, 2010):

> The Moon is said to deviate the most in either direction from the middle of the zodiac compared to other planets. Next comes Aphrodite, moving 5° in each direction in the planetary motion, Hermes 4°, Ares and Zeus $2\frac{1}{2}$°, Cronus 1° in each direction.

Eventually, a new geocentric model was developed by Apollonius of Perga (ca. 250–170 BC), Hipparchus of Nicaea (ca. 190–120 BC), and Claudius Ptolemy (ca. 100–170 AD). (O. Neugebauer, 1957)

However, in the new model of the world, the planets moving along epicycles are not in the ecliptic plane, but move along a toroidal surface. Therefore, interest in the torus remained, as Theon of Smyrna wrote (2nd century AD).

Perseus (2nd century BC) constructed a torus with a circular cross-section by the plane passing through the axis of the torus. But more general toroidal surfaces can also be obtained by rotating a circle around an axis when the axis of rotation does not lie in the plane of the circle. This surface can be defined within the framework of the ancient Greek tradition, although there is no explicit mention of this.

Later, **Theon of Smyrna** (2nd century AD) compared the motion of the planets to the spirals of a grapevine. (A. I. Schetnikov, 2009)

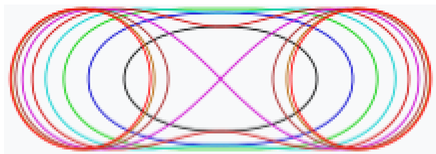
> The planets describe spirals by participation, (. . .) their motion is not straight along the zodiac, but around the sphere of the fixed stars, describing spirals between the parallels, like the spirals of a grapevine. This is like a cylinder wound with a belt from one end to the other, when the Laconian ephors wrapped their scytales with belts and wrote on them. The planets describe another spiral, not passing from one end of the cylinder to the other, but one that can be drawn on a plane. For an entire eternity they pass from one parallel to another and again return to this one, incessantly and endlessly, and if we depict the parallels as straight lines extended to infinity, then the planets will travel from one line to another, approaching now the summer tropic, now the winter, revealing to us to infinity the spirals they describe.

The spiric of Perseus is a toric section

Heron of Alexandria (1st century AD) and Proclus Diadochus (5th century AD), who worked in Athens, wrote that Perseus (2nd century BC) and Geminus, who lived after Perseus in Rhodes, considered plane sections of the torus known as spirics of Perseus (M. Chasles, 1837; J. Gow, 1884).

The spiric of Perseus is a quartic plane curve defined by the equation

$$(x^2 + y^2 + p^2 + d^2 - r^2)^2 = 4d^2(x^2 + p^2).$$



Special cases are the Cassini ovals, which attracted particular attention in the Modern Age. The Cassini oval has the remarkable property that the product of distances to two foci are constant.

Heron of Alexandria wrote the following:

Speira fit quando circulus aliquis centrum habensin circulo et erectus existens, ad planum ipsius circuli fuerit circumductus, et revertatur iterum unde coeperat moveri; illud ipsum figurae genus nominatur κρικος *orbis.* (M. Chasles, 1837)

A torus is formed when a circle with its center on a circle, standing upright, is drawn around [a straight line] in the plane of the circle itself and returns again to the place from which it started; this kind of figure is called a ring (κρικος) throughout the world.

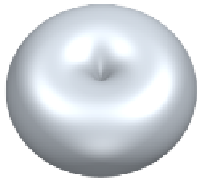
James Gow wrote in 1884:

The σπειρα is somewhat imperfectly described by Heron as the solid "produced by the revolution of a circle which has its center on the circumference of another circle and which is perpendicular to the plane of that other circle. This is also called a κρικος (ring)." This solid varies in form according to the ratios between the radii of the two circles. It may resemble an anchor-ring or a modern teacake, with a dimple at the center. Proclus describes three kinds of sections, which were obtained from it (. . .)

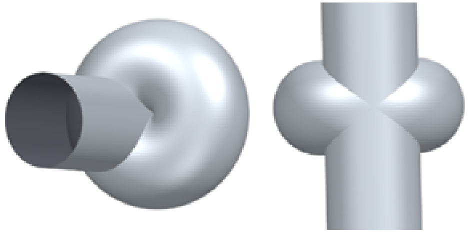
Elsewhere seems to suggest that Perseus had treated the spiral sections as Apollonius had treated the conics. From this, perhaps, it may be inferred that whereas one or two sections of the σπειρα were known before and were obtained from different forms of the solid, Perseus investigated all the sections and shewed that they could be obtained from one σπειρα.
(J. Gow, 1884, p. 270.)

Interpretation: **Archytas of Tarentum** used the torus. (I. G. Bashmakova, 1958) Let us consider a special case, which is important for doubling the cube. For given R, both $\sqrt[3]{4}R$ and $\sqrt[3]{2}R$ are of interest.

Let us consider the horn torus obtained by revolution of a circle of radius R around a tangent straight line. If this line is the coordinate axis z, then the torus is defined by the equation $\left(x^2+y^2+z^2\right)^2 = 4R^2\left(x^2+y^2\right)$.



Let us consider the cylinder $x^2+y^2=2Rx$ of the same radius R so that its axis is parallel to the axis of the horn torus. This cylinder passes through the center of the torus and touches the torus at another point. Both surfaces intersect along the Archytas curve. It is a space curve of the eighth order, which has a point of self-tangency (in the center of the torus) and another point of self-intersection (in the great circle of the torus).



The Archytas curve intersects the cone $x^2+y^2+z^2=4x^2$ at the origin and at four other points with real coordinates. The consequence of the system of equations

$$\begin{cases} x^2+y^2 &= 2Rx \\ x^2+y^2+z^2 &= 4x^2 \\ \left(x^2+y^2+z^2\right)^2 &= 4R^2\left(x^2+y^2\right) \end{cases}$$

is the equation $2x^4 = R^3x$. Therefore, the abscissa of the intersection point of three surfaces is equal to either zero or $\sqrt[3]{1/2}R$. The distance from the origin to another intersection point is equal to $\sqrt[3]{4}R$, and the distance from the origin to the projection of any of these points onto the coordinate plane $z=0$ is equal to $\sqrt[3]{2}R$. Therefore, using the intersection point different from the origin, it is easy to find the desired values. All three surfaces are surfaces of revolution with a circle or straight line as the generatrix.

This reconstruction differs from Archytas's own reasoning.

Eutocius of Ascalon explicitly mentioned only cylindrical surface κυλινδρικην επιφανειαν as well as conical surface κωνικην ποιησει επιφανειαν. But there is *no designation for the torus.*

Description of the Archytas curve by Eutocius

τοῦτο δὴ τὸ ἡμικύκλιον περιαγόμενον
ὡς ἀπὸ τοῦ Δ ἐπὶ τὸ Β μένοντος τοῦ Α
πέρατος τῆς διαμέτρου τεμεῖ τὴν
κυλινδρικὴν ἐπιφάνειαν ἐν τῇ περιαγωγῇ
καὶ γράψει ἐν αὐτῇ γραμμήν τινα.

Then, this semicircle drawn around from Δ towards B, while the extreme A of the diameter remains fixed, will cut the cylindrical surface in its being drawn around, and will trace a certain trace on it.
(Ramon Masià, 2016)

Muhammad, Al-Hasan, and Ahmad Banu Musa

Book on the Measurement of Plane and Spherical Figures. 16th.

Translated from Arabic by J. Al-Dabbagh, 1965.

This construction belongs to one of the ancients named Menelaus. He gave this construction in his book on geometry.

Implicit description of the Archytas curve:

Let us construct on the arc ACB a half of a right circular cylinder so that its generatrices are perpendicular to the plane of the circle ACB, and on the line AB construct a half of a circle so that its plane makes a right angle with the plane ABC, it is the arc AHE.

Fix the point A in the arc AHE in its place and, using it as the center, rotate the arc AHE around the center A in such a way that its plane makes a right angle with the plane ABC during the entire rotation and so that the arc AHE divides the surface of the half of the cylinder standing on the arc ABC.

Moreover, **Eutocius** wrote preceding the story about Eratosthenes:

> They say that Archytas of Tarentum obtained the solution with the help of semi-cylinders, and Eudoxus with the help of so-called curved καμπυλων) lines. However, it turned out that they only proved the possibility of such a construction, but they were unable to actually carry it out or give a practical method, if we do not count the short method of Menaechmus, and even that is difficult.

This text confirms that Archytas' construction is not in the plane, but in space because a semi-cylinder is mentioned.

Eutocius does not report Eudoxus' method for solving the same problem, but the mention of curved (καμπυλων) lines indicates that sections of some surface were used, therefore, Eudoxus also worked in space.

Motion and Spherical Geometry

Eudoxus of Cnidus significantly developed spherical geometry. In particular, Eudoxus created a device, the improvement of which led in the future to the creation of the medieval astrolabe. The astrolabe uses a stereographic projection. It does not follow that the properties of this projection were proven by Eudoxus. But Apollonius already knew the proof.

Spherical geometry is discussed in Euclid's *Phenomena*, which was probably written after the works of Autolycus of Pitane (refer to next page).

Theodosius of Bithynia (2nd century BC) wrote the three-volume treatise *Spherics*. He examined systems of parallel and intersecting circles on a sphere (G. P. Matvievskaya, 1982).

Geminus of Rhodes (1st century BC) wrote *Introduction to the Phenomena*. It is an astronomy textbook, which included some results of spherical geometry (A. I. Schetnikov, 2011).

Autolycus of Pitane (ca. 360–290 BC) wrote *On the Moving Sphere* and *On Risings and Settings* (I. P. Rushkin, 2017, 2023), where discuss circles on the sphere (meridians, parallels, ecliptic) and the application of spherical geometry in astronomy (G. P. Matvievskaya, 1982).

On the Moving Sphere contains the earliest definition of uniform motion:

1. A point is said to move uniformly if it covers equal and similar parts of the path in equal intervals of time.
2. When a point moving uniformly along a line passes through two sections of the path, the ratio of the time intervals during which the point passes through each section will be the same as the ratio of the sections.

On subsequent events

In 86 BC, Sulla's troops plundered Athens.

In 48–47 BC, during the siege of Alexandria by Julius Caesar's troops, part of the library burned down.

In 44 BC, Gaius Cassius' troops plundered Rhodes.

In 30 BC, Ptolemaic Egypt was annexed by the Roman Republic.

Ancient Greece had ended.

Menelaus of Alexandria (ca. 100 AD) wrote other three-volume treatise *Spherics*, which are known from translations into Arabic and then into Latin. In *Spherics*, Menelaus discusses the properties of spherical triangles, the relative positions of circles on a sphere, and the theorem on a complete quadrilateral, which is transferred to a sphere when centrally projected. Spherical geometry is also discussed by Claudius Ptolemy (ca. 100–170 AD).

One of the last geometers before the long decline was Pappus of Alexandria, who laid the foundations of projective geometry. Pappus also considered **helical surfaces and conical spirals** (H. Wieleitner, 1921).

Conclusion

Probably, the torus (σπειρα) had been known in Ancient Greece from the 4th century BC, but it was forgotten by Menelaus of Alexandria and Theon of Smyrna at the beginning of the 2nd century AD. The torus serves as a natural part of the reconstruction of Archytas' solution. Plane curves that serve as sections of the torus were studied by Perseus (2nd century BC) and Geminus (1st century BC). Moreover, the torus is useful for describing the motion of celestial bodies, which must have attracted geometers in Rhodes.

Remark

In Ancient Greece, almost all works on stereometry had applied significance.

The science of Ancient Greece was not dogmatic. Different methods and models were discussed. Kinematic curves were used. Later, the works of Plato, Aristotle, and Euclid began to be perceived dogmatically, distorting the image in accordance with the views of the Middle Ages.

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Σας ευχαριστώ

Thank you