system-dependent random matrix ensembles

Pragya Shukla

Department of Physics and Meteorology,
Indian Institute of Technology
Kharagpur

February 16, 2025

[1] Online Seminar QOART, 2025

Complexity: how it affects a physical property?

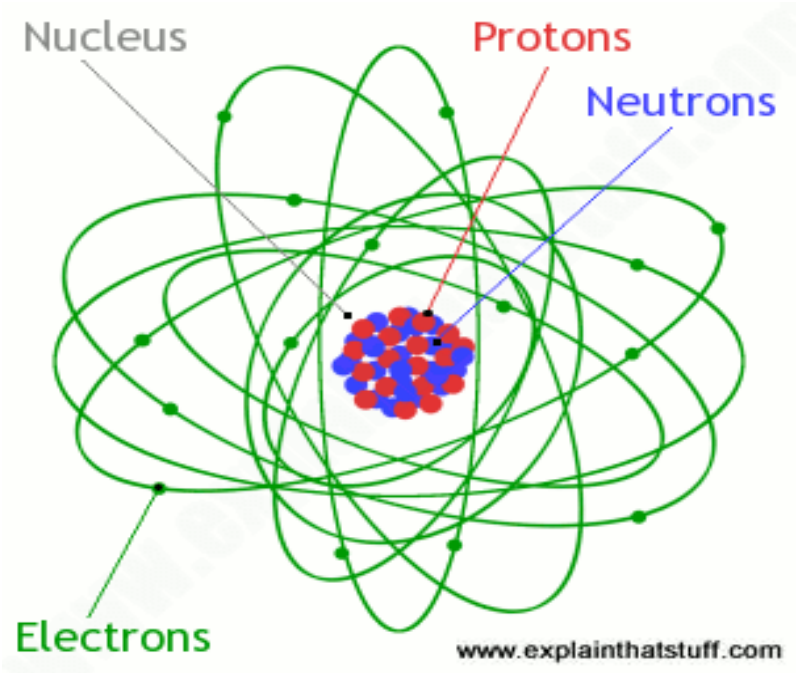
physical properties fluctuate due to complexity (even in a single sample).
Knowledge of their average behaviour is no longer sufficient.

- a knowledge of their distribution must

a possible tool

- find matrix representation of generator of the system (quantum/classical) in a physically relevant basis
- **study of the distributions of the eigenfunctions and eigenvalues and thereby physical properties**

connection with random matrices



How complexity leads to apperance of random matrices in physical systems?

How systems-specifcs manifest in the matrix?

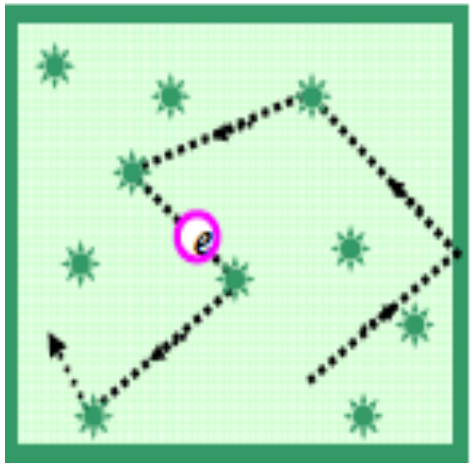
why system-dependent random matrices?

- lack of detailed knowledge about interactions manifests itself by randomization (partial / full) of various operators of the system.
- determination of some/ all matrix elements $\Rightarrow$ within an uncertainty $\Rightarrow$ matrix elements are at best described by a probability distribution.
- uncertainty associated with various matrix element can be of different types $\Rightarrow$ matrix elements can have different distributions.
- generator $\Rightarrow$ a random matrix (**some/ all matrix elements randomly distributed**)
- generator of complex system $\Rightarrow$ represented by an ensemble of matrices, each of them equally probable representative of system.
- **choice of a suitable random matrix model of a complex system is sensitive to nature of the complexity.**
- analysis of a wide range of random matrices required.

randomness due to disorder

Anderson Hamiltonian
simplest model of a disordered system, describes a particle of mass m moving in a random potential $V(r)$

$$\mathbf{H} = \frac{\mathbf{p^2}}{\mathbf{2m}} + \mathbf{V(r)}$$



tight binding approximation

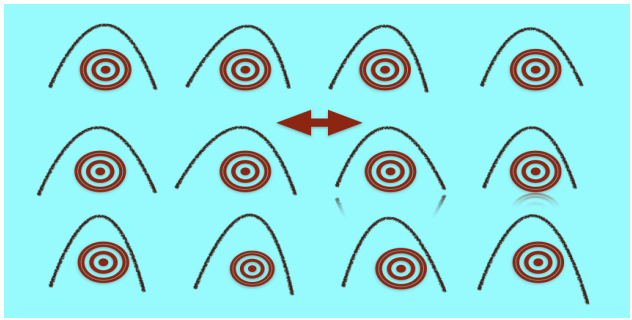
in atomic site basis of the lattice of arbitrary dimension

$$\mathbf{H} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} \, |\mathbf{k}\rangle\langle\mathbf{k}| + \sum_{\mathbf{k,l}} \mathbf{V_{kl}} \, |\mathbf{k}\rangle\langle\mathbf{l}|.$$

$\mathbf{V_{kl}} = \langle\mathbf{k}|\mathbf{V}|\mathbf{l}\rangle$ tunneling amplitude of an electron between atomic sites k and l.

system-dependence of H-matrix

site energies: type of randomness? site-dependent? independent?, correlated?
hopping: short/long range? isotropic/ anisotropic? random/ non-random?



$$\mathbf{H} = \begin{pmatrix} \varepsilon_1 & \mathbf{t} & \mathbf{0} & \mathbf{t} \\ \mathbf{t} & \varepsilon_2 & \mathbf{t} & \mathbf{0} \\ \mathbf{0} & \mathbf{t} & \varepsilon_3 & \mathbf{t} \\ \mathbf{t} & \mathbf{0} & \mathbf{t} & \varepsilon_4 \end{pmatrix}.$$

example site-energies $\mathbf{H_{kk}} = \epsilon_\mathbf{k}$ as independent Gaussian distributions

$$\rho_{kk}(H_{kk}) = \mathrm{e}^{-\frac{(\mathbf{H_{kk}}-\varepsilon_\mathbf{k})^2}{\mathbf{2h_{kk}}}}$$

hopping $\mathbf{V_{kl}}$: isotropic, non-random, nearest neighbor $\rho_{\mathbf{kl}}(\mathbf{H_{kl}}) = \delta(\mathbf{H_{kl}} - \mathbf{V_{kl}})$

$$\rho_{\mathbf{kl}}(\mathbf{H_{kl}}) = \lim_{v\to 0}\ \mathrm{e}^{-\frac{(H_{kl}-t)^2}{2v^2}} \quad {\scriptstyle k,l \atop \text{neighbor}}, \qquad \lim_{v\to 0}\ \mathrm{e}^{-\frac{H_{kl}^2}{2v^2}} \quad {\scriptstyle k,l \atop \text{not neighbors}}$$

Anderson ensemble

can be of various types,
dependent on nature of the disorder and hopping (isotropic or anisotropic, non-random or random), boundary conditions and dimensionality

for Gaussian randomness, probability density of matrix elements of H:

$$\rho(\mathbf{H}) = \prod_{\mathbf{k},\mathbf{l};\mathbf{k}\leq\mathbf{l}} \rho_{\mathbf{kl}}(\mathbf{H_{kl}})$$

$$= \mathcal{N} \exp\left[-\sum_{\mathbf{k}} \frac{(\mathbf{H_{kk}}-\mathbf{b_{kk}})^2}{\mathbf{2v_{kk}}}\right] \prod_{\mathbf{k},\mathbf{l};\mathbf{k}\neq\mathbf{l}} \delta(\mathbf{H_{kl}}-\mathbf{f_{kl}})$$

$\mathbf{f_{kl}} = \mathbf{t}$ if $\{k,l\}$ nearest neighbor sites and otherwise $\mathbf{f_{kl}} = \mathbf{0}$.

limiting Gaussian approximation of δ-function $\rightarrow$

$$\rho(H) = \mathcal{N} \exp\left[-\sum_{\mathbf{k}\leq\mathbf{l}} \frac{(\mathbf{H_{kl}}-\mathbf{b_{kl}})^2}{\mathbf{2v_{kl}}}\right]$$

Anderson Hamiltonian described by a multi-parametric Gaussian ensemble
P. Shukla, J. Phys.: Condens. Matter 17, (2005) 1653.

randomness due to interaction: with/without disorder

Consider 2 electrons in a lattice of $\mathcal{N}$ sites

$$\mathbf{H} = \sum_{\mathbf{k,l;s_1}} \mathbf{V_{kl}}\ \mathbf{c^{\dagger}_{ks_1}}\ \mathbf{c_{ls_1}} + \sum_{\mathbf{k,s_1,s_2}} \mathbf{U_k}\ \mathbf{c^{\dagger}_{ks_1}}\ \mathbf{c_{ks_1}}\ \mathbf{c^{\dagger}_{ks_2}}\ \mathbf{c_{ks_2}}$$

anti-symmetrized 2-particle product basis ($|ks_1\rangle$ single particle state at site k with spin s_1)

$$|\mu\rangle \equiv |\mathbf{ks_1; ls_2}\rangle = \frac{\mathbf{1}}{\sqrt{\mathbf{2}}}\left(|\mathbf{ks_1}\rangle.|\mathbf{ls_2}\rangle - |\mathbf{ls_2}\rangle.|\mathbf{ks_1}\rangle\right)$$

- $\mathbf{H}$ **is a** $\mathbf{N_2 \times N_2}$ **sparse Hermitian matrix with both random and non-random elements.**
- **sparsity of matrix depends on hopping/connectivity, dimensionality, boundary conditions**
- **randomness depends on** $\mathbf{U}$ **and** $\mathbf{V}$**, whether random or non-random**
- **many types of sparse random matrix ensemble possible for** $\mathbf{H}$

ensemble density for 2-body Hamiltonian

- **Gaussian distributed $\mathbf{U_k}$ $\rightarrow$ same for $\mathbf{H_{\mu\mu}}$ if $\mathbf{k = l, s_1 \neq s_2}$.**
- **non-random $\mathbf{H_{\mu\mu}}$ if $\mathbf{k = l, s_1 = s_2}$,**
- **non-random hopping i.e $\mathbf{H_{\mu\nu}}$ non-random,**

$$\rho(\mathbf{H}) = \prod_{\substack{\mu \\ \mathbf{k=l,} \\ \mathbf{s_1 \neq s_2}}} \mathrm{e}^{-\frac{(\mathbf{H_{\mu\mu}} - \mathbf{U_0})^2}{\mathbf{2w^2}}} \prod_{\substack{\mu \\ \mathbf{s_1 = s_2}}} \delta(\mathbf{H_{\mu\mu}}) \prod_{\substack{\mu,\nu \\ \mathbf{contd.}}}^{\mathbf{N}} \delta(\mathbf{H_{\mu\nu}} - \mathbf{t}) \prod_{\substack{\mu,\nu\neq \\ \mathbf{contd.}}}^{\mathbf{N}} \delta(\mathbf{H_{\mu\nu}})$$

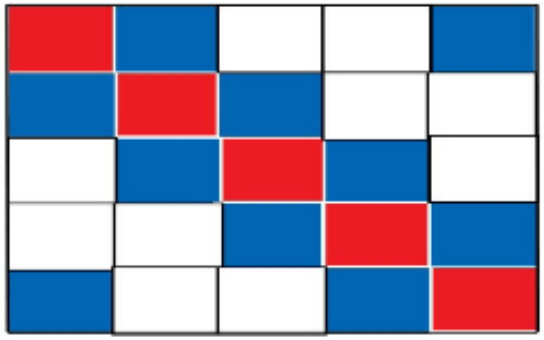
Replace δ-functions by limiting Gaussian distributions,

$$\rho(\mathbf{H}) = \mathcal{N} \exp\left[-\sum_{\mathbf{k,l,s}} \frac{\mathbf{1}}{\mathbf{2v_{\mu\nu;s}}} (\mathbf{H_{\mu\nu}} - \mathbf{b_{\mu\nu}})^{\mathbf{2}} \right]$$

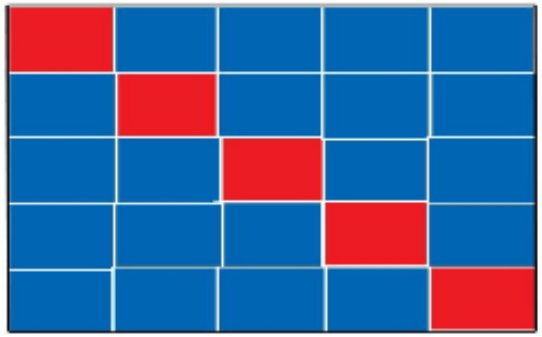
Pragya Shukla, Phys. Rev. B, 98, 054206, (2018).

effectively sparse random matrices

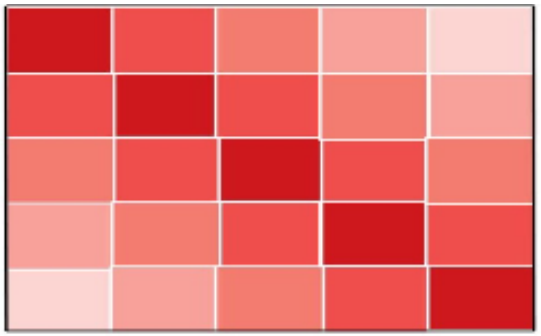
anderson



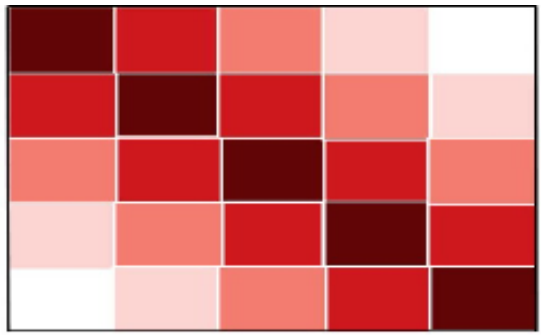
constant off-diagonal variance



power law decay variance



expoential decaying variance



a common mathematical formulation of various complex systems possible?

consider uncorrelated multiparametric Gaussian ensemble

$$\rho(\mathbf{H}) = \mathcal{N} \exp\left[-\sum_{\mathbf{k}\leq\mathbf{l}} \frac{(\mathbf{H_{kl}} - \mathbf{b_{kl}})^2}{\mathbf{2v_{kl}}}\right]$$

v_{kl}: variance of H_{kl} , zero for non-random cases, finite for random cases

b_{kl}: mean-value of H_{kl}

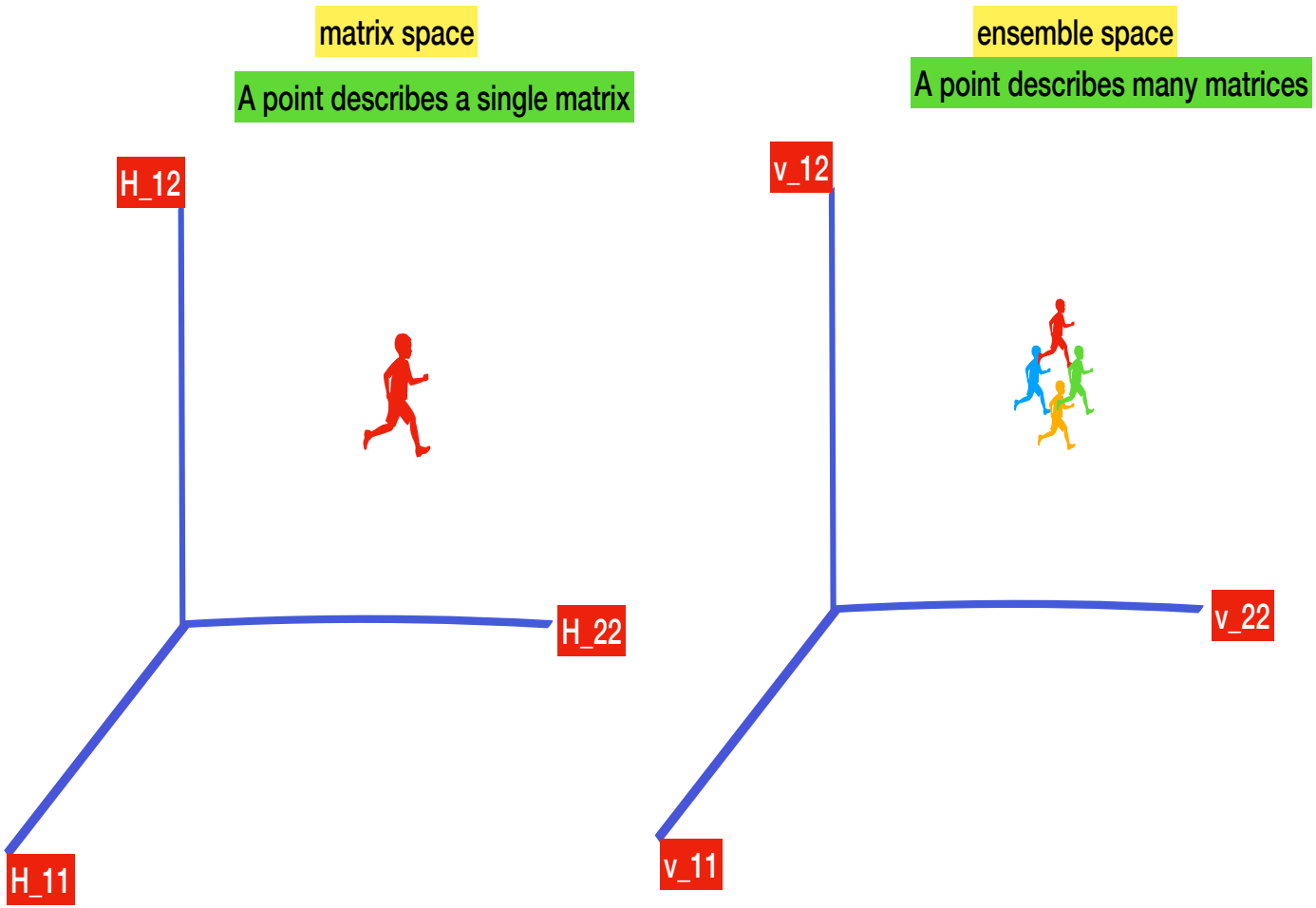
ensemble-parameters v, b influenced by system conditions:

- dimensionality due to b-dependence on localization length,
- boundary conditions due to their effect on interaction of basis-states close to boundaries,
- symmetry conditions, system-size, disorder and scattering conditions

system information is contained in the sets of parameters v, b

- different systems will correspond to different v, b sets.

matrix vs ensemble dynamics: $\rho(\mathrm{H};\mathrm{v},\mathrm{b})$ with $v = \langle H_{kl}^2 \rangle - \langle H_{kl} \rangle$, $b = \langle H_{kl} \rangle$



If an ensemble has to continue representing a single matrix throughout latter's variation, the dynamics of the matrix elements statistically must be mimicked by ensemble parameters describing a point in ensemble space

what if system conditions change?

- complexity due to changing complicated interactions/ disorder acts as noise!

  changing system conditions can lead to diffusion of matrix elements and of $\rho(\mathbf{H}, \mathbf{v}, \mathbf{b})$ in matrix space!

  P.Shukla, J.Phys. A, 41, 304023, (2008).

- ensemble parameters are functions of system parameters if the ensemble is a good representation of the system

$$\mathbf{H_{kl}} = \mathbf{func.}(\tau_1, \tau_2, ...) \rightarrow \ldots \rightarrow \langle \mathbf{H^m_{kl}} \rangle = \mathbf{func.}(\tau_1, \tau_2, ...)$$

changing system conditions $\tau_1, \tau_2, \ldots$ may change ensemble parameters too , $\Longrightarrow$ $\rho(\mathbf{H}, \mathbf{v}, \mathbf{b})$ evolves in ensemble space too!

Q1: are the two evolutions analogous?

Q2: is the evolution really governed by the system-details and is multiparametric?

ergodicity in non-equilibrium regime?

Q1: can diffusion of $\rho(\mathrm{H}, \mathrm{v}, \mathrm{b})$ in matrix space be exactly described by its evolution in ensemble space for all stages of evolution?

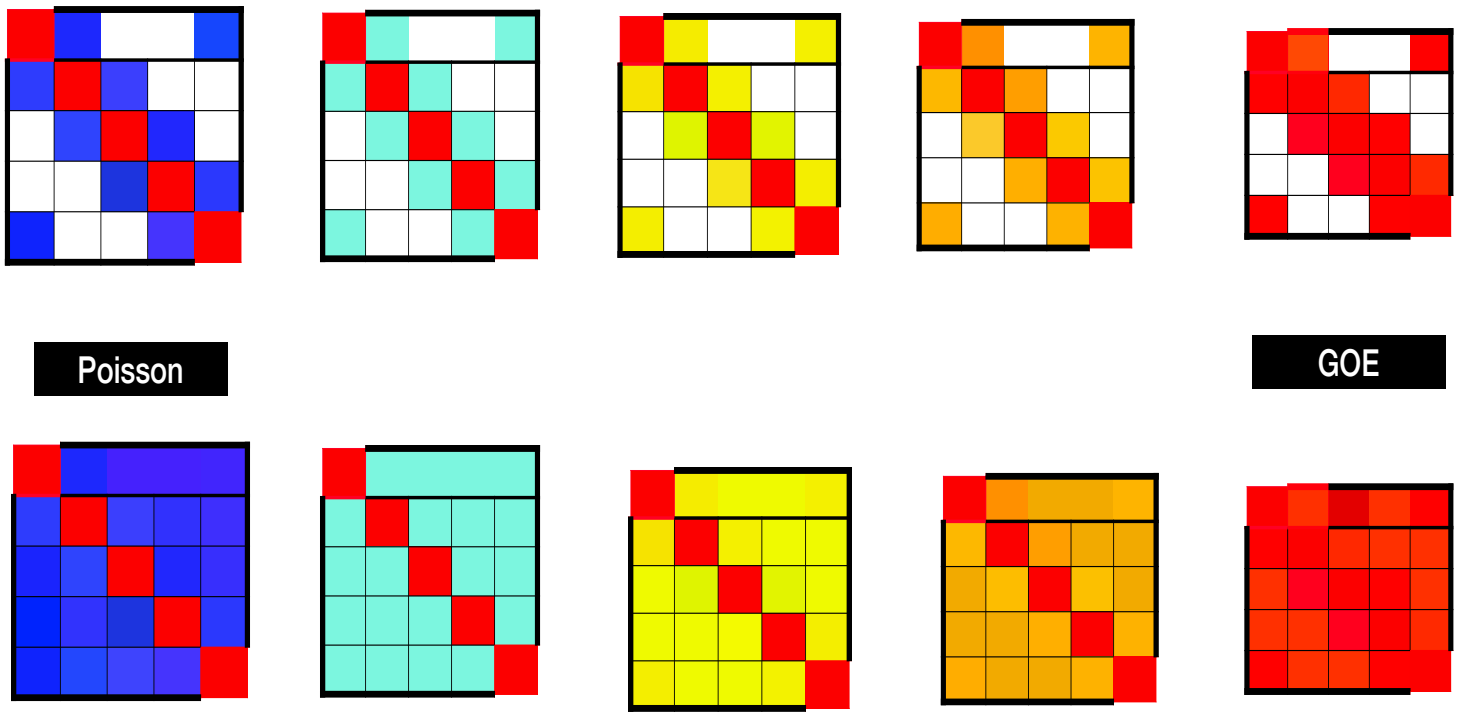
why?

because that would ensure each matrix in the ensemble is evolving in an analogus way,
matrices in the ensemble remains exact replicas of system even during evolution

two different routes of transition

Transition from Poisson to GOE limit.

Two different types of transition



single parametric evolution

Q2: is the evolution governed by a single parameter related to average complexity of the system and not its details?

why?

because system conditions change with respect to time,

evolution of system due to variation of multiple system conditions can be described in terms of a single variable "time"

what is analog of time in ensemble space?

can evolution of respresentative ensemble due to variying system conditions also be described in terms of a single variable ?

changing system conditions: role of accuracy

- Presence of complicated interactions :
  $\rightarrow$ determination of matrix elements within a certain accuracy, may vary from element to element (sensitivity to local environment).
- The accuracy fluctuates rapidly as the system conditions change
  $\rightarrow$ different *time-scales* of fluctuations for each matrix element possible
- Each matrix element undergoes a diffusive dynamics due to changing system conditions and rapidly fluctuating accuracy.

- elements have a natural tendency to oppose their change
  $\rightarrow$ dynamics subjected to local frictional forces too.

Dynamics described by the Langevin equation

accuracy driven Diffusion of matrix elements

matrix element H_{kl} changes at "times-scale" t_{kl} due to changing system conditions but rapidly fluctuating accuracy acts as a noise on the dynamics.

Dynamics of a single "particle" H_{kl} described by the Langevin equation

$$\frac{\mathrm{d}^2 H_{kl}}{\mathrm{d}t_{kl}^2} = -f\frac{\mathrm{d}H_{kl}}{\mathrm{d}t_{kl}} + V(H_{kl}) + A(t_{kl})$$

- $V(H)$: external potential (Gaussian case $V(H) = H$)
- f: friction coefficient, a measure of resistance to change
- $A(t)$ white noise due to a rapidly fluctuating accuracy due to varying parameter t (a measure of the scale for accuracy fluctuations).

P.Shukla, J.Phys. A, 41, 304023, (2008).

probability density of each element diffuses

assume Markovian dynamics:

- Langevin equation gives the moments for H_{kl}:

$$\langle \delta H_{kl} \rangle = -V(H_{kl})\ \delta t_{kl} \qquad \langle (\delta H_{kl})^2 \rangle = -g_{kl}\ \delta t_{kl}$$

- diffusion equation for the probability density ρ_{kl} of a "particle" H_{kl}

$$\frac{\partial \rho_{kl}}{\partial t_{kl}} = \frac{\partial}{\partial H_{kl}} \left[\frac{g_{kl}}{2} \frac{\partial}{\partial H_{kl}} + V(H_{kl}) \right] \rho_{kl}$$

- H: a collection of "particles" H_{kl}, undergoing independent Langevin dynamics

$$\sum_{k \le l} \frac{\partial \rho}{\partial t_{kl}} = \sum_{k \le l} \frac{\partial}{\partial H_{kl}} \left[\frac{g_{kl}}{2} \frac{\partial}{\partial H_{kl}} + V(H_{kl}) \right] \rho$$

Independent matrix elements: equation unaffected if $\rho_{kl} \to \rho$:

$$\rho(H_{11} \ldots H_{NN}; t_{11} t_{NN}) = \prod_{k \le l} \rho_{kl}(H_{kl}; t_{kl})$$

diffusion of all elements as a joint system

- H: generator of the dynamics of the system as a whole unit, natural to seek a common time scale Y in which all matrix elements evolve

$$\rho(H_{11}\dots H_{NN};t_{11}t_{NN}) = \rho(H_{11},H_{12},\dots,H_{NN};Y)$$

this would imply

$$\frac{\partial\rho}{\partial Y}\,\delta Y = \sum_{k\le l}\frac{\partial}{\partial H_{kl}}\left[\frac{1}{2}\frac{\partial}{\partial H_{kl}}\langle(\delta H_{kl})^2\rangle - \langle\delta H_{kl}\rangle\right]\rho$$

but accuracy driven Langevin equation gives the moments for H_{kl} as

$$\langle\delta H_{kl}\rangle = -V(H_{kl})\,\delta t_{kl} \qquad \langle(\delta H_{kl})^2\rangle = -g_{kl}\,\delta t_{kl}$$

If $\quad \delta t_{11} = \delta t_{12} = ... = \delta t_{NN} = ...\delta Y$

moments become $\quad \langle\delta H_{kl}\rangle = -V(H_{kl})\,\delta Y \quad \langle(\delta H_{kl})^2\rangle = -g_{kl}\,\delta Y$

leads to same diffusion equation as before but now governed by Y

Y can be defined

changing system conditions lead to diffusion of matrix elements!

changing system conditions change ensemble parameters

leads to evolution of $\rho(\mathbf{H}) \propto \mathrm{e}^{-\sum \frac{(\mathbf{H_{kl}}-\mathbf{b_{kl}})^2}{\mathbf{2v_{kl}}}}$ **(with** $g_{kl} = 1 + \delta_{kl}$**)**

$$\sum_{\mathbf{k\le l}} \left[(\mathbf{g_{kl}} - \mathbf{2v_{kl}}) \frac{\partial \rho}{\partial \mathbf{v_{kl}}} - \mathbf{b_{kl}} \frac{\partial \rho}{\partial \mathbf{b_{kl}}} \right] = \sum_{\mathbf{k\le l}} \frac{\partial}{\partial \mathbf{H_{kl}}} \left[\frac{\mathbf{g_{kl}}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{H_{kl}}} + \mathbf{H_{kl}} \right] \rho$$

possible to define a function Y **such that**

$$\frac{\partial \rho}{\partial \mathbf{Y}} = \sum_{\mathbf{k,l}} \frac{\partial}{\partial \mathbf{H_{kl}}} \left[\frac{\mathbf{g_{kl}}}{\mathbf{2}} \frac{\partial}{\partial \mathbf{H_{kl}}} + \mathbf{H_{kl}} \right] \rho$$

where

$$\mathbf{Y} = -\frac{\mathbf{1}}{\mathbf{N(N+1)}} \ \ln \left[\prod_{\mathbf{k\le l}} |\mathbf{g_{kl}} - \mathbf{2v_{kl}}| \quad |\mathbf{b_{kl}}|^{\mathbf{2}} \right] + \mathbf{const.}$$

multi-parametric diffusion of $\rho(H)$ **can be reduced to a single parameter**

Y= average distribution parameter, a measure of average uncertainty of system

P. Shukla, Phys. Rev. E, (71), (2005), 026226(1-13).

How Y is related to v_{kl} and b_{kl}?

- Diffusion equation for ρ with Y as diffusion parameter

$$\frac{\partial \rho}{\partial \mathbf{Y}} = \sum_{\mathbf{k \leq l}} \left[(\mathbf{g_{kl}} - \mathbf{2v_{kl}}) \frac{\partial}{\partial \mathbf{v_{kl}}} - \mathbf{b_{kl}} \frac{\partial}{\partial \mathbf{b_{kl}}} \right]$$

write $\mathbf{t}_\mu = \ln|\mathbf{g_{kl}} - \mathbf{2v_{kl}}|$ and $\mathbf{t}_\mu = \ln|\mathbf{b_{kl}}|^{\mathbf{2}}$

$$\Rightarrow \quad \frac{\partial \rho}{\partial Y} = \sum_\mu \frac{\partial \rho}{\partial t_\mu}$$

satisfied if $\mathbf{dt_1} = \mathbf{dt_2} = ... = \mathbf{dt_M} = ...\mathbf{dY}$

$$\mathbf{Y} = \frac{\mathbf{1}}{\mathbf{M}} \sum_\mu \mathbf{t}_\mu$$

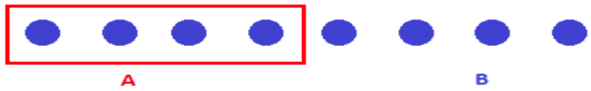
eigenfunctions

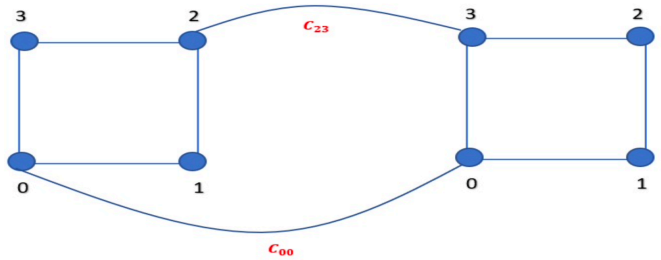
randomization of the operator also mainfests in eigenfunctions expressed in any physically useful basis

leads to ensemble of state matrices (for each eigenstate)

bipartite entanglement of a pure state

Pure state of system in product basis of two subsystems A and B





$$|\Psi\rangle = \sum_{\mathrm{i,j}} \mathrm{C_{ij}} \, |\mathrm{a_i}\rangle \, |\mathrm{b_j}\rangle$$

C: state matrix

matrix elements as components of $|\Psi\rangle$ in product basis

rectangular $\mathrm{N_a} \times \mathrm{N_b}$ if A and B of size $\mathrm{N_a}$ and $\mathrm{N_b}$

moments of eigenfunctions from perturbation theory, P.S, Phys. Rev. E, 75, 051113, (2007), J. Phys A (IOP), 50, 435003, (2017)

moments of eigen components by second order perturbation theory

$\mathbf{H} \mid \mathbf{\Psi_n}\rangle = \mathbf{e_n} \mid \mathbf{\Psi_n}\rangle$ an arbitrary basis $\mid \mathbf{1}\rangle, \ldots, \mid \mathbf{N}\rangle$

$$\mathbf{C_{\mu n}} \equiv \langle \mu \mid \mathbf{\Psi_n}\rangle, \qquad \mid \mu\rangle \equiv \mid a_k b_l\rangle$$

$$\delta \mathbf{C_{\mu n}} = \sum_{\substack{\mathbf{m=1,} \\ \mathbf{m \neq n}}}^{\mathbf{N}} \frac{|\delta \mathbf{H_{mn}}|}{(\mathbf{e_n - e_m})} \mathbf{C_{\mu m}} - \frac{\mathbf{1}}{\mathbf{2}} \sum_{\mathbf{m=1, m \neq n}}^{\mathbf{N}} \frac{|\delta \mathbf{H_{mn}}|}{(\mathbf{e_n - e_m})^2} \mathbf{C_{\mu n}} +$$

$$+ \sum_{\substack{\mathbf{m,m' \neq n} \\ \mathbf{m \neq m'}}}^{\mathbf{N}} \frac{|\delta \mathbf{H_{mn}}||\delta \mathbf{H_{m'n}}|}{(\mathbf{e_n - e_m})(\mathbf{e_n - e_{m'}})} \mathbf{C_{\mu m} C_{\mu m'}}$$

substitute moments from diffusion equation of matrix elements

$$\langle \delta H_{kl} \rangle = -H_{kl}\ \delta Y$$

$$\langle (\delta H_{kl})^2 \rangle = g_{kl}\ \delta Y$$

moments of the components of n^{th} eigenfunction

diffusion of components of an eigenfunction

consider an eigenfunction $|\ \Psi\rangle$ in the N-dimensional bipartite basis $|\ \mathbf{1}\rangle, \ldots, |\ \mathbf{N}\rangle$

$$\mathbf{C_{kl}} \equiv \mathbf{C}_\mu \equiv \langle \mu \mid \mathbf{\Psi} \rangle$$

Y-governed moments of state matrix

$$\Rightarrow \boxed{\langle \delta C_{kl} \rangle = C_{kl} \delta\Lambda}$$

$$\Rightarrow \boxed{\langle \delta C_{kl}\ \delta C_{mn} \rangle = (\delta_{kl,mn} - C_{kl} C^*_{mn}) \delta\Lambda}$$

$$\boxed{\Lambda = \frac{\chi_0 (Y - Y_0)}{\Delta^2_{local}}}$$

moments of reduced density matrix $\rho_{\mathbf{A}} = \mathrm{C}.\mathrm{C}^{\dagger}$

perturbation of state matrix $\mathbf{C}$ manifests in reduced density matrix $\rho_{\mathbf{A}}$.
matrix elements $\rho_{\mathbf{A};\mathbf{mn}} = \sum_{\mathbf{k}=\mathbf{1}}^{\mathbf{N_a}} \mathbf{C}^{*}_{\mathbf{km}} \mathbf{C}_{\mathbf{kn}}$ undergo diffusive dynamics too,

$$\langle \delta\rho_{A;mm} \rangle = -(N-1)\, \rho_{A;mm} \delta\Lambda \quad .$$

$$\langle \delta\rho_{A;mn}\ \delta\rho^{*}_{A;mn} \rangle = \left[\beta \left(\rho_{A;mm} + \rho_{A;nn}\right) - 4\rho_{A;mm}\rho_{A;nn}\right]\ \delta\Lambda$$

$$\langle \delta\rho_{A;mm}\ \delta\rho_{A;nn} \rangle = -4\, \rho_{A;mm}\, \rho_{A;nn}\ \delta\Lambda$$

moments of Schmidt eigenvalues

Second order spectral perturbation theory of the Hermitian matrices gives

$$\delta\lambda_{\mathbf{n}} = \delta\rho_{\mathbf{A};\mathbf{nn}} + \sum_{\mathbf{m}(\neq\mathbf{n})} \frac{(\delta\rho_{\mathbf{A};\mathbf{mn}})^{\mathbf{2}}}{\lambda_{\mathbf{n}} - \lambda_{\mathbf{m}}} + \mathcal{O}(\delta\rho^{\mathbf{3}}_{\mathbf{A};\mathbf{mn}}).$$

on ensemble averaging of the above leads to

$$\boxed{\langle \delta\lambda_n \rangle = \beta \left[-(N-1)\,\lambda_n + \sum_{m(\neq n)} \frac{\lambda_n+\lambda_m}{\lambda_n-\lambda_m} \right] \delta\Lambda},$$

$$\boxed{\langle \delta\lambda_n \delta\lambda_m \rangle = 4\lambda_n \ (\delta_{mn} - \lambda_m) \ \delta\Lambda}$$

diffusion of JPDF P of Schmidt eigenvalues

assuming Markovian dynamics, standard Fokker-Planck equation is

$$\frac{\partial \mathbf{P}}{\partial \mathbf{\Lambda}}\,\delta\mathbf{\Lambda} = \frac{\mathbf{1}}{\mathbf{2}}\sum_{\mathbf{m,n}}\frac{\partial^{\mathbf{2}}}{\partial\lambda_{\mathbf{m}}\partial\lambda_{\mathbf{n}}}\left(\langle\delta\lambda_{\mathbf{n}}\delta\lambda_{\mathbf{m}}\rangle\mathbf{P}\right) - \sum_{\mathbf{n}}\frac{\partial}{\partial\lambda_{\mathbf{n}}}\left(\langle\delta\lambda_{\mathbf{n}}\rangle\mathbf{P}\right).$$

Substitution of the moments of Schmidt eigenvalues leads to

$$\frac{\partial \mathbf{P}}{\partial \mathbf{\Lambda_e}} = \sum_{\mathbf{m,n}}\frac{\partial^{\mathbf{2}}(\lambda_{\mathbf{n}}(\delta_{\mathbf{nm}}-\lambda_{\mathbf{m}})\mathbf{P}}{\partial\lambda_{\mathbf{m}}\partial\lambda_{\mathbf{n}}} - \beta\sum_{\mathbf{n=1}}^{\mathbf{N}}\frac{\partial}{\partial\lambda_{\mathbf{n}}}\left(\sum_{\substack{\mathbf{m=1,}\\ \mathbf{m\neq n}}}^{\mathbf{N}}\frac{\lambda_{\mathbf{n}}}{\lambda_{\mathbf{n}}-\lambda_{\mathbf{m}}} + \nu - \eta\lambda_{\mathbf{n}}\right)\mathbf{P}$$

diffusion of entanglement entropy

$R_1 = -\sum_n \lambda_n \log \lambda_n$

$$\langle \mathbf{R_1} \rangle = \int \left(-\sum_{\mathbf{n}} \lambda_{\mathbf{n}} \log \lambda_{\mathbf{n}} \right) \mathbf{P}(\lambda_1, \ldots, \lambda_{\mathbf{N}})\, \mathrm{D}\lambda.$$

-
- Differentiate the above equation with respect to Λ_e,
- substitute diffusion equation for P_λ,
- simplify by partial integration, leads to

$$\frac{\partial \langle \mathbf{R_1} \rangle}{\partial \mathbf{\Lambda}} = (\mathbf{2} - \mathbf{2N_a^2} + \mathbf{2N_a}\nu + \mathbf{N_a N_b}) + \frac{\mathbf{N_b}}{\mathbf{2}} \langle \mathbf{R_0} \rangle - \frac{\mathbf{N}}{\mathbf{2}} \langle \mathbf{R_1} \rangle.$$

$$\langle R_1(\Lambda) \rangle = R_{1,\infty}(1 - e^{-\frac{N\Lambda}{2}})$$

$\Lambda \to \infty$: $\langle R_1 \rangle \to R_{1,\infty} \approx \log N_A - \frac{N_A}{2N_B}$, the Page limit for the ergodic states

diffusion of purity $S_2 = \sum_{n=1}^{N} \lambda_n^2$

$$\langle \mathbf{S_2} \rangle = \int \left(\sum_{\mathbf{n=1}}^{\mathbf{N}} \lambda_{\mathbf{n}}^{\mathbf{2}} \right) \mathbf{P}(\lambda_{\mathbf{1}}, \ldots, \lambda_{\mathbf{N}})\, \mathrm{D}\lambda$$

$$\frac{\partial \langle \mathbf{S_2} \rangle}{\partial \mathbf{\Lambda}} = \mathbf{a} - \mathbf{b} \langle \mathbf{S_2} \rangle; \qquad \mathbf{a} = \mathbf{N_a} + \mathbf{N_b} + \mathbf{1},\ \mathbf{b} = \mathbf{N_a N_b} + \mathbf{2}.$$

For separability initial condition $\langle S_2 \rangle (\Lambda = 0) = 1$ at $\Lambda = 0$, the solution is

$$\langle S_2(\Lambda) \rangle = \frac{a}{b} + \frac{b-a}{b} e^{-b\Lambda_e}$$

random energy model in a random external field

one dimensional lattice of L spin $1/2$ particles in a random external field,

$$\mathbf{H} = \sum \mathbf{J_{k_1 \ldots k_L}}\, \sigma_{\mathbf{k_1}} \ldots \sigma_{\mathbf{k_L}} + \sum_{\mathbf{k}} \mathbf{\Gamma_k} \sigma^{\mathbf{x}}_{\mathbf{k}}$$

H_{REM}: **energies** $\{E_k\}$

$$P(E_k; h, L) = \frac{1}{\sqrt{\pi L}}\, e^{-\frac{E^2}{L}}$$

field strength random

$$\langle \delta\Gamma^2_{kl} \rangle = \frac{1}{1+(\frac{k-l}{b})^2} \quad \langle \Gamma_{kl} \rangle = 0$$

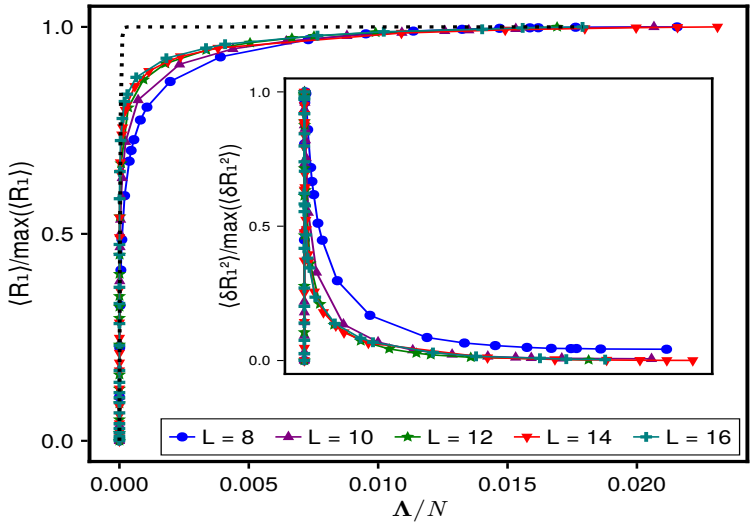
joint probability density of matrix elements H_{kl}

$$\rho_1(H) \propto \prod_k \exp\left(-\frac{H^2_{kk}}{L}\right) \prod'_{k<l} \exp\left[-\left(1+\left(\frac{k-l}{b}\right)^2\right)\frac{H^2_{kl}}{2}\right]$$

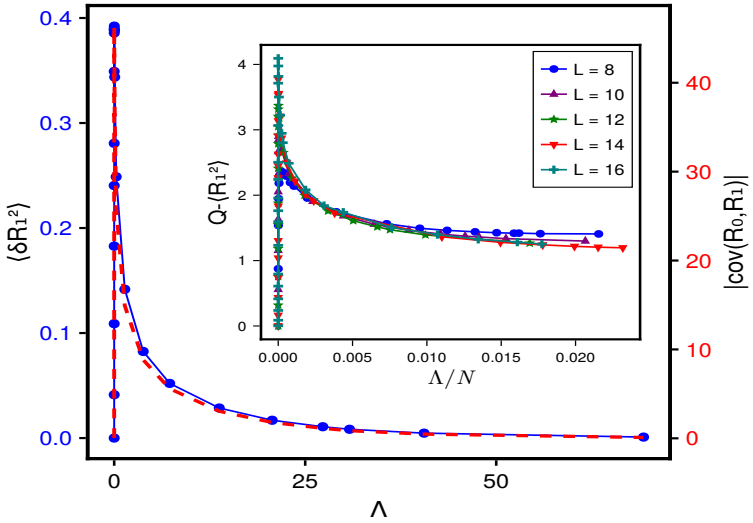
$\prod'_{k<l}$:product over basis elements k and l at a unit Hamming distance.

random energy model: entanglement

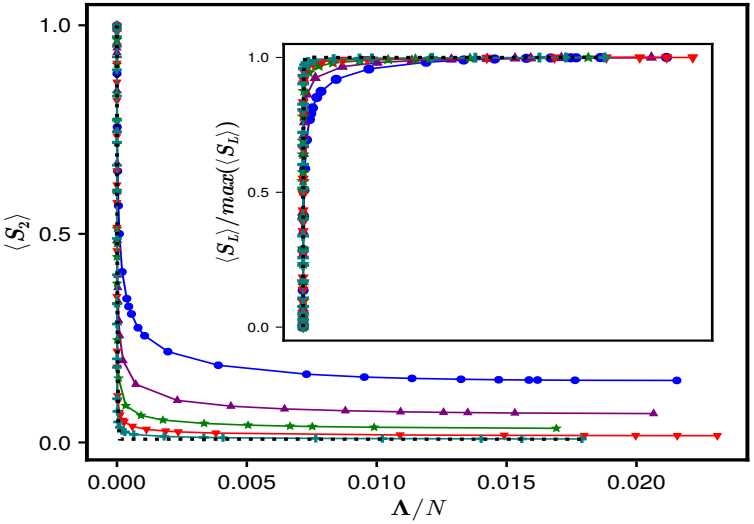
average von Neumann entropy



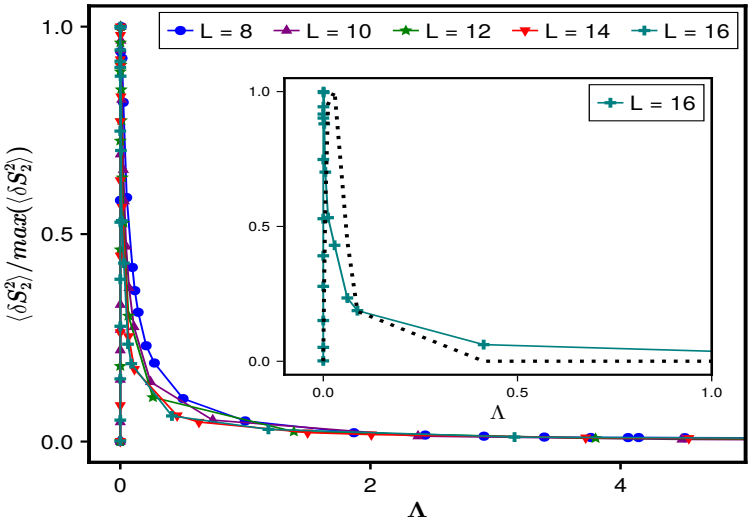
variance of Von Neumann



average purity



variance of purity



what have we discussed so far?

- a complex system: best described by an ensemble $rho(H)$ of Hamiltonians H in a physically motivated basis,
- for an ensemble to well-represent a system, ensemble parameters must be functions of system parameters
- started with a multiparametric gaussian ensemble of Hamiltonian matrices
- **defined a functional Y as a function of ensemble parameters that governs the evolution of $\rho(H)$ in the matrix space, (Λ: rescaled Y)**
  $\Longrightarrow$$\Lambda$-governed diffusion of state matrix ensemble for an eigenstate,
  $\Longrightarrow$$\Lambda$-governed diffusion of reduced density matrix ensemble,
  $\Longrightarrow$ Λ-governed diffusion of JPDF of Schmidt eigenvalues,
  $\Longrightarrow$ Λ-governed evolution of average entanglement entropy $\langle R_1 \rangle$ and purity S_2,
- **both $\langle R_1 \rangle$ and $\langle S_2 \rangle$ expressed as functions of Λ and N, system information entering through Λ and N and not on ensemble parameters set,**

2-parameter based universality classes of the entanglement statistics of the non ergodic states of a physical system

what if we engineer a multivariate gaussian state

a random quantum state with Gaussian distributed components subjected to normalization condition

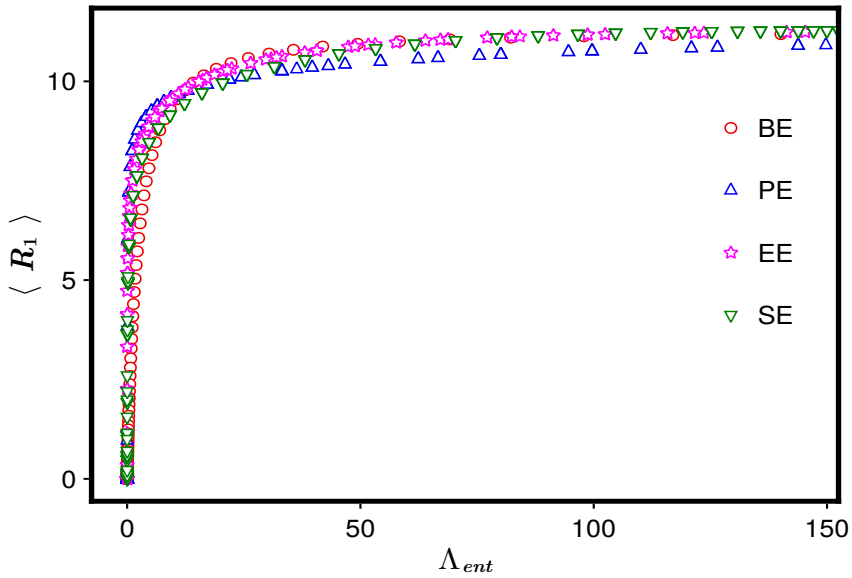
$$\rho_{\mathbf{cn}}(\mathbf{C};\mathbf{h},\mathbf{b}) \propto \exp\left[-\sum_{\mathbf{k,l,s}} \frac{\mathbf{1}}{\mathbf{2h_{kl;s}}}\left(\mathbf{C_{kl;s}}-\mathbf{b_{kl;s}}\right)^{\mathbf{2}}\right]\delta\left(\sum|\mathbf{C_{kl}}|^{\mathbf{2}}-\mathbf{1}\right)$$

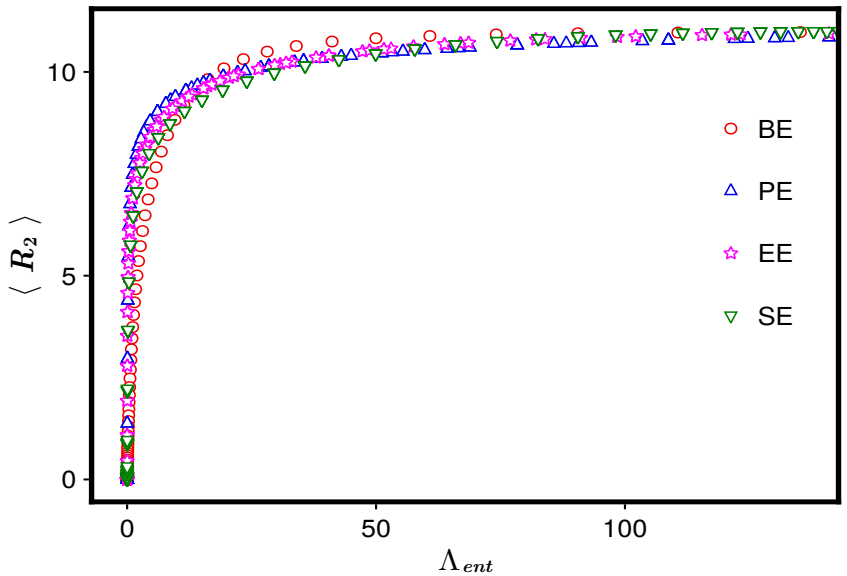
- notwithstanding the normalization condition, the components can still be statistically uncorrelated and described by independent distributions.
- can arise for a case in which C_{kl} are best known up to their mean and variances only Based on maximum entropy hypothesis,
- Berry's Gaussian random wave hypothesis conjecturing that almost all components of a typical state of a quantum chaotic system can be described by independent, identical Gaussian distributions subjected to normalization condition,
- hypothesis corroborated by the success of well-known Porter Thomas distribution of local intensity for quantum chaotic states,

multivariate gaussian state ensembles

Devanshu Sekhar and P.S., J.Phys. A, 56, 265303, (2023);

arxiv:2310.12796, 2402.01102





(i) **BE:**

$$\mathbf{h_{k1} = 1, h_{kl} = (1+\mu)^{-1},\ l}$$

(ii) **PE:**

$$\mathbf{h_{kl} = \left(1 + \frac{k(l-1)}{\mu}\right)^{-1}},$$

(iii) **EE:**

$$\mathbf{h_{kl} = \exp\left(-\frac{k|l-1|}{\mu}\right)}$$

(iv) **SE:**

$$\mathbf{h_{kl} = \exp\left[-\frac{k(l-1)}{w^2}\right]}$$

for $d(k,l) = 0$,

$$\mathbf{h_{kl} = \exp\left[-\frac{k(l-1)}{w_s^2}\right]},$$

for $0 < d(k,l) \leq n$

$d(k,l)$: hemming distance

Implications of complexity parametric formulation

- a common mathematical formulation for a broad range of non-ergodic states with Gaussian components (independent or pairwise correlated, subjected to normalization constraint).

- an arbitrary ensemble can be more efficiently realized in physical systems $\longrightarrow$,

  the knowledge that we only need to increase the complexity parameter resulting in its entanglement entropy approach that of Haar ensemble, is useful for multiple quantum engineering application.

- **complexity parameter Λ is a function of all ensemble parameters and thereby a functional of all system parameters.**

- **Λ can be changed in multiple ways, the formulation provides information how to change the ensemble parameters in a controlled way to increase Λ and thereby approach Haar limit**

- as entanglement entropy depends on all moments of an ensemble, the knowledge of how and when it approaches that of Haar ensemble far more relevant then the search for the tools to form quantum state k-designs.

Implications of Universality

- Λ formulation helps in determination of critical value of ensemble parameters above which the arbitrary state becomes Haar type.
- different states with same Λ and N, although different ensemble parameters, will have same Schmidt eigenvalue statistics (valid for same global constraint class).
  $\Longrightarrow$ Possible to classify states into continuum of universality classes characterized by Λ and N
- the knowledge that Haar limit can be achieved for different state ensembles through a common universal route has important implications for quantum state engineering.
- **the formulation suggests a deep rooted universality and a web of connection hidden underneath the world of random states.**

Thank you

References

- **P. Shukla, Phys. Rev. E, 62, 2098, (2000).**
  (Common mathematical formulation for Gaussian ensembles of Hermitian operators)
- **P. Shukla, Phys. Rev. Lett. (2001).**
  (for Gaussian ensembles of non-Hermitian operators)
- **P. Shukla, Phys. Rev. E. 71, 026226, (2005).**
  (for non-Gaussian ensembles of Hermitian operators)
- **P.Shukla, J.Phys. C: Condens. Matter, 17, 1653, (2005).**
  (application to disordered systems)
- **P. Shukla and I.P.Batra, Phys. Rev. B, (2005).**
  (for transfer matrices)
- **P. Shukla, Phys. Rev. E, 75, 051113, (2007).**
  (for eigenfunction correlations)
- **Rina Dutta and P. Shukla, Phys. Rev. E,76, 051124, (2007).**
  (application to spin systems)
- **Rina Dutta and P.Shukla, Phys. Rev. E 78, 031115 (2008).**
  (application to kicked rotor)
- **P.Shukla, J.Phys. A, 41, 304023, (2008).**
  (accuracy approach)
- **M V Berry and P. Shukla, J. Phys. A, 42, 485102, (2009).**
  (2×2 case)
- **Pragya Shukla, Phys. Rev. B, 98, 054206, (2018).**
  (many body case)
- **T. Mondal and P. Shukla, Phys. Rev. E 102, 032131 (2020)**
  (chiral ensemble)
- **P. Shukla, J. Phys A (IOP), 50, 435003, (2017).**
  (wishart ensemble)