Hurewicz homomorphism of C^*-algebras

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Noncommutative geometry and topology

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One has the Hurewicz homomorphism $h_1 : \pi_1(\mathcal{X}, x_0) \to K^1(\mathcal{X})$ such that

$$h^{\mathrm{top}}_{K^1} : \pi_1(\mathcal{X}, x_0) \cong \left[S^1, s_0; \mathcal{X}, x_0\right] \xrightarrow{K^1}$$

$$\mathrm{Hom}\left(K^1\left(C\left(S^1\right)\right), K^1\left(C\left(\mathcal{X}\right)\right)\right) \xrightarrow{\phi} K^1\left(C\left(\mathcal{X}\right)\right).$$

Let us describe $h^{\mathrm{top}}_{K^1}$ in details. The map K^1 is a functor of K^1-homology. If the C^*-algebra $C\left(S^1\right)$ be a C^*-algebra generated by a single unitary element u, then the group $K^1\left(S^1\right)$ is generated by an element $[u]$ which is represented by u. $K^1\left(S^1\right)$ is a free Abelian group generated $\left[K^1_{S^1}\right]$ which corresponds to the identical homomorphism

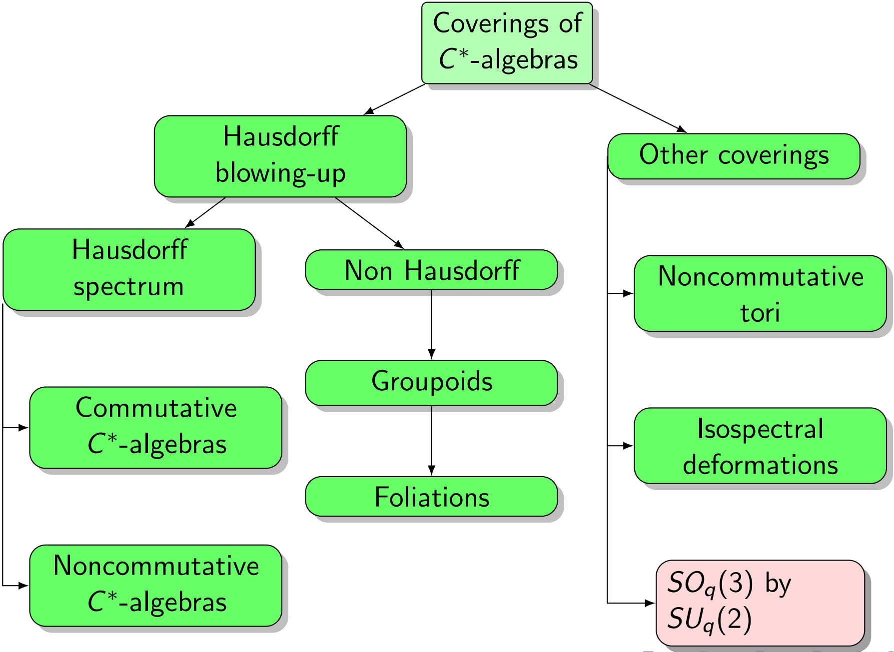
$$\mathrm{Id}_{\mathbb{Z}} \in \mathrm{Hom}\left(K^1\left(C\left(S^1\right) \cong \mathbb{Z}[u]\right), \mathbb{Z}\right).$$

The Hurewicz homomorphism is given by

$$h^{\mathrm{top}}_{K^1} : \pi_1(\mathcal{X}, x_0) \to K^1\left(C\left(\mathcal{X}\right)\right),$$

$$[\omega] \mapsto K^1(\omega)\left(\left[K^1_{S^1}\right]\right).$$

Known C^*-algebras admitting fundamental group



Finite-fold coverings

Theorem

Alexander Pavlov, Evgenij Troitsky. Suppose both $\mathcal{X}$ and $\mathcal{Y}$ are compact Hausdorff connected spaces and $p : \mathcal{Y} \to \mathcal{X}$ is a continuous surjection. If $C(\mathcal{Y})$ is a projective finitely generated Hilbert module over $C(\mathcal{X})$ with respect to the action

$$(f\xi)(y) = f(y)\xi(p(y)),\ f \in C(\mathcal{Y}),\ \xi \in C(\mathcal{X}),$$

then p is a finite-fold covering.

It is naturally to define a finite-fold covering of C^*-algebras as an injective $*$-homomorphisms $A \hookrightarrow \widetilde{A}$ such that $\widetilde{A}$-is a finitely generated Hilbert module over A. However this definition does not gives good generalizations of results related to topological coverings.

Definition

We say that a C^*-algebra A is connected if it cannot be represented as a direct sum $A \cong A' \oplus A''$ of nontrivial C^*-algebras A' and A''.

Definition

A connected closed two-sided ideal A of C^*-algebra B is said to be a connected component of B is there is a direct sum $B = A \oplus A'$ of C^*-algebras.

Definition

Pet Ivankov. Let $\pi : A \hookrightarrow \widetilde{A}$ be an injective *-homomorphism of connected C^*-algebras such that following conditions hold:

(a) If $\mathrm{Aut}\left(\widetilde{A}\right)$ is a group of *-automorphisms of $\widetilde{A}$ then the group $G \overset{\mathrm{def}}{=} \left\{g \in \mathrm{Aut}\left(\widetilde{A}\right) \,\middle|\, g\pi\left(a\right) = \pi\left(a\right);\ \ \forall a \in A\right\}$ is finite.

(b)

$$\pi\left(A\right) = \widetilde{A}^G \overset{\mathrm{def}}{=} \left\{a \in \widetilde{A} \,\middle|\, a = ga;\ \forall g \in G\right\}.$$

We say that the quadruple $\left(A, \widetilde{A}, G, \pi\right)$ and/or *-homomorphism $\pi : A \to \widetilde{A}$ is a noncommutative finite-fold pre-covering.

Definition

Petr Ivankov Let $\left(A, \widetilde{A}, G, \pi\right)$ be a noncommutative finite-fold pre-covering. Suppose both A and $\widetilde{A}$ are unital. We say that $\left(A, \widetilde{A}, G, \pi\right)$ is an unital noncommutative finite-fold covering if $\widetilde{A}$ is a finitely generated projective A-module.

Lemma

Petr Ivankov, Alexander Pavlov, Evgenij Troitsky. If $\mathcal{X}$ is a connected, compact, Hausdorff space then there is a natural 1-1 correspondence

$$\left(p : \widetilde{\mathcal{X}} \to \mathcal{X}\right) \leftrightarrow \left(C\left(\mathcal{X}\right), C\left(\widetilde{\mathcal{X}}\right), G\left(\widetilde{\mathcal{X}}\middle|\, \mathcal{X}\right), C_0\left(p\right)\right).$$

between finite-fold transitive coverings of $\mathcal{X}$ and unital noncommutative finite-fold coverings of $C\left(\mathcal{X}\right)$.

A covering $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ is transitive if for all $x \in \mathcal{X}$ the group $G\left(\widetilde{\mathcal{X}}\middle|\, \mathcal{X}\right)$ transitively acts on $p^{-1}\left(x\right)$.

Definition

Let $\left(A, \widetilde{A}, G, \mathfrak{lift}\right)$ be a noncommutative finite-fold pre-covering of C^*-algebras A and $\widetilde{A}$ such that following conditions hold:

(a) There are unitizations $A \hookrightarrow B$ and $\widetilde{A} \hookrightarrow \widetilde{B}$;

(b) There is a unital noncommutative finite-fold quasi-covering $\left(B, \widetilde{B}, G, \mathfrak{lift}^B\right)$ such that $\mathfrak{lift} = \mathfrak{lift}^B|_A$ (or, equivalently $\widetilde{A}$ is the generated by A hereditary subalgebra of $\widetilde{B}$) and the action $G \times \widetilde{A} \to \widetilde{A}$ comes from the $G \times \widetilde{B} \to \widetilde{B}$ one.

We say that the triple $\left(A, \widetilde{A}, G\right)$ and/or the quadruple $\left(A, \widetilde{A}, G, \mathfrak{lift}\right)$ and/or $*$-homomorphism $\mathfrak{lift} : A \hookrightarrow \widetilde{A}$ is a noncommutative finite-fold covering with unitization.

Roughly speaking the above Definition is an approximation of any covering by coverings with compact spaces. In result one has the following theorem.

Theorem

Petr Ivankov. Let $\mathcal{X}$ be a connected, locally compact, Hausdorff space. If the quadruple $\left(C_0\left(\mathcal{X}\right), \widetilde{A}, G, \pi\right)$ is a noncommutative finite-fold covering then there is a connected space $\widetilde{\mathcal{X}}$ and a transitive finite-fold covering $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ such that $\left(C_0\left(\mathcal{X}\right), \widetilde{A}, G, \pi\right)$ is equivalent to $\left(C_0\left(\mathcal{X}\right), C_0\left(\widetilde{\mathcal{X}}\right), G\left(\widetilde{\mathcal{X}} \middle| \mathcal{X}\right), \pi\right)$

This Theorem has a Hausdorff blowing-up generalization.

Infinite coverings

Let $\widetilde{\mathcal{X}}$ be a topological space with an action $G \times \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}$ of residually finite group G of properly discontinuous group of homeomorphisms. Let $\mathcal{X} \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{X}}/G$ and $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ be a natural covering. For any finite factor group $G_\lambda = G/H_\lambda$ we define a space $\mathcal{X}_\lambda \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{X}}/H_\lambda$. Then there is a category of topological spaces and finite-fold transitive coverings given by

$$\mathfrak{S}_p \stackrel{\mathrm{def}}{=} \left\{ \{\mathcal{X}_\lambda\}_{\lambda\in\Lambda}, \{p^\mu_\nu : \mathcal{X}_\mu \to \mathcal{X}_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\geq\nu}} \right\}. \tag{1}$$

Usage of the functor C_0 yields a category of C^*-algebras and $*$-homomorphisms given by

$$\mathfrak{S}_{C_0(p)} \stackrel{\mathrm{def}}{=}$$

$$\left\{\left\{C_0\left(p_\lambda\right) : C_0\left(\mathcal{X}\right) \hookrightarrow C_0\left(\mathcal{X}_\lambda\right)\right\}, \left\{C_0\left(p^\mu_\nu\right) : C_0\left(\mathcal{X}_\mu\right) \hookrightarrow C_0\left(\mathcal{X}_\nu\right)\right\}\right\}.$$

If $\widehat{G} \stackrel{\mathrm{def}}{=} \varprojlim_{\lambda \in \Lambda} G\left(\mathcal{X}_\lambda \mid \mathcal{X}\right)$ is an inverse limit of finite groups then the group $\widehat{G}$ is profinite. One has $\mathcal{X}_\lambda \stackrel{\mathrm{def}}{=} \widetilde{\mathcal{X}} / \ker\left(G\left(\widetilde{\mathcal{X}} \,\middle|\, \mathcal{X}\right) \to G\left(\mathcal{X}_\lambda \mid \mathcal{X}\right)\right)$ and there is an inverse limit $\widehat{\mathcal{X}} = \varprojlim_{\lambda \in \Lambda} \mathcal{X}_\lambda$ of topological spaces. There is a natural continuous map $\widetilde{\widehat{p}} : \widetilde{\mathcal{X}} \to \widehat{\mathcal{X}}$. If we consider a final with respect to the family of maps $\left\{g \circ \widehat{p}\right\}_{g \in \widehat{G}}$ topology on $\widehat{\mathcal{X}}$ then we obtain a topological space $\overline{\mathcal{X}}$.

Lemma

Under the above hypotheses the following conditions hold.

(i) *If* $\left\{g_\iota G\left(\widetilde{\mathcal{X}}\,\middle|\,\mathcal{X}\right)\right\}_{\iota\in I}$ *is a set of all left cosets of* $G\left(\widetilde{\mathcal{X}}\,\middle|\,\mathcal{X}\right)$ *in* $\widehat{G}$ *then there is a natural homeomorphism*

$$\overline{\mathcal{X}} \cong \bigsqcup_{\iota\in I} g_\iota \widetilde{\mathcal{X}}.$$

(ii) *The natural map* $\widetilde{\widehat{p}} : \widetilde{\mathcal{X}} \to \widehat{\mathcal{X}}$ *yields a natural inclusion* $\widetilde{\mathcal{X}} \subset \overline{\mathcal{X}}$ *such that* $\widetilde{\mathcal{X}}$ *is a quasi-component of* $\overline{\mathcal{X}}$.

(iii) *For any a quasi-component* $\widetilde{\mathcal{X}}' \subset \overline{\mathcal{X}}$ *there is* $g \in \widehat{G}$ *such that* $\widetilde{\mathcal{X}}' = g\widetilde{\mathcal{X}}$.

(iv) *For any* $\lambda \in \Lambda$ *the natural surjective map* $\widehat{p}_\lambda : \widehat{\mathcal{X}} \to \mathcal{X}_\lambda$ *yields a covering* $\overline{p}_\lambda : \overline{\mathcal{X}} \to \mathcal{X}_\lambda$ *such that* $\mathcal{X}_\lambda \cong \overline{\mathcal{X}} / \ker\left(\widehat{G} \to G_\lambda\right)$.

(v) *There is a natural bijective continuous map* $\overline{\widehat{p}} : \overline{\mathcal{X}} \to \widehat{\mathcal{X}}$.

Definition

Under the hypotheses of the above Lemma we say that the map $\overline{p} : \overline{\mathcal{X}} \to \mathcal{X}$ is the disconnected covering of $p : \widetilde{\mathcal{X}} \to \mathcal{X}$. The topological $\overline{\mathcal{X}}$-$\widehat{G}$-category $\mathfrak{S}_p$ is the finite covering category of $p : \widetilde{\mathcal{X}} \to \mathcal{X}$. Write

$$\mathfrak{S}_p \stackrel{\mathrm{def}}{=} \left\{ \{\mathcal{X}_\lambda\}_{\lambda\in\Lambda}, \{p^\mu_\nu : \mathcal{X}_\mu \to \mathcal{X}_\nu\}_{\substack{\mu,\nu\in\Lambda \\ \mu\ge\nu}} \right\}.$$

We say that $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ is the covering inverse limit of $\mathfrak{S}_p$ and we write

$$\widetilde{\mathcal{X}} \stackrel{\mathrm{def}}{=} \varprojlim \mathfrak{S}_p$$

If $\widehat{G}$ is a profinite group then $\widehat{G} \stackrel{\mathrm{def}}{=} \varprojlim_{\lambda \in \Lambda} G_\lambda$ is an inverse limit of finite groups. The set Λ is directed. Indeed Λ is the $\widehat{G}$-set. Let $\overline{A}$ be a C^*-algebra with an action $\widehat{G} \times \overline{A} \to \overline{A}$ such that any $g \in \widehat{G}$ yields an $*$-automorphism of $\overline{A}$. Suppose that for any element $\overline{a} \in K\left(\overline{A}\right)$ of the Pedersen's ideal of $\overline{A}$ a series

$$\sum_{g \in \widehat{G}} g\overline{a}$$

is convergent with respect to the strict topology of $M\left(\overline{A}\right)$. For any $\lambda \in \Lambda$ denote by A_λ a generated by elements

$$a_\lambda = \beta\text{-} \sum_{g \in \ker\left(\widehat{G} \to G_\lambda\right)} g\overline{a} \qquad (2)$$

C^*-subalgebra of $M\left(\overline{A}\right)$, where β-$\sum$ means a convergence with respect to the strict topology of $M\left(\overline{A}\right)$.

Lemma

Under the above hypotheses all $\mu, \nu \in \Lambda$ *such that* $\nu \geq \mu$ *there is a natural noncommutative finite-fold quasi-covering* $\left(A_\mu, A_\lambda, G_\nu / G_\mu, \pi^\mu_\nu\right)$.

Definition

Under the above hypotheses $\lambda_{\min} \in \Lambda$ is the minimal element and $A \stackrel{\mathrm{def}}{=} A_{\lambda_{\min}}$ then we say that the triple $\left(A, \overline{A}, \widehat{G}\right)$ is an infinite quasi-covering. We say that A_λ is the λ-descent of $\overline{A}$. The natural injective $*$-homomorphism $\mathfrak{lift}_\lambda : A_\lambda \hookrightarrow M\left(\overline{A}\right)$ is the λ-lift.

Definition

It is proven that under the above hypotheses for all $\lambda \in \Lambda$ there is a natural homomorphism of A_λ-A_λ-bimodules given by

$$\mathfrak{desc}_\lambda : K\left(\overline{A}\right) \to K\left(A_\lambda\right),$$
$$\overline{a} \mapsto \beta\text{-} \sum_{g \in \ker\left(\widehat{G} \to G_\lambda\right)} g\overline{a}$$

where β-$\sum$ means the convergence with respect to the strict topology of $M\left(\overline{A}\right)$. We denote this homomorphism as $\mathfrak{desc}_\lambda$ and we say that it is the λ-descent.

Definition

The above category is said to be an algebraical finite covering category if one has:

(a) any $\mathfrak{S}$-morphism $\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu$ is a noncommutative finite-fold covering,

(b) for all $\lambda \in \Lambda$ is the λ-descent $\mathfrak{desc}_\lambda : K\left(\overline{A}\right) \to K\left(A_\lambda\right)$ is surjective, i.e. $\mathfrak{desc}_\lambda\left(K\left(\overline{A}\right)\right) = K\left(A_\lambda\right)$.

We write

$$\mathfrak{S} \stackrel{\text{def}}{=} \left\{ \left\{A_\lambda\right\}_{\lambda\in\Lambda}, \left\{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\right\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\} \tag{3}$$

Moreover the given above infinite quasi-covering $\left(A, \overline{A}, \widehat{G}\right)$ is said to be a pre-covering of the algebraical finite covering category $\mathfrak{S}$.

It is not clear whether pre-covering of the algebraical finite covering category is always unique. So one needs the following definition.

Definition

Roughly speaking the disconnected infinite noncommutative covering of $\mathfrak{S} = \left\{ \{A_\lambda\}_{\lambda\in\Lambda}, \{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\}$ is the union of all pre-coverings.

Theorem

For any algebraical finite covering category $\mathfrak{S} = \left\{ \{A_\lambda\}_{\lambda\in\Lambda}, \{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\}$ *there is the unique disconnected infinite noncommutative covering.*

Let $\left(A, \overline{A}, \widehat{G}\right)$ be a disconnected infinite noncommutative covering of $\mathfrak{S} = \left\{ \{A_\lambda\}_{\lambda\in\Lambda}, \{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\}$. If $\widetilde{A}$ is a connected component of $\overline{A}$, i.e. $\overline{A} = \widetilde{A} \oplus \widetilde{A}^\perp$, and

$$G\left(\widetilde{A} \,\middle|\, A\right) \stackrel{\text{def}}{=} \left\{ g \in \widehat{G} \middle| \forall \widetilde{a}^\perp \in \widetilde{A}^\perp \quad g\widetilde{a}^\perp = \widetilde{a}^\perp \right\}$$

then there is a natural action

$$G\left(\widetilde{A} \,\middle|\, A\right) \times \widetilde{A} \to \widetilde{A}.$$

Definition

A disconnected infinite noncommutative covering $\left(A, \overline{A}, \widehat{G}\right)$ be of $\mathfrak{S}$ is good if following conditions hold:

(a) if both $\widetilde{A}'$ and $\widetilde{A}''$ are connected components of $\overline{A}$ then there is $g \in \widehat{G}$ such that $g\widetilde{A}' = \widetilde{A}''$,

(b) if $\widetilde{A}$ is a connected component of $\overline{A}$ then for any $\lambda \in \Lambda$ the restriction $h_\lambda|_{\widetilde{A}}$ is an epimorphism, i. e. $h_\lambda\left(G\left(\widetilde{A} \,\middle|\, A\right)\right) = G\left(A_\lambda \mid A\right)$.

Definition

If $\left(A, \overline{A}, \widehat{G}\right)$ is a good disconnected infinite noncommutative covering of $\mathfrak{S} = \left\{ \{A_\lambda\}_{\lambda\in\Lambda}, \{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\}$ then a connected component $\widetilde{A} \subset \overline{A}$ is said to be the inverse noncommutative limit of $\mathfrak{S} = \left\{ \{A_\lambda\}_{\lambda\in\Lambda}, \{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\}$. The group $G\left(\widetilde{A}\;\middle|\;A\right)$ is said to be the covering transformation group. The triple

$$\left(A, \widetilde{A}, G\left(\widetilde{A}\;\middle|\;A\right)\right)$$

is said to be the infinite noncommutative covering or the covering of

$$\mathfrak{S} = \left\{ \{A_\lambda\}_{\lambda\in\Lambda}, \{\pi^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}} \right\}.$$

Theorem

If one has

- *the disconnected* $\overline{p} : \overline{\mathcal{X}} \to \mathcal{X}$ *covering of a covering* $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ *with connected* $\widetilde{\mathcal{X}}$ *and a residually finite covering group* $G\left(\widetilde{\mathcal{X}} \,\middle|\, \mathcal{X}\right)$,
- *the finite covering category* $\mathfrak{S}_p \stackrel{\mathrm{def}}{=} \left\{\{\mathcal{X}_\lambda\}_{\lambda\in\Lambda}, \{p^\mu_\nu : \mathcal{X}_\mu \to \mathcal{X}_\nu\}\right\}$ *of* $p : \widetilde{\mathcal{X}} \to \mathcal{X}$,

then the given by

$$\mathfrak{S}_{C_0(p)} \stackrel{\mathrm{def}}{=} \left\{\{C_0(\mathcal{X}_\lambda)\}_{\lambda\in\Lambda}, \{C_0(p^\mu_\nu) : C_0(\mathcal{X}_\mu) \hookrightarrow C_0(\mathcal{X}_\nu)\}\right\}$$

algebraic finite covering category is good and the triple

$$\left(C_0(\mathcal{X}), C_0\left(\widetilde{\mathcal{X}}\right), G\left(\widetilde{\mathcal{X}} \,\middle|\, \mathcal{X}\right)\right)$$

is the infinite noncommutative covering of $\mathfrak{S}_{C_0(p)}$.

There is Hausdorff blowing-up generalization of this theorem.

Definition

Let A be a connected C^*-algebra, and let $\left(A, \widetilde{A}, G\left(\widetilde{A} \middle| A\right)\right)$ be the infinite noncommutative covering of

$$\mathfrak{S} = \left\{ \left\{A_\lambda\right\}_{\lambda \in \Lambda}, \left\{\mathfrak{lift}^\mu_\nu : A_\mu \hookrightarrow A_\nu\right\}_{\substack{\mu,\nu \in \Lambda \\ \mu \le \nu}} \right\}$$

such that $A = A_{\lambda_{\min}}$. Suppose that $\mathfrak{S}$ contains all classes of isomorphisms of noncommutative finite-fold coverings of A. Then the triple $\left(A, \widetilde{A}, G\left(\widetilde{A} \middle| A\right)\right)$ of $\mathfrak{S}$ is said to be the universal covering of A. The group $G\left(\widetilde{A} \middle| A\right)$ is said to be the fundamental group of A. We use the following notation

$$\pi_1(A) \stackrel{\text{def}}{=} G\left(\widetilde{A} \middle| A\right).$$

Definition

Let P be a property of noncommutative finite-fold coverings. Let A be a C^*-algebra, and let $\left(A, \widetilde{A}, G\left(\widetilde{A}\middle| A\right)\right)$ be the infinite noncommutative covering of

$$\mathfrak{S} = \left\{\{A_\lambda\}_{\lambda\in\Lambda}, \{\mathfrak{lift}^\mu_\nu : A_\mu \hookrightarrow A_\nu\}_{\substack{\mu,\nu\in\Lambda\\ \mu\le\nu}}\right\}$$

such that $A = A_{\lambda_{\min}}$. Suppose that $\mathfrak{S}$ contains all classes of isomorphisms of noncommutative finite-fold coverings of A which possess the property P. Assume that for all $\mu, \nu \in \Lambda$ such that $\mu \le \nu$ the finite-fold noncommutative cornering $\mathfrak{lift}^\mu_\nu : A_\mu \hookrightarrow A_\nu$ possesses the property P. Then the triple $\left(A, \widetilde{A}, G\left(\widetilde{A}\middle| A\right)\right)$ of $\mathfrak{S}$ is said to be the P-universal covering of A. The group $G\left(\widetilde{A}\middle| A\right)$ is said to be the P-fundamental group of A. We use the following notation

$$\pi^P_1(A) \stackrel{\mathrm{def}}{=} G\left(\widetilde{A}\middle| A\right).$$

Let P be a property of noncommutative finite-fold coverings such that

$$\left(A, \widetilde{A}, G\left(\widetilde{A}\;\middle|\;A\right)\right) \in P \quad \Leftrightarrow \quad G\left(\widetilde{A}\;\middle|\;A\right) \quad \text{is an Abelian group}$$

and let $\pi_1^{\mathrm{ab}}(A) \overset{\mathrm{def}}{=} \pi_1^{P}(A)$ be the P-fundamental group. There are two homomorphisms

$$h^{\mathrm{free}}_{K^1} : \pi_1^{\mathrm{ab}}(A)_{\mathrm{free}} \overset{\mathrm{def}}{=} \pi_1^{\mathrm{ab}}(A) / \pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \to \mathrm{Hom}\left(K_1(A), \mathbb{Z}\right),$$
$$h^{\mathrm{tors}}_{K^1} : \pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \to \mathrm{Ext}^1\left(K_0(A), \mathbb{Z}\right).$$

If A is N-algebra then there is an exact sequence

$$0 \to \mathrm{Ext}^1\left(K_0(A), \mathbb{Z}\right) \xrightarrow{\psi} K^1(A) \xrightarrow{\varphi} \mathrm{Hom}(K_1(A), \mathbb{Z})) \to 0.$$

Using the above equations we will prove that under some hypothesis ones has:

1. If A is a N -algebra then the above invariants yield the natural homomorphism
$$h_{K^1} : \pi_1^{\mathrm{ab}}(A) \to K^1(A)$$
2. If $A \cong C(\mathcal{X})$ then h_{K^1} is the topological Hurewicz homomorphism,

Let A be an unital C^*-algebra with unitary element $u \in A$, and let

$$\left(A, \widetilde{A}, \mathbb{Z}_n \cong G\left(\widetilde{A}\middle| A\right), \mathfrak{lift}\right)$$

be an unital noncommutative finite-fold covering. Suppose that there are $u \in A$ and $v \in \widetilde{A}$ such that

$$v^n = u,$$

$$\widetilde{A} \stackrel{\text{def}}{=} \oplus_{j=0}^{n-1} \mathfrak{lift}(A)\, v^j$$

where $\oplus$ means a direct sum of left A-modules. Assume that

$$G\left(\widetilde{A}\middle| A\right) \stackrel{\text{def}}{=} \left\{g \in \operatorname{Aut}\left(\widetilde{A}\right) \middle| \forall a \in \mathfrak{lift}(A) \quad ga = a\right\} \cong \mathbb{Z}_n,$$

$$\forall m \in \mathbb{Z}, \quad \forall \overline{k} \in G\left(\widetilde{A}\middle| A\right) \cong \mathbb{Z}_n, \quad \forall a \in \mathfrak{lift}(A) \quad \overline{k} \cdot (av^m) = av^m e^{\frac{2\pi imk}{n}}.$$

where k is a representative of $\overline{k}$.

Definition

Under the above hypothesis the noncommutative finite-fold covering with unitization $\left(A, \widetilde{A}, \mathbb{Z}_n \cong G\left(\widetilde{A}\middle| A\right), \mathfrak{lift}\right)$ is a (u, v, n)-covering.

Let ϕ_n is a Borel $n^{\rm th}$ root ϕ_n of identity map on the set $\{z \in \mathbb{C} | \, |z| = 1\}$, i.e.

$$(\phi_n)^n = \mathrm{Id}_{\{z \in \mathbb{C} | |z| = 1\}}$$

In particular ϕ_n can be given by

$$\phi_n(\varphi) = e^{\frac{i\varphi}{n}}$$

where $\varphi \in (0, 2\pi]$ is the angular parameter on $\{z \in \mathbb{C} | \, |z| = 1\}$.

Definition

Let A be a C^*-algebra, and let be a property P_{ab} of noncommutative finite-fold coverings such that

$$\left(A, \widetilde{A}, G, \mathfrak{lift}\right) \in P_{\mathrm{ab}} \quad \Leftrightarrow G \text{ is an Abelian group.}$$

then the P_{ab}-fundamental group of A is the Abelian fundamental group. denoted by

$$\pi_1^{ab}(A).$$

Definition

Let A be an unital C^*-algebra with a faithful nondegenerate representation $\pi : A \to B(\mathcal{H})$. An nontrivial element $x \in K_1(A)_{\mathrm{free}} \overset{\mathrm{def}}{=} K_1(A) / K_1(A)_{\mathrm{tors}}$ is admissible if it can be represented by an unitary element $u \in A$ such that there is a set $\{v_n\}_{n \in \mathbb{N}} \subset B(\mathcal{H}) \setminus A$ of unital elements with

$$v_n^n = \pi(u),$$
$$\forall n, l \in \mathbb{N} \quad v_n^l = v_{nl}.$$

and for any $n \in \mathbb{N}$ there is an (u, v_n, n)-covering.

$$\left(A, \widetilde{A}_n, \mathbb{Z}_n \cong G\left(\widetilde{A}_n \middle| A\right), \mathfrak{lift}_n\right).$$

If $V^{\mathrm{adm}} \subset K_1(A)_{\mathrm{free}} \otimes \mathbb{Q}$ is a generated by admissible elements subspace then V^{adm} is isomorphic to the factor-space of $K_1(A)_{\mathrm{free}} \otimes \mathbb{Q}$. Similarly if $K_1^{\mathrm{adm}}(A)_{\mathrm{free}} \stackrel{\mathrm{def}}{=} V^{\mathrm{adm}} \cap K_1(A)_{\mathrm{free}}$ then $K_1^{\mathrm{adm}}(A)$ is isomorphic to a factor-group of a $K_1(A)_{\mathrm{free}}$. Indeed

$$K_1^{\mathrm{adm}}(A) \cong \mathbb{Z}x_1 \oplus ... \oplus \mathbb{Z}x_p$$

where x_j is admissible for any $j \in \{1, ..., p\}$ and there is a (non unique) surjective homomorphism $K_1(A)_{\mathrm{free}} \to \mathbb{Z}x_j$. Any $x \in \{x_1, ..., x_p\}$ can be represented by $u \in A$ satisfying to the above conditions. For any $n \in \mathbb{N}$ let $(A, A_n, \mathbb{Z}_n, \mathfrak{lift}_n)$ For any $n \in \mathbb{N}$ let $(A, A_n, \mathbb{Z}_n, \mathfrak{lift}_n)$ be the required by the above definition unital finite-fold noncommutative covering. If $h_n : \pi_1^{\mathrm{ab}}(A) \to \mathbb{Z}_n$ is the natural homomorphism then any $g \in \pi_1(A)$ yields a character

$$\chi_n^g : \mathbb{Z}x \to \mathcal{U}(1),$$
$$kx \mapsto \frac{h_n(g)\, v_n^k}{v_n} = e^{\frac{2\pi iks}{n}}$$

where and $s \in \mathbb{Z}$ is a representative of $h_n(g) \in \mathbb{Z}_n$.

Moreover one has

$$\forall g, g_2 \in \pi_1(A) \cong \mathbb{Z}^m \quad \chi_n^{g_1+g_2} = \chi_n^{g_1} \chi_n^{g_2}.$$

If $\mathbb{Q}x \overset{\mathrm{def}}{=} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}x$ then there is a character

$$\chi_{\mathbb{Q}}^{g} : \mathbb{Q}x \to \mathcal{U}(1),$$

$$\forall a \in \mathbb{Z} \quad \forall b \in \mathbb{N} \quad \chi_{\mathbb{Q}}^{g}\left(\frac{a}{b}x\right) \overset{\mathrm{def}}{=} \left(\chi_b^{g}(x)\right)^a$$

such that

$$\forall \frac{a}{b} \in \mathbb{Z} \quad \chi_{\mathbb{Q}}^{g}\left(\frac{a}{b}x\right) = 1.$$

The map

$$\phi^{\mathrm{ab}}_{(u_j, v_j, j)} : \pi_1(A) \to \chi(\mathbb{Q}x),$$

$$g \mapsto \chi_{\mathbb{Q}}^{g}$$

is a homomorphism of groups.

For any locally compact Abelian group G one can define its topological dual G^* as a group of continuous characters. For any vector space V over field K there is algebraic dual space V' of K-linear functionals.

Theorem

Let K be a non-discrete locally compact field, and V a left vector-space of finite dimension n over K; let χ be a non-trivial character of the additive group of K. Then the topological dual V^* of V is a right vector-space of dimension n over K; the formula

$$\langle v, v^* \rangle_V = \chi\left([v, v']\right)$$

defines a bijective mapping $v' \mapsto v^*$ of the algebraic dual V' of V onto V^*.

From the Theorem it turns out that if we consider the standard character

$$\chi_{\text{standard}} : \mathbb{R} \to \mathcal{U}(1), \quad x \mapsto e^{2\pi i x}$$

then since $\mathbb{R}$ is a locally compact field any character $\chi : \mathbb{R} \to U(1)$ uniquely defines a functional $f_{\mathbb{R}} : \mathbb{R} \to \mathbb{R}$ with

$$\chi = \chi_{\text{standard}} \circ f_{\mathbb{R}}.$$

In particular the explained above character $\chi_{\mathbb{Q}}^{q} : \mathbb{Q}x \to U(1)$ is continuous, so it can be uniquely extended up to the character $\chi_{\mathbb{R}}^{q} : \mathbb{R}x \stackrel{\text{def}}{=} \mathbb{Q}x \otimes_{\mathbb{Q}} \mathbb{R} \to U(1)$. There is the unique functional such $f_{\mathbb{R}}^{g} : \mathbb{R}x \to \mathbb{R}$ such that $\chi_{\mathbb{R}}^{g} = \chi_{\text{standard}} \circ f_{\mathbb{R}}^{g}$. From the our construction it turns out that $\chi_{\mathbb{R}}^{q}(\mathbb{Z}) = \{1\}$, so $f_{g}^{\mathbb{R}}(\mathbb{Z}x) \subset \mathbb{Z}$ and the functional $f_{\mathbb{R}}^{g}$ yields a homomorphism $\phi_{g} \in \mathrm{Hom}(\mathbb{Z}x, \mathbb{Z})$.

From the direct sum $K_1^{\mathrm{adm}}(A) \cong \mathbb{Z}x_1 \oplus ... \oplus \mathbb{Z}x_p$ one can deduce a non unique direct sum $K_1(A)_{\mathrm{free}} \cong \mathbb{Z}x_1 \oplus ... \oplus \mathbb{Z}x_p \oplus K_1^{\perp}(A)_{\mathrm{free}}$ Using it one can construct a homomorphism

$$f_x^g : K_1(A)_{\mathrm{free}} \to \mathbb{Z}$$

and from the above equations it follows that f_x^g linearly depends on g, i.e.

$$\forall g, g_2 \in \pi_1^{\mathrm{ab}}(A) \cong \mathbb{Z}^m \quad f_x^{g_1+g_2} = f_x^{g_1} + f_x^{g_2}.$$

The formula

$$f^g = f_{x_1}^g + ... + f_{x_p}^g$$

yields an element of $\mathrm{Hom}(K_1(A)_{\mathrm{free}}, \mathbb{Z})$.

In result one has a group homomorphism

$$h^{\text{free}}_{K^1} : \pi_1 (A) \to \mathrm{Hom} (K_1 (A) , \mathbb{Z}) ,$$
$$g \mapsto f^g$$

Definition

The homomorphism $h^{\text{free}}_{K^1}$ is the free noncommutative Hurewicz homomorphism.

Let G be a finite Abelian group, and let

$$\left(A, \widetilde{A}, G = G\left(\widetilde{A} \middle| A\right), \mathfrak{lift}\right)$$

be an unital finite-fold covering. Consider a category of finitely generated projective $\widetilde{A}$ - G-modules, i.e. $\widetilde{A}$-modules with equivariant action of G. According to the well known result this category Morita equivalent to both:

- Category of finitely-generated projective $\widetilde{A} \rtimes G$-modules where $\widetilde{A} \rtimes G$ is a crossed product.
- Category of finitely-generated projective A-modules.

So there are natural isomorphisms

$$K_0^G\left(\widetilde{A}\right) \cong K_0\left(\widetilde{A} \rtimes G\right) \cong K_0\left(A\right).$$

If Q is a projective finitely generated $\widetilde{A}$ - G-module and $Q^G \overset{\text{def}}{=} \{q \in Q \,|\forall g \in G \quad gq = q\}$ then there is a natural direct sum

$$Q = Q^G \oplus Q^{\perp}$$

since any $q \in Q$ equals to the sum $q^G + q^{\perp}$ where

$$q^G \overset{\text{def}}{=} \frac{1}{|G|} \sum_{g \in G} gq \in Q^G,$$

$$q^{\perp} \overset{\text{def}}{=} q - q^G \in P^{\perp}.$$

Similarly if $r : G \to U(1)$ is an irreducible representation then $Q^{\perp} = Q_r \oplus Q_r^{\perp}$ since any $p \in P^{\perp}$ equals to the sum $q_r + q_r^{\perp}$ where

$$q_r \overset{\text{def}}{=} \frac{1}{|\ker r|} \sum_{g \in \ker r} gq$$

$$q_r^{\perp} \overset{\text{def}}{=} q - q_r$$

It follows that any projective finitely generated $\widetilde{A}$-G-module Q is represented by direct sum

$$Q = Q^G \bigoplus \left(\bigoplus_{r \in R} Q_r \right)$$

where R is a set of irreducible representations of G. It turns out that

$$K_0^G\left(\widetilde{A}\right) = \left(K_0^G\left(\widetilde{A}\right)\right)^G \bigoplus \left(\bigoplus_{r \in R} K_0^G\left(\widetilde{A}\right)_r \right)$$

For any $r \in R$ there is a prime number $p_r \in \mathbb{N}$ such that $\mathrm{im}\ r = e^{\frac{2\pi i \mathbb{Z}}{p_r}}$. There is $g \in G$ with

$$r\left(g\right) = e^{\frac{2\pi i}{p_r}},$$
$$\forall r' \in R \setminus \{r\} \quad r'(g) = 1.$$

If $x_1, x_2 \in K_0^G\left(\widetilde{A}\right)_r$ are such that $\chi_{x_1}(g) = \chi_{x_2}(g) = e^{\frac{2\pi ik}{p_r}}$ with $k \in \mathbb{N}$ then one has

$$\forall g \in G \quad \chi_{x_1 - x_2}(g) = \{1\} \quad x_1 - x_2 \in \left(K_0^G\left(\widetilde{A}\right)\right)^G$$

it is possible if and only if $x_1 - x_2 = 0$. From our construction there is an isomorphism

$$\phi_r : \mathbb{Z}_{p_r} \cong K_0^G\left(\widetilde{A}\right)_r$$

such that

$$\forall \overline{k} \in \mathbb{Z}_{p_r} \quad \chi_{\phi_r\left(\overline{k}\right)}(g) = e^{\frac{2\pi ik}{p_r}}$$

where $k \in \mathbb{Z}$ is a representative of $\overline{k}$. Moreover any $g \in G$ yield a character

$$\chi_r : K_0^G\left(\widetilde{A}\right)_r \to U(1). \tag{4}$$

Following Lemma is a consequence of the above construction and the isomorphism .

Lemma

If R is a set of irreducible representations of G then there is a decomposition

$$K_0(A) = K_0(A)^{\perp} \bigoplus \left(\bigoplus_{r \in R} K_0(A)_r \right).$$

If $\mathrm{im}\ r = e^{\frac{2\pi i \mathbb{Z}}{p r}}$ *then* $K_0(A)_r$ *is trivial, or there is an isomorphism* $K_0(A)_r \cong \mathbb{Z}_{p^r}$.

The decomposition of the lemma yield a map from G to the set of characters of $K_0(A)$

$$g \mapsto \left(x^{\perp} + \sum_{r \in \mathbb{R}} x_r \mapsto \prod_{r \in R} \chi_r(x_r) \right)$$

From

$$\forall g', g'' \in G \quad \chi_r(g') \chi_r(g'') = \chi_{P_j}(g' g'').$$

Using it one can construct a homomorphism

$$h^{\mathrm{tors}}_{K^1} : G \to \mathrm{Ext}^1_{\mathbb{Z}}(K_0(A), \mathbb{Z})$$

If A belongs to class N then one has a homomorphism

$$h^{\mathrm{tors}}_{K^1} : G \to K^1(A)$$

Definition

The above map is the torsion of noncommutative Hurewicz homomorphism.

Definition

An unital C^*-algebra A admits Hurewicz homomorphism if one has:

(a) All Abelian groups $\pi_1^{\mathrm{ab}}(A)$, $K_0(A)$ and $K_1(A)$ are finitely generated.

(b) If $\pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \subset \pi_1^{\mathrm{ab}}(A)$ is the torsion subgroup then there an unital finite-fold noncommutative covering $\left(A, \widetilde{A}, G\left(\widetilde{A}\middle| A\right), \mathfrak{lift}\right)$ such that the composition $\pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \hookrightarrow \pi_1(A) \to G\left(\widetilde{A}\middle| A\right)$ is isomorphism.

If an unital C^*-algebra A admits Hurewicz homomorphism then $\pi_1^{\mathrm{ab}}(A)$ is the direct sum of groups

$$\begin{split}\pi_1^{\mathrm{ab}}(A) \cong \pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \oplus \pi_1^{\mathrm{ab}}(A) / \pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \cong \\ G\left(\widetilde{A}\middle| A\right) \oplus \pi_1^{\mathrm{ab}}\left(\widetilde{A}\right) \cong \\ \cong \pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \oplus \pi_1^{\mathrm{ab}}(A)_{\mathrm{free}}\,.\end{split} \tag{5}$$

There are the free and the torsion Hurewicz homomorphisms

$$h^{\mathrm{free}}_{K^1} : \pi_1^{\mathrm{ab}}\left(\widetilde{A}\right) \to \mathrm{Hom}\left(K_1\left(\widetilde{A}\right), \mathbb{Z}\right),$$
$$h^{\mathrm{tors}}_{K^1} : G\left(\widetilde{A}\middle| A\right) \cong \pi_1^{\mathrm{ab}}\left(A\right)_{\mathrm{tors}} \to \mathrm{Ext}^1_{\mathbb{Z}}\left(K_0\left(\widetilde{A}, \mathbb{Z}\right)\right).$$

The inclusion yields a homomorphism $\iota : K_0\left(A\right) \to K_0\left(\widetilde{A}\right)$ so there are homomorphisms

$$r_1 : \mathrm{Hom}\left(K_1\left(\widetilde{A}\right), \mathbb{Z}\right) \to \mathrm{Hom}\left(K_1\left(A\right), \mathbb{Z}\right),$$
$$r_2 : \mathrm{Ext}^1_{\mathbb{Z}}\left(K_0\left(\widetilde{A}\right), \mathbb{Z}\right) \to \mathrm{Ext}^1_{\mathbb{Z}}\left(K_0\left(A\right), \mathbb{Z}\right),$$

On the other hand there are subjective homomorphism $s_1 : \pi_1^{\mathrm{ab}}\left(A\right) \to G\left(\widetilde{A}\middle| A\right) = \pi_1^{\mathrm{ab}}\left(A\right)_{\mathrm{tors}}$ and $s_2 : \pi_1^{\mathrm{ab}}\left(A\right) \to \pi_1^{\mathrm{ab}}\left(\widetilde{A}\right)$.

Definition

If A admits Hurewicz homomorphism then a pair of homomorphisms

$$\begin{aligned} h^1_{K^1} &\stackrel{\mathrm{def}}{=} r_1 \circ s_1 : \pi_1^{\mathrm{ab}}(A) \to \mathrm{Hom}\left(K_1(A), \mathbb{Z}\right), \\ h^2_{K^1} &\stackrel{\mathrm{def}}{=} r_2 \circ s_2 : \pi_1^{\mathrm{ab}}(A) \to \mathrm{Ext}^1_{\mathbb{Z}}\left(K_0(A), \mathbb{Z}\right) \end{aligned} \tag{6}$$

is the Hurewicz pair.

Definition

If A is an N-algebra then both direct sum and exact sequence yield the following diagram

$$\begin{array}{ccccc}
\pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} & \longrightarrow & \pi_1^{\mathrm{ab}}(A)_{\mathrm{tors}} \oplus \pi_1^{\mathrm{ab}}(A)_{\mathrm{free}} & \longrightarrow & \pi_1^{\mathrm{ab}}(A)_{\mathrm{free}} \\
\downarrow h^1_{K^1} & & \downarrow h^A_{K^1} \stackrel{\mathrm{def}}{=} h^1_{K^1} + h^2_{K^1} & & \downarrow h^2_{K^1} \\
\mathrm{Ext}^1(K_0(A), \mathbb{Z}) & \xrightarrow{\psi} & K^1(A) & \xrightarrow{\varphi} & \mathrm{Hom}(K_1(A), \mathbb{Z}))
\end{array}$$

So there is the unital Hurewicz homomorphism given by

$$h^A_{K^1} \stackrel{\mathrm{def}}{=} h^1_{K^1} + h^2_{K^1} : \pi_1^{\mathrm{ab}}(A) \to K^1(A).$$

Hurewicz homomorphism for commutative C^*-algebras

If $\widetilde{\mathcal{X}} \to \mathcal{X}$ is an universal covering then the Hurewicz homomorphism looks like

$$h^{C(\mathcal{X})}_{K^1} : G\left(\widetilde{\mathcal{X}}\middle|\,\mathcal{X}\right) \to K^1\left(C\left(\mathcal{X}\right)\right)$$

If $\widetilde{\mathcal{X}}$ is not path connected then it is possible that $\pi_1\left(\mathcal{X}, x_0\right)$ is trivial but $G\left(\widetilde{\mathcal{X}}\middle|\,\mathcal{X}\right)$ is not trivial the Hurewicz homomorphism of C^*-algebras is more informative. There is the weak fundamental group $\pi_1^{\mathrm{w}}\left(\mathcal{X}, x_0\right)$ such that $\pi_1^{\mathrm{w}}\left(\mathcal{X}, x_0\right) \cong \pi_1\left(\mathcal{X}, x_0\right)$ is $\mathcal{X}$ is path connected a semilocally 1-connected. However it is possible that $\pi_1\left(\mathcal{X}, x_0\right)$ is trivial but $\pi_1^{\mathrm{w}}\left(\mathcal{X}, x_0\right)$ is not trivial. Moreover for any Abelian group A one can define a Hurewicz homomorphism

$$\pi_1^{\mathrm{w}}\left(\mathcal{X}, x_0\right) \to \check{H}_1\left(\mathcal{X}, A\right)$$

to Čech homology. Above homomorphism have new early unknown type. There are examples nontrivial homomorphisms with trivial $\pi_1\left(\mathcal{X}, x_0\right)$.

Let $\mathcal{X}$ be a compact, connected topological space such that:

- The groups $\pi_1^{\mathrm{ab}}(\mathcal{X}, x_0)$, $K_0(C(\mathcal{X})) \cong K^0(\mathcal{X})$ and $K_1(C(\mathcal{X})) \cong K^1(\mathcal{X}))$ are finitely generated Abelian groups,
- There is the universal covering $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ with the natural isomorphism $\pi_1(\mathcal{X}, x_0) \cong G\left(\widetilde{\mathcal{X}} \middle| \mathcal{X}\right)$.

The (classical) Hurewicz homomorphism $h^{\text{sing}} : \pi_1(\mathcal{X}, x_0) \to H_1(\mathcal{X})$ into singular homology is an isomorphism. If $\omega : (S^1, s_0) \to (\mathcal{X}, x_0)$ represents an element $[\omega] \in \pi_1(\mathcal{X}, x_0)$ is such that

$$\pi_1(\mathcal{X}, x_0) = \mathbb{Z}[\omega] \oplus \pi_1(\mathcal{X}, x_0)^{\perp}.$$

There is a surjective homomorphism $\phi_H : H_1(\mathcal{X}) \to \mathbb{Z}$ with $\phi_H(\mathbb{Z}y) = \mathbb{Z}$ and $\phi_H\left(H_1(\mathcal{X})^{\perp}\right) = \{0\}$. This homomorphism yields an element $z \in H^1(\mathcal{X}, \mathbb{Z})$ with $H^1(\mathcal{X}, \mathbb{Z}) = \mathbb{Z}z \oplus H^1(\mathcal{X}, \mathbb{Z})^{\perp}$. There is a representative $\varphi_z : \mathcal{X} \to K(\mathbb{Z}, n) = S^1$ of z. The composition $\varphi_z \circ \omega : S^1 \to S^1$ yields an isomorphism of cohomology of S^1 so it is a homotopy equivalence. It follows that there is a surjective homomorphism

$$\pi_1(\varphi_z) : \pi_1(\mathcal{X}, x_0) \to \pi_1(S^1, s_0) \cong \mathbb{Z}.$$

Free case

For any $n \in \mathbb{N}$ thee is a there are a finite index subgroup, topological space and two transitive coverings given by

$$\begin{aligned} H_n &\overset{\text{def}}{=} \pi_1^{-1}(\varphi_z)\left(n\pi_1\left(S^1, s_0\right)\right) \\ \widetilde{\mathcal{X}}_n &\overset{\text{def}}{=} \widetilde{\mathcal{X}}/H_n, \\ \widetilde{p}_n &: \widetilde{\mathcal{X}} \to \widetilde{\mathcal{X}}_n, \\ p_n &: \widetilde{\mathcal{X}}_n \to \mathcal{X}. \end{aligned} \tag{7}$$

Since $S^1 \cong U(1)$ the map φ_z yields an unitary element $u \in U(C(\mathcal{X}))$. If $\pi : C(\mathcal{X}) \to B(\mathcal{H}_a)$ be an atomic representation and ϕ_n is given is defined above then for any $n > 1$ there is a generally discontinuous map $v_n \overset{\text{def}}{=} \phi_n \circ \varphi_z : \mathcal{X} \to U(1) \cong S^1$ which can be regarded as an element of $B(\mathcal{H}_a)$. If v_n is continuous map then v_n represents an element $z_n \in H^1(\mathcal{X}, \mathbb{Z})$ with $nz_n = z$. It is impossible since x is not divisible, so $v_n \notin C(\mathcal{X})$. It follows that

$$v_n^n = \pi(u),$$

$$\forall n, l \in \mathbb{N} \quad v_n^l = v_{nl}.$$

If $\widetilde{A}_n$ is a C^*-subalgebra of $B(\mathcal{H}_a)$ generated by the union $C(\mathcal{X}) \cup \{v_n\}$ then $\widetilde{A}$ is a subalgebra of maps from $\mathcal{X} \to \mathbb{C}$, so it is commutative, so from the Theorem Gelfand theorem it turns out that $\widetilde{A}_n \cong C\left(\widetilde{\mathcal{X}}'_n\right)$. Moreover

$$\begin{aligned} v^n &= u, \\ C\left(\widetilde{\mathcal{X}}'_n\right) &= \oplus_{j=0}^{n-1} \pi\left(C\left(\mathcal{X}\right)\right) v^j \end{aligned} \tag{8}$$

where $\oplus$ means a direct sum of left A-modules, i.e. there is an (u, v_n, n)-covering

$$\left(C\left(\mathcal{X}\right), C\left(\widetilde{\mathcal{X}}'_n\right), \mathbb{Z}_n \cong G\left(C\left(\widetilde{\mathcal{X}}'_n\right) \middle| C\left(\mathcal{X}\right)\right), C\left(p'_n\right)\right)$$

where $p'_n : \widetilde{\mathcal{X}}'_n \to \mathcal{X}$ is a covering induced by an inclusion $C\left(\mathcal{X}\right) \hookrightarrow C\left(\widetilde{\mathcal{X}}'_n\right)$.

From $v^n = u$ it turns out that

$$\pi_1\left(\varphi_z \circ p'_n\right)\left(\pi_1\left(\widetilde{\mathcal{X}}'_n\right)\right) = n\pi_1\left(\mathcal{X}, x_0\right) = \pi_1\left(\varphi_z \circ p_n\right)\left(\pi_1\left(\widetilde{\mathcal{X}}_n\right)\right)$$

and from the above equation it follows that the covering $p'_n : \widetilde{\mathcal{X}}'_n \to \mathcal{X}$ is equivalent to the $p_n : \widetilde{\mathcal{X}}_n \to \mathcal{X}$ one. If u represents a nonzero element $[u] \in K_1\left(C\left(\mathcal{X}\right)\right)$ then from the above equations it turns out that $[u]$ is admissible. For any $n \in \mathbb{N}$ the specialization of the character explained in general theory character

$$\chi_n^{[\omega]} : \mathbb{Z}\left[u\right] \to \mathcal{U}\left(1\right),$$
$$k\left[u\right] \mapsto e^{\frac{2\pi i k}{n}}.$$

and from the above equation it follows that the covering $p'_n : \widetilde{\mathcal{X}}'_n \to \mathcal{X}$ is equivalent to the $p_n : \widetilde{\mathcal{X}}_n \to \mathcal{X}$ one. Clearly a set $\left\{v_n\right\}_{n\in\mathbb{N}}$ satisfies to the conditions of the definition of admissible element, i.e. u is admissible.

Here we drop analogs manipulations below the equation and obtain a specialization

$$f_{[u]}^{[\omega]} : K_1\left(A\right)_{\mathrm{free}} \to \mathbb{Z}$$

of the given by the equation free Hurewicz homomorphism, i,e. $h_{K^1}^{\mathrm{free}}$ maps ω onto the image of $f_{[u]}^{[\omega]}$ Using this fact one can prove that free part of classical free Hurewicz homomorphism coincides with noncommutative one,

Torsion case

Let $p \in \mathbb{N}$ be a prime number and $\mathcal{X}$ is path connected and $\omega : \left(S^1, s_0\right) \to (\mathcal{X}, x_0)$ is a representative of an element $[\omega] \in \pi_1(\mathcal{X}, x_0)$ with $p[\omega] = 0$. Suppose that there is a p-listed covering $\theta_p : \widetilde{\mathcal{X}} \to \mathcal{X}$ such that the composition

$$\mathbb{Z}[\omega] \cong \mathbb{Z}_p \to \pi_1(\mathcal{X}, x_0) \to G\left(\left.\widetilde{\mathcal{X}}\right| \mathcal{X}\right)$$

is isomorphism of Abelian groups. It turns out that the composition $\omega \circ \theta_p$ represents a trivial element $[\omega \circ \theta_p] = p[\omega] \in \pi_1(\mathcal{X}, x_0)$. So there is a homotopy $\Phi : S^1 \times [0,1] \to \mathcal{X}$ with

$$\begin{aligned} \forall s \in S^1 \quad \Phi(s, 0) = \omega \circ \theta_p(s); \\ \Phi(s, 1) = x_0, \\ \forall t \in [0,1] \quad \Phi(x_0, t) = x_0 \end{aligned}$$

Let $C'\omega$ be the mapping cone defined by the following way:

- there is a mapping cylinder M_{θ_p} (cf. Definition **??**),
- $C'\omega \overset{\text{def}}{=} M_{\theta_p}/j\left(S^1\right)$.

then map Φ yields a composition

$$S^1 \to C'\omega \to \mathcal{X}.$$

If m_0 corresponds to a base point of $C'\omega$ then there is a decomposition

$$S^1 \to C'\omega \setminus \{m_0\} \to \mathcal{X}.$$

It follows that one has

$$K^1\left(C\left(S^1\right)\right) \to K^1\left(C\left(C'\omega \setminus \{m_0\}\right)\right) \to K^1\left(C\left(\mathcal{X}\right)\right)$$

On the other hand $\left(C\left(C'\omega \setminus \{m_0\}\right)\right.$ is the mapping cone $C_{C(\theta_p)}$ of the homomorphism $C\left(\theta_p\right) : C\left(S^1\right) \hookrightarrow C\left(S^1\right)$. There is a following exact sequence

$$0 \to SC\left(S^1\right) \xrightarrow{\iota} C_{C(\theta_p)} \xrightarrow{P} C\left(S^1\right) \to 0.$$

From the Puppe sequences

$$KK\left(SC\left(S^1\right),SC\left(S^1\right)\right)\xrightarrow{KK\left(\mathrm{Id}_{SC(S^1)},SC(\theta_p)\right)}KK\left(SC\left(S^1\right),SC\left(S^1\right)\right)$$
$$\xrightarrow{KK\left(\mathrm{Id}_{SC(S^1)},\iota\right)}KK\left(SC\left(S^1\right),C\left(C_{\theta_p}\right)\right)\xrightarrow{KK\left(\mathrm{Id}_{SC(S^1)},P\right)}KK\left(SC\left(S^1\right),C\left(S^1\right)\right)$$
$$\xrightarrow{KK\left(\mathrm{Id}_{SC(S^1)},C(\theta_p)\right)}KK\left(SC\left(S^1\right),C\left(S^1\right)\right);$$

$$KK\left(C\left(S^1\right),C\left(S^1\right)\right)\xrightarrow{KK\left(C(\theta_p),\mathrm{Id}_{C(S^1)}\right)}KK\left(C\left(S^1\right),C\left(S^1\right)\right)\xrightarrow{KK\left(P,\mathrm{Id}_{C(S^1)}\right)}$$
$$\to KK\left(C_\phi,C\left(S^1\right)\right)\xrightarrow{KK\left(\iota,\mathrm{Id}_{C(S^1)}\right)}KK\left(SC\left(S^1\right),C\left(S^1\right)\right)$$
$$\xrightarrow{KK\left(SC(\theta_p),\mathrm{Id}_{C(S^1)}\right)}KK\left(SC\left(S^1\right),C\left(S^1\right)\right).$$

it follows that

$$K_0\left(SC\left(S^1\right)\right) \xrightarrow{K_0(SC(\theta_p))} K_0\left(SC\left(S^1\right)\right) \xrightarrow{K_0(\iota)} K_0\left(C\left(C_{\theta_p}\right)\right) \xrightarrow{K_0(p)} K_0\left(C\left(S^1\right)\right) \xrightarrow{K_0(C(\theta_p))} K_0\left(C\left(S^1\right)\right),$$

$$K^1\left(C\left(S^1\right)\right) \xrightarrow{K^1(C(\theta_p))} K^1\left(C\left(S^1\right)\right) \xrightarrow{K^1(\iota)} K^1\left(C\left(C_{\theta_p}\right)\right) \xrightarrow{K^1(p)} K^1\left(SC\left(S^1\right)\right) \xrightarrow{K^1(SC(\theta_p))} K^1\left(SC\left(S^1\right)\right),$$

So one has

$$K_0\left(C\left(C_{\theta_p}\right)\right) \cong K^1\left(C\left(C_{\theta_p}\right)\right) \cong \mathbb{Z}_p.$$

The decomposition

$$S^1 \to C'\omega \setminus \{m_0\} \to \mathcal{X}$$

yields the following homomorphisms

$$K^1\left(C\left(S^1\right)\right) \to K\left(C_0\left(C'\omega \setminus \{m_0\}\right)\right) \to K^1\left(C\left(\mathcal{X}\right)\right)$$

Using the above homomorphisms one can prove the coincidence of classical and noncommutative Hurewicz homomorphism.