A new family of identities for Bessel functions and its relation to one mixed functional differential equation

E. B. Velilaev[1], A. E. Rassadin[2].

Keywords: the Cauchy problem; integral of motion; the Fourier transform.

MSC2020 codes: 33C10

Introduction

Consideration of various differential equations often leads to the establishment of identities for various special functions. Mixed functional differential equations are just beginning to be intensively studied (see [1] and references therein), so their consideration makes it possible to discover new very nontrivial identities.

Introduction

Let $u_0 \in L_2(\mathbb{R})$ and $a \in \mathbb{R}$. Further, let one construct the following infinite-dimensional Hermitian matrix:

$$c_{nm}(a) = \frac{\int_{-\infty}^{+\infty} u_0(x - n\,a)\, u_0^*(x - m\,a)\, dx}{\int_{-\infty}^{+\infty} |u_0(x)|^2\, dx}\,, \qquad n, m \in \mathbb{Z}, \tag{1}$$

then the next identity is valid:

$$\sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} c_{nm}(a)\, J_n(t)\, J_m(t) = 1\,, \qquad t \geq 0. \tag{2}$$

where $J_n(t)$ is the Bessel function of the first kind with index n.

To prove this identity let one consider the next Cauchy problem:

$$\frac{\partial u(x,t)}{\partial t} = -\frac{u(x+a,t) - u(x-a,t)}{2}\,, \qquad u(x,0) = u_0(x). \tag{3}$$

[1]HSE University, Department of Fundamental Mathematics, Russia, Nizhny Novgorod. Email: ebvelilaev@edu.hse.ru.

[2]HSE University, Department of Fundamental Mathematics, Russia, Nizhny Novgorod. Email: aerassadin@hse.ru.

Under small a this mixed functional differential equation turns into a linear transfer equation.

Using the Fourier transform, it is easy to show that the general solution of the Cauchy problem (3) is:

$$u(x,t) = \sum_{n=-\infty}^{+\infty} J_n(t)\, u_0(x - n\, a), \tag{4}$$

and that equation (3) has the following integral of motion:

$$\int_{-\infty}^{+\infty} |u(x,t)|^2\, dx = \int_{-\infty}^{+\infty} |u_0(x)|^2\, dx. \tag{5}$$

Substituting the sum (4) into the integral of motion (5), we obtain the identity (2) with the coefficients (1).

References

1. *Myshkis A.* Mixed functional differential equations. // Journal of Mathematical Sciences. — 2005. — Vol. 129, no. 5. — P. 4111–4226.