Ординальные инварианты гомоморфных предпорядков k-размеченных лесов

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WPOs: examples

WPO (well partial order) is a partial order that has neither infinite descending chains nor infinite antichains. Terminology applies to preorders (WQOs), meaning the corresponding quotient orders. WPOs appear in several fields of mathematics and informatics and are often useful. Some examples:

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1. If Q is WQO then $(Q^*; \leqslant^*)$ is WQO where Q^* is the set of finite sequences in Q and $(x_1, \ldots, x_m) \leqslant^* (y_1, \ldots, y_n)$ means that for some strictly increasing $\varphi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ we have $x_i \leqslant y_{\varphi(i)}$ (Higman).
2. If Q is WQO then $(\mathcal{T}_Q; \leqslant_h)$ is WQO where $\mathcal{T}_Q$ is the set of finite Q-labeled trees, and $(T, c) \leqslant_h (S, d)$ if there is a monotone function $\varphi : (T, \sqsubseteq) \to (S, \sqsubseteq)$ such that $c(t) \leqslant d(\varphi(t))$ (S. after Kruskal).
3. The finite graphs with the graph-minor relation (Robertson-Seymour).
4. The countable linear orders with the embeddability relation (Laver).

WPOs: complexity measures

1. Computational complexity.
Natural countable WQOs P are often computably presentable w.r.t. their natural encodings. What is the computational complexity of these presentations? In particular, which P are PTIME presentable w.r.t. their natural encodings?

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2. Logical complexity.
Logical approach to classifying WPOs P consists in estimating the complexity of first-order theory of P or in biinterpreting P with a well known structure (say, with the standard model of first-order arithmetic). This is closely related to a good understanding of the definable predicates in P.

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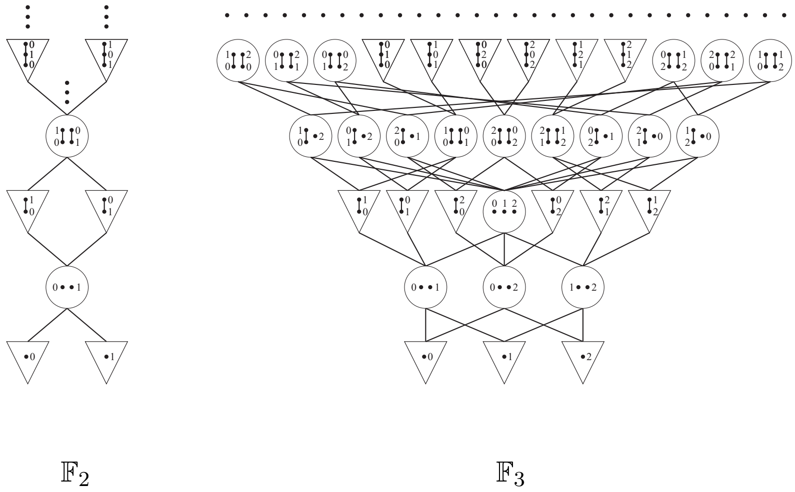
3. Combinatorial complexity.
The idea is to measure the complexity of $(P; \leqslant)$ by natural ordinals ("ordinal invariants") which reflect some aspects of P. These include *height* $h(P)$, *width* $w(P)$, and *maximal ordinal type* $o(P)$. The latter is the supremum of order types of linear orders (equivalently, well orders) on P that extend $\leqslant$.

h-Preorders of k-labeled forests



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In this work, we describe the distinction in combinatorial complexity, by calculating the three ordinal invariants of $\mathbb{F}_k$ for each $k \geqslant 3$. For $k = 2$ the invariants are obvious. The height of $\mathbb{F}_k$ for each $k \geqslant 2$ is known to be ω, hence it remains to calculate the width and the maximal order type for each $k \geqslant 3$.

Main rezult

THEOREM. For any integer $k \geqslant 3$ we have:
$o(\mathbb{F}_k) = w(\mathbb{F}_k) = \varphi_{k-1}(0)$, where $\{\varphi_n\}$ is the sequence of Veblen's unary functions on ordinals defined by induction as follows:
$\varphi_0(\alpha) = \omega^\alpha$, and $\varphi_{n+1}(\alpha)$ is the α-th fixed point of φ_n.

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This result fits well to the popular programme of WQO-theory which aims to calculate the ordinal invariants of naturally arising WPOs. This task is often non-trivial and requires technically involved proofs and notions. In our case, the ordinal invariants distinguish all structures $\mathbb{F}_k$ for different $k \geqslant 2$ by their combinatorial complexity, in contrast with the computational and logical complexity.

Future work

There are several natural and useful variations of h-preorders on the labeled trees (and forests) for which the ordinal invariants remain unknown.
In particular, this is the case for the preorder $(\mathcal{T}(Q); \leqslant_h)$, where Q is a finite WPO which is not an antichain. We do not know the answer even for the particular case $Q = (P(X); \subseteq)$ where X is a finite set with at least 3 elements.

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Another interesting series of WQOs is obtained by iterating the construction $Q \mapsto \mathcal{T}(Q)$ starting with $Q = \bar{k}$. Namely, we define the sequence $\{\mathcal{T}_m(\bar{k})\}_{m<\omega}$ of h-preorders as follows: $\mathcal{T}_0(\bar{k}) = \bar{k}$ and $\mathcal{T}_{m+1}(\bar{k}) = \mathcal{T}(\mathcal{T}_m(\bar{k}))$. The ω-th iterate $\mathcal{T}_\omega(\bar{k})$ and the h-preorder on it may also be naturally defined. For such iterates (with the exception of the zero and first iterates), and also for the forest variants of them, all three ordinal invariants are currently not known.

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Finally, all the mentioned variations of h-preorders have important extensions to the infinite well founded trees and forests which are intimately related to the Wadge hierarchy and its extensions (S., Kihara-Montalban).

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Among these variations, the simplest are the structures of countable k-labeled forests introduced in S.2007. We hope that the ideas and methods of this paper could help to compute the ordinal invariants of these structures (so far, only the height was computed in S.2007). For the other variations, the problem is widely open and looks challenging.

Preliminary results were obtained by us recently on some of the mentioned questions.

Maximal order type

DEFINITION. Maximal order type of $(P, \leqslant)$ is the supremum of order types of linear orders on P that extend $\leqslant$.

DEFINITION. $L_Q(x) = \{y \mid x \nleqslant y\}$, $l_Q(x) = o(L_Q(x))$

THEOREM (de Jongh-Parikh). The supremum is attained.

COROLLARY (de Jongh-Parikh).
$o(Q) \leqslant \alpha \iff \forall x \in Q(l_Q(x) < \alpha)$

The following is also due to de Jongh and Parikh.

1. $o(A \sqcup B) = o(A) \oplus o(B)$
2. $o(A \cup B) \leqslant o(A) \oplus o(B)$

Upper bound: finite subsets

If A is WQO then $(\mathcal{S}(A), \leqslant)$ is WQO where $\mathcal{S}(A)$ is the set of finite subsets of A and $X \leqslant Y \iff \forall a \in X \exists b \in Y (a \leqslant b)$.

It is easy to see that $\mathbb{F}_k(A) \cong \mathcal{S}(\mathcal{T}_k(A))$.

LEMMA 1. 1. If $o(A) \leqslant \alpha$, then $o(\mathcal{S}(A)) \leqslant \omega^\alpha$.
2. If $X \in \mathcal{S}(A)$ and $\forall x \in X (l_A(x) < \alpha)$, then $l_{\mathcal{S}(A)}(X) < \omega^\alpha$.

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PROOF. We argue by induction on α. (2) and Corollary imply (1), hence it suffices to prove (2).

For $X = \varnothing$, we have $l(X) = 0 < \omega^\alpha$.

For $X = \{x\}$, we have $L(X) = \mathcal{S}(L_A(x))$. Since $o(L_A(x)) = l_A(x) < \alpha$, we get by induction $l(X) = o(\mathcal{S}(L_A(x)) \leqslant \omega^{l_A(x)} < \omega^\alpha$.

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For $X = \{x_1, \dots, x_n\}$, we have $L(X) = L(\{x_1\}) \cup \dots \cup L(\{x_n\})$. So $l(X) \leqslant l(\{x_1\}) \oplus \dots \oplus l(\{x_n\}) = (< \omega^\alpha) \oplus \dots \oplus (< \omega^\alpha) < \omega^\alpha$.

Trees with restrictions

Let $f \colon \overline{k} \to P(\mathcal{S}(\mathcal{T}_m)) \cup \{\perp\}$ and let $f(i) = \perp$.
We define the set $\mathcal{T}^i(f)$ of $\max(k,m)$-labeled trees with the following restrictions:
1. Root label is distinct from i.
2. If all predecessors of some vertex have labels in $f^{-1}(\perp)$, then the label x of this vertex is in $\overline{k}$.
3. If also $f(x) \neq \perp$, then the successors of this vertex form a set from $f(x)$.

For example, if we have a tree $a = mb$, and we want to have a set of trees t with restriction $a \not\leq t$, we may take $f(m) = L(b)$.

Upper bound: main lemma

LEMMA 2. $o(\mathcal{T}^i(f)) \leqslant \varphi_{|f^{-1}(\perp)|-1}\left(\bigoplus_{f(x)\neq\perp} o(f(x)) + 1\right) = \alpha$.

PROOF. Induction on α. It suffices to show that $l(a) < \alpha$ for each $a = mb \in \mathcal{T}^i(f)$. Induction on a.

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If $f(m) = \perp$, then

$$L(a) \subseteq \mathcal{T}^i\left(f + \left(m \mapsto L_{\mathcal{S}(\mathcal{T}^m(f))}(b)\right)\right)$$

By Lemma 1 and induction, $l_{\mathcal{S}(\mathcal{T}^m(f))}(b) < \omega^\alpha = \alpha$. Then

$$l(a) \leqslant \varphi_{|f^{-1}(\perp)|-2}\left(l_{\mathcal{S}(\mathcal{T}^m(f))}(b) \oplus \bigoplus_{f(x)\neq\perp} o(f(x)) + 1\right)$$
$$= \varphi_{|f^{-1}(\perp)|-2}((<\alpha) \oplus (<\alpha)) < \alpha$$

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Since $l_{f(m)}(b) < o(f(m))$, we have

$$l(a) \leqslant \varphi_{|f^{-1}(\perp)|-1}\left(l_{f(m)}(b) \oplus \bigoplus_{\substack{x\neq m\\ f(x)\neq\perp}} o(f(x)) + 1\right)$$

$$= \varphi_{|f^{-1}(\perp)|-1}\left(< \bigoplus_{f(x)\neq\perp} o(f(x)) + 1\right) < \alpha$$

Upper bound

THEOREM. $o(\mathcal{T}_k) \leqslant \varphi_{k-1}(0)$.

PROOF. We will show $l(a) \leqslant \varphi_{k-1}(0)$ by induction on $a = mb$.

$$L(a) = \bigcup_{i \in \overline{k} \setminus \{m\}} \mathcal{T}^i \left(m \mapsto L_{\mathcal{S}(\mathcal{T}_k)}(b) \right)$$

By Lemma 1 and induction, $l(b) < \omega^{\varphi_{k-1}(0)} = \varphi_{k-1}(0)$. Then

$$l(a) \leqslant \bigoplus_{i \in \overline{k} \setminus \{m\}} o\left(\mathcal{T}^i(f)\right) \leqslant \bigoplus_{i \in \overline{k} \setminus \{m\}} \varphi_{k-2}\left(l_{\mathcal{S}(\mathcal{T}_k)}(b) + 1\right)$$
$$= \varphi_{k-2}(< \varphi_{k-1}(0)) \cdot (k-1) < \varphi_{k-1}(0)$$

Lower bound: finite subsets

LEMMA. Let $h(A) \leqslant \omega$ and let $o(A) = \omega \cdot \alpha + n$ for some $n < \omega$. Then $o(\mathcal{S}(A)) \geqslant \omega^\alpha \cdot (n+1)$.

PROOF. We argue by induction on $o(A)$.

For $o(A) = 0$, we have $\varnothing \in \mathcal{S}(A)$, so $o(\mathcal{S}(A)) \geqslant 1 = \omega^0$.

For $n = 0$, we have $o(\mathcal{S}(A)) \geqslant \sup\{o(\mathcal{S}(L_A(x))) \mid x \in A\} \geqslant \sup\{\omega^\beta \cdot m \mid \beta < \alpha, m < \omega\} = \omega^\alpha$.

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If $n > 0$, then A has a maximal element x. As the set $M = \{y \in A \mid y \leqslant x\}$ is finite, $o(A \setminus M) \geqslant \omega \cdot \alpha$. We extend the order on $\mathcal{S}(A)$ by saying that if $x \in X \setminus Y$ then $Y \leqslant X$. Then

$$o(\mathcal{S}(A)) \geqslant o(\mathcal{S}(A \setminus \{x\})) + o(\mathcal{S}(A \setminus M)) \geqslant \omega^\alpha \cdot n + \omega^\alpha = \omega^\alpha \cdot (n+1)$$

Lower bound: main lemma

LEMMA. Let $k \geqslant 3$, $A \subseteq \mathcal{S}(\mathcal{T}_m)$, $i \neq j \in \overline{k}$, and $f = (j \mapsto A)$. Then $o(\mathcal{T}^i(f)) \geqslant \omega \cdot 2$ for $o(A) = 1$, and $o(\mathcal{T}^i(f)) \geqslant \varphi_{k-2}(\alpha)$ for $o(A) \geqslant 2 + \alpha$.

PROOF. We argue by induction on k, and on $o(A)$. It suffices to prove the assertion for $j = k - 1$, $i = 0 = \operatorname{minarg}_{x \neq j} o(\mathcal{T}^x(f))$.

Let first $A = \{a\}$ for some forest a. The order $1 < 10 < 101 < \cdots < 1ja < 10ja < 101ja < \cdots$ on the displayed subset of $\mathcal{T}^0(f)$ extends the h-preorder on this subset, hence $o(\mathcal{T}^0(f)) \geqslant \omega \cdot 2$.

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Let now $o(A) = 2 + \alpha$. Then $o(\mathcal{T}^0(f)) > o(\mathcal{T}^0(j \mapsto L_A(b))) \geqslant \varphi_{k-2}((-2) + l_A(b))$ and since $l_A(b)$ approaches to $2 + \alpha$, it is enough to proof that it is φ_{k-2}-ordinal.

For $k = 3$, we have $\mathcal{T}^0(f) \supseteq 1(\mathcal{S}(\mathcal{T}^1(f)))$, hence $o(T^0(f)) \geqslant \omega^{o(\mathcal{T}^1(f))} \geqslant \omega^{o(\mathcal{T}^0(f))}$, so $o(T^0(f))$ is an ε-ordinal.

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For $k > 3$, we have $L_{\mathcal{T}^0(f)}(1b) \supseteq \mathcal{T}^0 \left(1 \mapsto L(b) \Big|_{\overline{k-1}} \right)$, so $o(\mathcal{T}^0(f)) \geqslant \sup\{\varphi_{k-2}(\beta) + 1 \mid \beta < o(\mathcal{T}^0(f))\}$.

Lower bound

THEOREM. $o(\mathcal{T}_k) \geqslant \varphi_{k-1}(0)$.

PROOF. Let $\omega^\gamma \leqslant o(\mathcal{T}_k) < \omega^{\gamma+1}$.

Let X_j be the set of trees whose root label is distinct from j. Since $\bigcup_j X_j = \mathcal{T}_k$, we get that $o(X_j) = \omega^\gamma$ for some j.

Then for $i \neq j$ we have

$$\omega^{\gamma+1} > o(\mathcal{T}_k) \geqslant o(\mathcal{T}^i(j \mapsto X_j)) \geqslant \varphi_{k-2}(o(X_j)) \geqslant \varphi_{k-2}(\omega^\gamma)$$

Then $\omega^\gamma = \varphi_{k-2}(\omega^\gamma)$, hence $o(\mathcal{T}_k) \geqslant \omega^\gamma \geqslant \varphi_{k-1}(0)$.

Спасибо за внимание!