PRIFYSGOL ABERYSTWYTH UNIVERSITY 1872

Non-Markovian Quantum Stochastic Models

QOART Seminar, Moscow (13 February 2025)

John Gough

jug@aber.ac.uk

Aberystwyth University

Papers

- *Quantum feedback control of a two-level atom network closed by a semi - infinite waveguide*

  H. Ding, G. Zhang, Mu-Tian Cheng, Guoqing Cai arXiv:2306.06373v2

- *Quantum coherent and measurement feedback control based on atoms coupled with a semi -infinite waveguide*

  H. Ding, N.H. Amini, G. Zhang, JG

  SIAM Journal on Control and Optimization, pp. 231-257.

- *On the control of non-Markovian quantum dynamics based on cavity-QED systems*

  H. Ding, N.H. Amini, G. Zhang, JG **arXiv:2408.09637**

- *Reproducing Kernel Hilbert Space Approach to Non-Markovian Quantum Stochastic Models*

  JG, H. Ding, N.H. Amini (to appear Journal of Mathematical Physics) arXiv:2407.07231v1

Open Quantum Systems

- Master Equation (Hamiltonian and Dissipation)

$$\frac{d}{dt}\rho_t = i[\rho_t, H] + \sum_l \left(L_k \rho_t L_k^* - \frac{1}{2} L_k^* L_k \rho_t - \frac{1}{2} \rho_t L_k^* L_k \right), \qquad \rho_0 = |\psi_0\rangle\langle\psi_0|.$$

- Unravellings

Let $(\mathcal{F}_t)$ be a filtration of sigma-algebras for a probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Adapted vector-state valued process : $(t, \omega) \mapsto |\Psi_t(\omega)\rangle$.

$$\rho_t = \mathbb{E}\big[|\Psi_t\rangle\langle\Psi_t|\big].$$

Stochastic Differential Form

- It is useful to consider a *linear* Ito SDE of the form

$$d|\chi_t\rangle = A\,|\chi_t\rangle\,dt + \sum_k B_k\,|\chi_t\rangle\,dW_k(t),$$

where $A, (B_k)$ are linear operators and the $(W_k(t))$ are martingales.

- The *normalized* state giving the unravelling is then

$$|\Psi_t\rangle = \frac{1}{\sqrt{\langle\chi_t|\chi_t\rangle}}|\chi_t\rangle.$$

Motivation

- Quantum Monte-Carlo
  (H. Carmichael, K. Mølmer, etc.)

- Quantum Filtering
  (V.P. Belavkin, A.S. Holevo, R.L. Stratonovich, etc.)

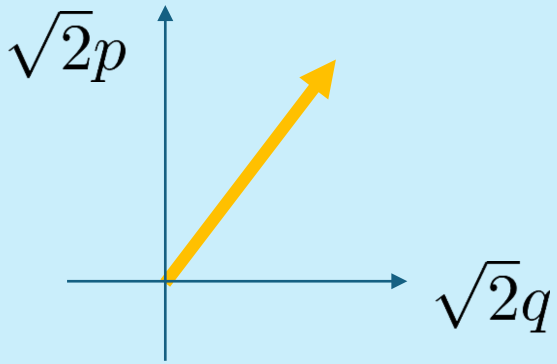
- Theory of Decoherence and «collapse of the wave-function»
  (Ghiradi, Rimini, Pearle, Gisin, Diósi, Percival, etc.)

Markov vs Non-Markov

- Markovianity is only ever an approximation in physical models, but the assumption is made for convenience whenever viable.
- For cavity QED, it is justified to the point of being mandatory.
- Systems with feedback loop implemented via time-delayed photonic waveguides are clearly non-Markovian (H. Ding and G. Zhang).
- May be of interest to include a non-markovian bath in an otherwise markovian model.

Mathematical Concepts

Complex amplitude $\alpha = \alpha' + i\alpha''$



- Quantum Harmonic Oscillator

$$[a, a^*] = 1, \qquad a\,|\mathrm{vac}\rangle = 0.$$

- **Bargmann/ Exponential vectors** (normalized = coherent states)

$$|\alpha\rangle = e^{\alpha a^*}\,|\mathrm{vac}\rangle = \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}}|n\rangle.$$

- Properties

$$(a-\alpha)\,|\alpha\rangle = 0, \qquad \langle\alpha|\beta\rangle = e^{\alpha^*\beta}, \qquad \int_{\mathbb{C}} e^{-|\alpha|^2}\frac{d\alpha'\,d\alpha''}{\pi}\,|\alpha\rangle\langle\alpha| = I.$$

Microscopic Model - Bath

- Bose Fock space $\mathfrak{H}_B = \Gamma(\mathfrak{f}_B)$ e.g., $\mathfrak{f}_B = L^2(\mathbb{R}_+, d\omega)$.
- We refer to $\mathfrak{f}_B$ as the **one-particle space**.
- Creation/annihilation operators

$$[\hat{a}_\omega, \hat{a}^*_{\omega'}] = \delta(\omega - \omega').$$

- For $g \in \mathfrak{f}_B$,

$$\hat{a}(g)^* = \int_0^\infty g(\omega)\hat{a}^*_\omega \, d\omega, \qquad \hat{a}(g) = \int_0^\infty g(\omega)^* \hat{a}_\omega \, d\omega$$

- For a given vector $|f\rangle \in \mathfrak{f}_B$, its *exponential vector* is defined as

$$|e^f\rangle_B = e^{\hat{a}(f)^*}|\mathrm{vac}\rangle_B \equiv |\mathrm{vac}\rangle \oplus |f\rangle \oplus \big(\frac{1}{\sqrt{2!}}|f\rangle \otimes |f\rangle\big) \oplus \cdots .$$

- In particular, the vacuum state corresponds to $|\mathrm{vac}\rangle_B \equiv |e^0\rangle_B$.

- The exponential vectors form a total subset in $\mathfrak{H}_B$ and we note that

$$\langle e^f|e^g\rangle_B = e^{\langle f|g\rangle}.$$

- The **Complex Wave representation** of a vector $|\Psi\rangle \in \mathfrak{H}_B$ is given by

$$\tilde{\Psi}\left(f\right) = \langle e^f \left|\Psi\right\rangle.$$

- The mapping $f \mapsto \langle e^f | \Psi \rangle$ is naturally anti-holomorphic.

- The pre-measure $\tilde{\mathbb{P}}$ on the one-particle space $\mathfrak{f}_B$ by

$$\int_{\mathfrak{f}_B} e^{\langle g_1 | f \rangle} \, e^{\langle f | g_2 \rangle} \, \tilde{\mathbb{P}}\left[df\right] = e^{\langle g_1 | g_2 \rangle}.$$

- For the case where the bath is a finite assembly of oscillators, $\tilde{\mathbb{P}}$ will define a Gaussian measure:

$$\mathfrak{f}_B \cong \mathbb{C}^N, \qquad f(\omega) = f(\omega)' + i f(\omega)''$$

$$\tilde{\mathbb{P}}\,[df] = \prod_{\omega} \left(\frac{1}{\pi} e^{-|f(\omega)|^2} df(\omega)' df(\omega)'' \right).$$

- In the infinite-dimensional case, we should extend $\tilde{\mathbb{P}}$ to a σ-additive measure over a larger space $\mathfrak{f}_B^{>}$. See J.Kupsch and O.G. Smolyanov (Doklady Math., April 2009) or the book by J.G. and J. Kupsch.

- In general, we have

$$\langle \Phi | \Psi \rangle_B = \int_{\mathfrak{f}_B} \tilde{\Phi}(f)^* \tilde{\Psi}(f) \, \tilde{\mathbb{P}}[df] \, .$$

- We also have the resolution of identity

$$\int_{\mathfrak{f}_B} |e^f\rangle\langle e^f| \, \tilde{\mathbb{P}}[df] = \hat{I}_B .$$

System-Bath Interaction

- Total Hamiltonian

$$\hat{H}_{\mathrm{Tot.}} = \hat{H} \otimes \hat{I}_B + \hat{I}_S \otimes \hat{H}_B + i\hat{L} \otimes \hat{a}(g)^* - i\hat{L}^* \otimes \hat{a}(g).$$

- The bath Hamiltonian is just oscillatory

$$e^{it\hat{H}_B}\hat{a}(g)e^{-it\hat{H}_B} \equiv \hat{a}(g_t), \text{ where } |g_t\rangle = e^{it\hat{h}_B}|g\rangle.$$

- The bath correlation function (**kernel**) is

$$K(t,s) \triangleq \langle g_t|g_s\rangle = \langle g|e^{-i(t-s)\hat{h}_B} g\rangle.$$

- Move into the interaction picture with respect to the bath

$$\frac{d}{dt}\hat{U}_t = -i\hat{\Upsilon}_t\hat{U}_t, \qquad -i\hat{\Upsilon}_t = -i\hat{H}\otimes\hat{I}_B + \hat{L}\otimes\hat{Z}(t) - \hat{L}^*\otimes\hat{Z}(t)^*.$$

- Here we introduce the "***bath processes***"

$$\hat{Z}(t) = \hat{a}(g_t)^*.$$

- The commutation relations are

$$[\hat{Z}(t)^*, \hat{Z}(s)] = K(t,s)\,\hat{I}_B.$$

Input-System-Output Model

- It is convenient to introduce the following operators:

$$\begin{aligned} j_t(\hat{X}) &= \hat{U}_t^*(\hat{X} \otimes \hat{I}_B)\hat{U}_t, \\ \hat{Z}_{\rm in}(t) &= \hat{I}_S \otimes \hat{Z}(t), \\ \hat{Z}_{\rm out}(t) &= \hat{U}_t^* \hat{Z}_{\rm in}(t) \hat{U}_t. \end{aligned}$$

- $j_t(\hat{X})$ commutes with both $\hat{Z}_{\rm out}(t)$ and $\hat{Z}_{\rm out}(t)^*$.

- Input-output relation

$$\hat{Z}_{\rm out}(t)^* = \hat{Z}_{\rm in}(t)^* + \int_0^t K(t,\tau) j_\tau(\hat{L}) d\tau.$$

- “Ehrenfest Equation” with memory!!!

$$\frac{d}{dt}\langle j_t(\hat{X})\rangle = \langle j_t(\frac{1}{i}[\hat{X},\hat{H}])\rangle + \int_0^t d\tau\, K(t,\tau)\langle j_t([\hat{L}^*,\hat{X}]) j_\tau(\hat{L})\rangle + \int_0^t d\tau\, K(t,\tau)^* \langle j_\tau(\hat{L}^*) j_t([\hat{X},\hat{L}])\rangle.$$

Markov Limit

- E.g., Weak Coupling Limit

$$K(t,s) \to \delta(t-s).$$

- Bath processes replaced by singular quantum white noises

$$[\hat{b}_t, \hat{b}_s^*] = \delta(t-s)$$

$$d\hat{B}_{\text{in}}(t) = \hat{b}_t \, dt$$

- Input-output relation

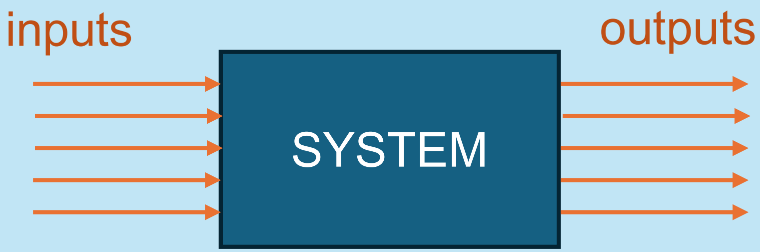
$$d\hat{B}_{\text{out}}(t) = d\hat{B}_{\text{in}}(t) + j_t(\hat{L})\, dt.$$

- "Ehrenfest Equation" becomes

$$\frac{d}{dt}\langle j_t(\hat{X})\rangle = \langle j_t(\frac{1}{i}[\hat{X},\hat{H}])\rangle + \langle j_t(\mathcal{L}X)\rangle.$$

$$\mathcal{L}\hat{X} = \tfrac{1}{2}[\hat{L}^*, \hat{X}]\hat{L} + \tfrac{1}{2}\hat{L}^*[\hat{X}, \hat{L}].$$

SLH Models



Stratonovich Form

$$\frac{d}{dt}\widehat{U}_t = -i\widehat{\Upsilon}_t\,\widehat{U}_t$$

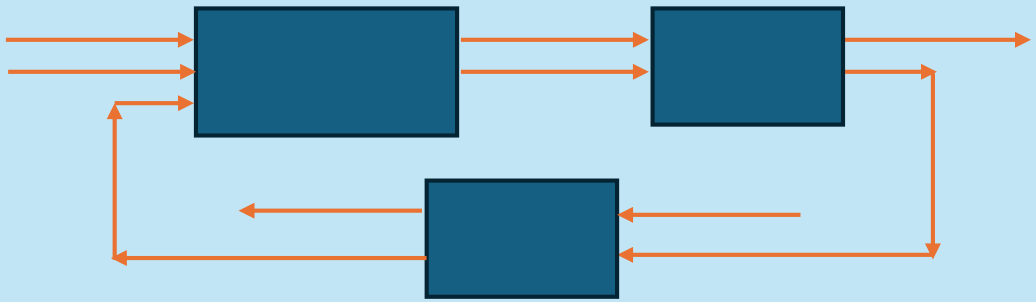
$$\widehat{\Upsilon}_t = \hat{E}_{11}\otimes\hat{b}_t^*\hat{b}_t + \hat{E}_{10}\otimes\hat{b}_t^* + \hat{E}_{01}\otimes\hat{b}_t + \hat{E}_{00}\otimes\hat{I}$$

Ito Form (Husdon-Parthasarathy)

$$\hat{S} = \frac{\hat{I}-\frac{i}{2}\hat{E}_{11}}{\hat{I}+\frac{i}{2}\hat{E}_{11}}, \quad \hat{L} = -\frac{i}{2}\frac{1}{\hat{I}+\frac{i}{2}\hat{E}_{11}}\hat{E}_{10}, \quad \hat{H} = \hat{E}_{00} + \frac{1}{2}\hat{E}_{01}Re\left(\frac{1}{\hat{I}+\frac{i}{2}\hat{E}_{11}}\right)\hat{E}_{10}.$$

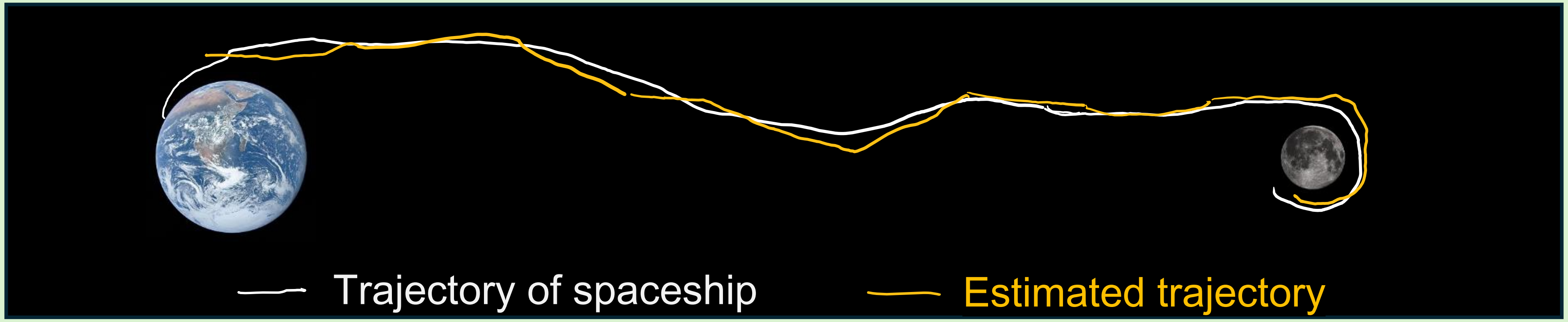
Quantum Feedback Networks (JG + Matthew James)

$$dB_{\text{out}}(t) = j_t(\hat{S})\,dB_{\text{in}}(t) + j_t(\hat{L})\,dt.$$

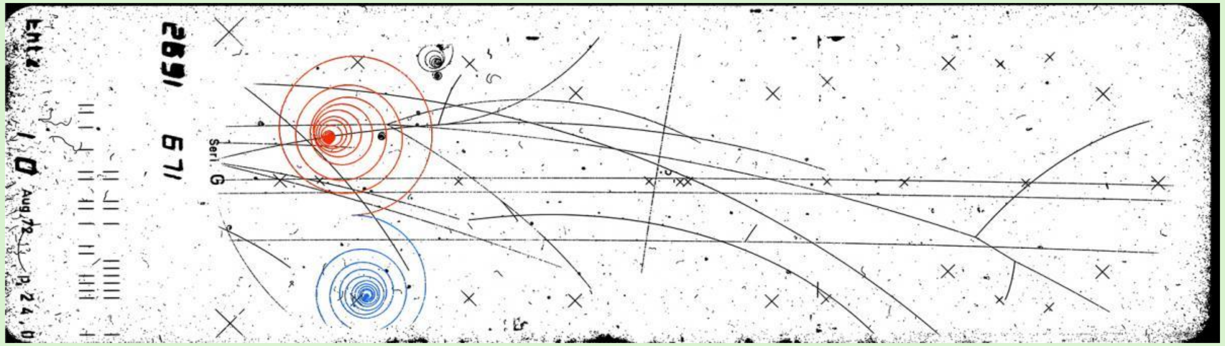


The Filtering Problem (Stratonovich, Kalman, …)

- Estimating the state of a (noisy) dynamical system based on (noisy) partial information.



- Does this apply to quantum systems?



Quantum Markov Filter: Homodyne Detection

- Measure the output quadrature

$$\hat{Y}(t) = \hat{B}_{\text{out}}(t) + \hat{B}_{\text{out}}(t)^*.$$

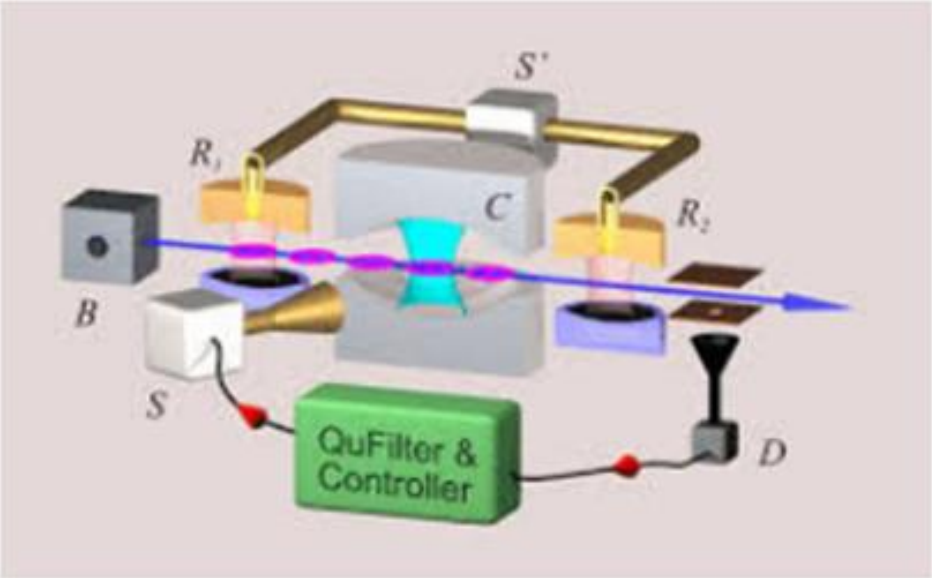
- The quadratures form a *self-commuting* process.
- The observables $j_t(\hat{X})$ commute with the output $\hat{Y}(s)$ for $t \geq s$.
- There is a best estimate $\pi_t(\hat{X})$ for $j_t(\hat{X})$ belonging to the algebra generated by the quadrature process up to time *t.*

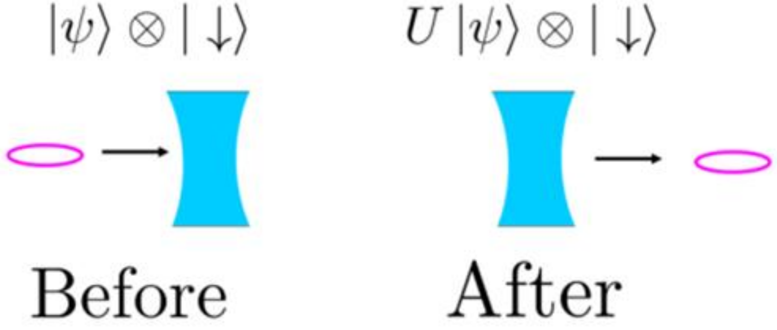
$$\pi_t(\hat{X}) = \frac{\langle \chi_t | \hat{X} | \chi_t \rangle}{\langle \chi_t | \chi_t \rangle}$$

$$d|\chi_t\rangle = \hat{L}|\chi_t\rangle dY(t) - (\tfrac{1}{2}\hat{L}^*\hat{L} + i\hat{H})|\chi_t\rangle dt.$$

Paris Photon-Box Experiment

I. Dotsenko, M. Mirrahimi, M. Brune, S. Haroche, J.-M. Raimond, and P. Rouchon
Quantum feedback by discrete quantum nondemolition measurements: Towards on-demand generation of photon-number states
Phys. Rev. A 80, 013805 (2009)



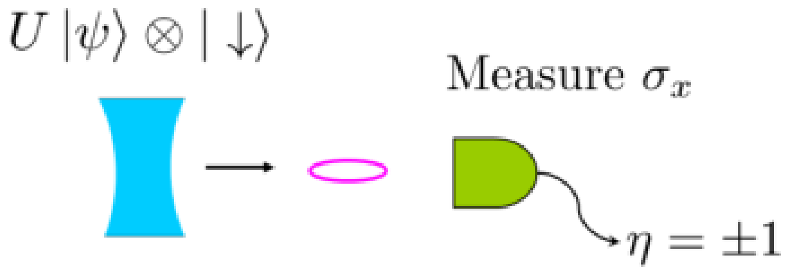


We take the interaction time τ to be very small and assume that the unitary has the form

$$
\begin{aligned}
U &= \exp\{\sqrt{\tau}L \otimes \sigma^* - \sqrt{\tau}L^* \otimes \sigma - i\tau H \otimes I_2\} \\
&\simeq 1 + \sqrt{\tau}L \otimes \sigma^* - \sqrt{\tau}L^* \otimes \sigma - \tau(\frac{1}{2}L^*L + iH) \otimes I_2 + \cdots .
\end{aligned}
$$

We now measure the spin σ_x of the qubit to obtain the values $\eta = \pm 1$ corresponding to the eigenvectors

$$|+\rangle = \frac{1}{\sqrt{2}}|\downarrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\rangle, \quad |-\rangle = \frac{1}{\sqrt{2}}|\downarrow\rangle - \frac{1}{\sqrt{2}}|\uparrow\rangle.$$



The probabilities for detecting $\eta = \pm 1$ are

$$p_{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{\tau}\langle\psi|L + L^*|\psi\rangle + \cdots .$$

After measurement, the system state becomes (up to normalisation!)

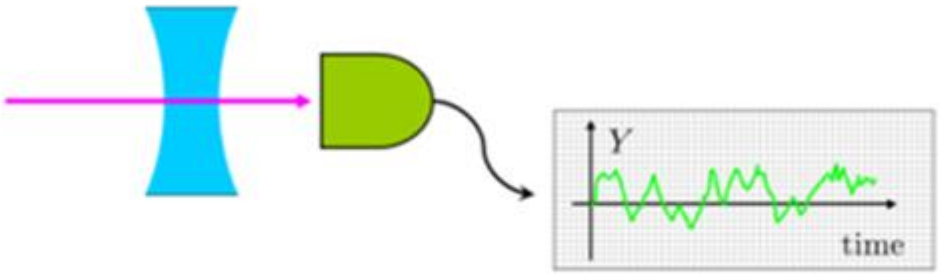
$$|\psi_\eta\rangle \propto |\psi\rangle + \sqrt{\tau}L|\psi\rangle\,\eta - \tau(\frac{1}{2}L^*L + iH)|\psi\rangle + \cdots .$$

We may take a continuum limit we have (central limit effect)

$$\tau \hookrightarrow dt \quad \sqrt{\tau}\eta \hookrightarrow dY$$

The limit equation is (quantum Zakai equation)

$$d|\chi_t\rangle = L|\chi_t\rangle\, dY_t - (\frac{1}{2}L^* L + iH)|\chi_t\rangle\, dt.$$



For the normalized state $|\psi_t\rangle = |\chi_t\rangle / \, \|\chi_t\|$ we find **(Stochastic Schrödinger equation)**

$$d|\psi_t\rangle = -\, iH|\psi_t\rangle\, dt - \frac{1}{2}(L - \lambda_t)^*(L - \lambda_t)|\psi_t\rangle dt + (L - \lambda_t)|\psi_t\rangle\, dI_t.$$

$$dI_t \;\; = \;\; dY_t - \;\lambda_t\, dt.$$

(**Innovations** – Wiener process)

Quantum State Diffusion

- In the Markov case, this is very similar to the quantum filtering/ Monte Carlo problem.
- But the processes are complex Wiener Processes!
- There is a non-Markovian version due to L. Diósi, W. Strunz (and later N. Gisin).

- We will derive their equation in this report.

Quantum State Diffusion (Markovian)

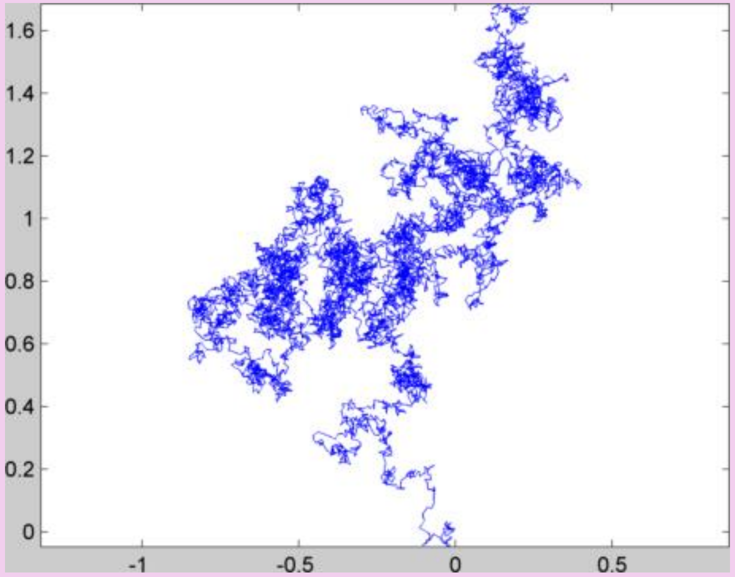
$$d|\chi_t\rangle = -(\frac{1}{2}L^*L + iH)|\chi_t\rangle\, dt + L\,|\chi_t\rangle\, d\xi(t),$$

Complex Wiener process

$$\xi(t) = \tfrac{1}{\sqrt{2}}\big(B_1(t) + iB_2(t)\big)$$

$$(d\xi)^2 = \tfrac{1}{2}(dB_1)^2 - \tfrac{1}{2}(dB_2)^2 + idB_1dB_2 = 0 + i0,$$

$$d\xi^*\, d\xi = \tfrac{1}{2}(dB_1)^2 + \tfrac{1}{2}(dB_2)^2 + i0 = dt.$$



Quantum State Diffusion (Non-Markovian)

- Postulated by W. Strunz and L. Diósi
- Replaces the complex Wiener process: $z_t \equiv \dot{\xi}_t$

$$\mathbb{E}[z_t z_s] = 0, \quad \mathbb{E}[z_t^* z_s] = K(t,s) = K(s,t)^*.$$

- Stochastic equation with memory (**causal?**)

$$\frac{d}{dt}|\Psi_t\rangle = -iH|\Psi_t\rangle + L|\Psi_t\rangle z(t) - L^* \int_0^t K(t,s) \frac{\delta}{\delta z_s} |\Psi_t\rangle \, ds.$$

Back to a Microscopic Model

- Recall that $|g_t\rangle = e^{it\hat{h}_B}|g\rangle,$

$$Z_t^* = e^{it\hat{H}_B}\hat{a}(g)e^{-it\hat{H}_B} \equiv \hat{a}(g_t).$$

For each $f \in \mathfrak{f}$, we define its associated **complex trajectory** over the time interval $\mathbb{T}$ to be the function $\zeta(f) : \mathbb{T} \mapsto \mathbb{C} : t \mapsto \zeta_t(f)$ where

$$\zeta_t(f) = \langle f|g_t\rangle_{\mathfrak{f}}.$$

The space of complex trajectories will be denoted as $\mathscr{C}_K(\mathbb{T}, dt)$.

Back to a Microscopic Model

- Recall $\hat{Z}(t) = \hat{a}(g_t)^*.$

- We then have the eigen-relation

$$\left(\hat{Z}_t^* - \zeta_t(f)^*\right)|e^f\rangle_B = 0.$$

- Need to go from the frequency domain (bath modes) to the time domain!

RKHS Formalism

- Let $\mathbb{T}$ be an interval in $\mathbb{R}$, and $K : \mathbb{T} \times \mathbb{T} \mapsto \mathbb{C}$ a positive definite kernel.
- For each $t \in \mathbb{T}$, a function $\mathsf{k}_t : \mathbb{T} \mapsto \mathbb{C}$ is then defined by

$$\mathsf{k}_t(\cdot) \triangleq K(t, \cdot).$$

- A Hilbert space $\mathscr{H}$ of complex-valued functions on $\mathbb{T}$ forms a **reproducing kernel Hilbert space (RKHS)** for the kernel K if $\mathsf{k}_t \in \mathscr{H}$, for each $t \in \mathbb{T}$, and we have the *reproducing property*

$$\langle \mathsf{k}_t, f \rangle_{\mathscr{H}} = f(t)$$

for all $t \in \mathbb{T}$ and all $f \in \mathscr{H}$.

RKHS Formalism

- A kernel is said to be derivable from a *feature map* if there exists a Hilbert space $\mathfrak{f}$ (called the **feature space**) and a function $g : \mathbb{T} \mapsto \mathbb{C}$ (called the **feature map**) such that

$$K(t, s) \equiv \langle g_t, g_s \rangle_{\mathfrak{f}}.$$

- The Bose bath model supplies us with these naturally:

  feature space = one-particle space for the bath;

  feature map = free evolution of one-particle states.

- Mercer kernel assumption implies the existence of a RKHS.

The Hilbert Space of Complex Trajectories

- **Theorem**

The set of complex trajectories, $\mathscr{C}_K(\mathbb{T}, dt)$, forms a Hilbert subspace of the RKHS $\mathscr{H}_K(\mathbb{T}, dt)$ (inheriting the same inner product) and the map ζ is a conjugate-linear isometry into the feature space $\mathfrak{f}_B$.

$$\langle \zeta(f_1)^*, \zeta(f_2)^* \rangle_{\mathscr{H}} = \langle f_1 | f_2 \rangle_{\mathfrak{f}}.$$

The space $\mathscr{C}_K(\mathbb{T}, dt)$ is conjugate-isomorphic to $\mathfrak{f}_B$, assuming that g is faithful.

Frequency/Time Domain Transformation

- **Change of Variable** Suppose that g is faithful so that the mapping $\zeta : \mathfrak{f}_B \mapsto \mathscr{C}_K(\mathbb{T}, dt)$ is invertible. Then, for each Bargmann function $\tilde{\Psi}$, we define the functions Ψ by

$$\Psi(\cdot) = \tilde{\Psi} \circ \zeta^{-1}.$$

Additionally, we can endow $\mathscr{C}_K(\mathbb{T}, dt)$ with the pull-back pro-measure $\mathbb{P} = \tilde{\mathbb{P}} \circ \zeta^{-1}$. We may extend $\mathbb{P}$ to a probability measure as outlined earlier.

$$\langle \Phi | \Psi \rangle = \int_{\mathfrak{f}_B} \langle \tilde{\Phi}(f) | \tilde{\Psi}(f) \rangle_S \, \tilde{\mathbb{P}}\,[df] = \int_{\mathscr{C}_K(\mathbb{T}, dt)} \langle \Phi(\zeta) | \Psi(\zeta) \rangle_S \, \mathbb{P}\,[d\zeta] \, .$$

- In the Bargmann representation, $\hat{Z}(t)$ corresponds to multiplication by $\zeta_t(\cdot)$:

$$\hat{Z}(t)\,\tilde{\Psi}(f) = \zeta_t(f)\,\tilde{\Psi}(f).$$

- With the change of variable, $\hat{Z}(t)$ is multiplication by $\zeta_t(\cdot)$!

- For the case where $\mathbb{T} = \mathbb{R}$, the adjoint operator $\hat{Z}(t)^*$ is

$$\hat{Z}(t)^* \equiv \int_{-\infty}^{\infty} d\tau\, K\left(t,\tau\right) \frac{\delta}{\delta\zeta_\tau}.$$

Complex Trajectory Unravellings

- **Causality?**

Let $|\Psi_t(\zeta)\rangle_S$ be the hybrid complex wave representation of $|\Psi_t\rangle = \hat{U}_t\,|\phi \otimes \mathrm{vac}\rangle$, then

$$\frac{\delta}{\delta\zeta_\tau}|\Psi_t(\zeta)\rangle_S = 0, \quad \text{whenever} \quad \tau \notin [0,t].$$

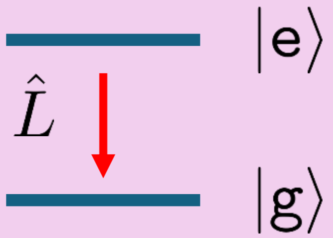
- **Theorem**

An unravelling is given on the space $\mathscr{C}_K(\mathbb{T}, dt)$ of complex trajectories with measure $\mathbb{P}$. The unravelling $\zeta \mapsto |\Psi_t(\zeta)\rangle_S$ satisfies the Diósi-Strunz equation with initial condition $|\Psi_0(\zeta)\rangle_S = |\phi\rangle_S$ for all ζ.

Exactly Solvable Model

- Jaynes-Cummings model: 2-level atom coupled to a bosonic bath

$$\hat{H} = \omega_0 |\mathsf{e}\rangle\langle\mathsf{e}| \text{ and } \hat{L} = |\mathsf{g}\rangle\langle\mathsf{e}|.$$



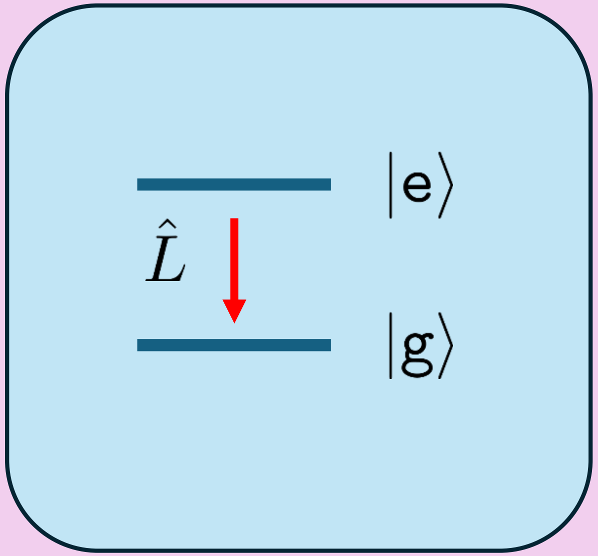
- Expand as a Dyson series and re-sum

$$|\Psi_t(\zeta)\rangle_S = |\phi\rangle_S + \langle\mathsf{e}|\phi\rangle_S \left((\lambda(t) - 1)|\mathsf{e}\rangle_S + \int_0^t \zeta_\tau \lambda(\tau)\, d\tau\, |\mathsf{g}\rangle_S \right),$$

where $\lambda(t) = \sum_{k=0}^{\infty} (-1)^k I_k(t)$ with $I_0(t) = 1$ and, for $k \geq 1$,

$$I_k(t) = \int_{t \geq \tau_{2k} > \tau_{2k-1} > \cdots > \tau_1 \geq 0} d\tau_{2k} \cdots d\tau_1\, K(\tau_{2k}, \tau_{2k-1}) \cdots K(\tau_2, \tau_1).$$

Exactly Solvable Model



- We note that

$$\dot{\lambda}(t) = -\int_0^t K(t,\tau)\lambda(\tau)\, d\tau.$$

- By substitution, the solution satisfies the Strunz-Diósi equation.
- For the “Markov” case we have

$$K(t,s) = \gamma\, \delta(t-s),$$

here we have $I_k(t) = \frac{1}{k!}(\gamma/2)^k$, and so $\lambda(t) = e^{-\gamma t/2}$.

Conclusions

- Quantum Filtering is fundamentally Markovian
- The complex trajectories in the non-Markovian Quantum State Diffusion models do *not* involve measurement. In fact, the complex processes are themselves not self-commuting.
- The apparent randomness is due to the background Gaussian measure appearing in the Bargmann-Segal representation.
- However, the model proposed by Diósi and Strunz is correct and naturally causal.