Tomography and Radon transform

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Mathematical tomography is a section of mathematics that describes methods for determining the structure of an object using scattering data. Such problems arise in microscopy, medicine, and technical control. In this report, we will write out the basic differential equation of X-ray tomography. In the course of its solution, the Radon transform arises. We will establish a connection between the Fourier transform and the Radon transform, and get acquainted with the methods of its inversion.

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3. *acoustic tomography* uses sound and ultrasound waves as a probing instrument.

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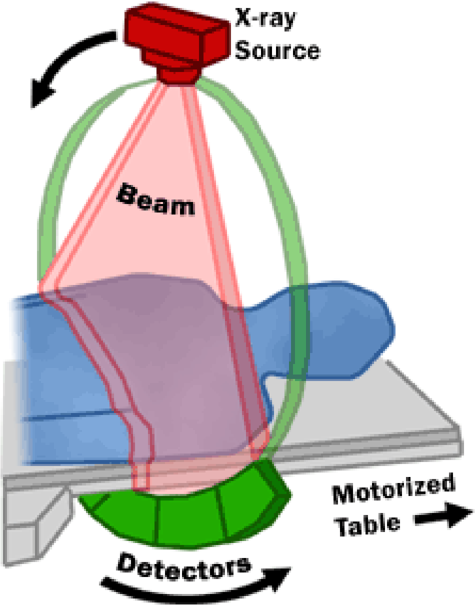
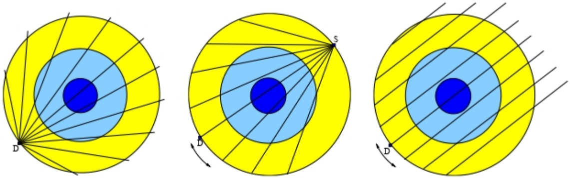
4. nuclear physics.


X-ray
Source
Beam
Motorized
Table
Detectors

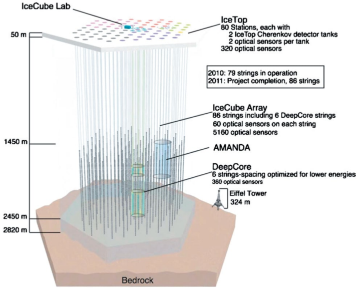


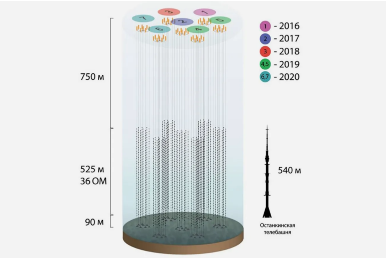
a) Isotropic flux

b) High-energy neutrino beam

c) Cosmic point source

Fig. 1 Three different approaches to "Whole Earth Tomography" using neutrino absorption. The lines refer to different baselines.


IceCube Lab
IceTop
80 Stations, each with
2 IceTop Cherenkov detector tanks
2 optical sensors per tank
320 optical sensors
2010: 79 strings in operation
2011: Project completion, 86 strings
IceCube Array
86 strings including 6 DeepCore strings
60 optical sensors on each string
5160 optical sensors
AMANDA
DeepCore
6 strings-spacing optimized for lower energies
360 optical sensors
Eiffel Tower
324 m
50 m
1450 m
2450 m
2820 m
Bedrock


1 - 2016
2 - 2017
3 - 2018
4,5 - 2019
6,7 - 2020
750 м
525 м
36 OM
90 м
540 м
Останкинская
телебашня

X-ray tomography

The word *tomography* comes from the Greek words $\tau o \mu o \varsigma$ - section, cross-section, layer, and $\gamma \rho \alpha \phi o$ – image, drawing. In tomography, the reconstruction of the structure of an object occurs, as a rule, layer by layer. At the mathematical level, in X-ray tomography, we are talking about the reconstruction of the *coefficient* $a(x)$, $x \in \mathbb{R}^3$, *the absorption of the radiation of X-ray photons*.

X-ray tomography

Example

Let us consider a model of a section of the human body at the level of the lungs. Let $D \subseteq R^2$ be a convex compact (section of the body), $D_1 \subseteq D$, $D_2 \subseteq D$ be convex non-intersecting compacts (sections of the lungs). The absorption coefficient in the lungs and in water is slightly higher than zero and can be approximately considered equal to zero. In the part of the section of the body that does not contain the lungs, the attenuation coefficient of X-ray photons can be considered approximately equal to unity, which corresponds to the approximate value of the absorption coefficient in water.

X-ray tomography

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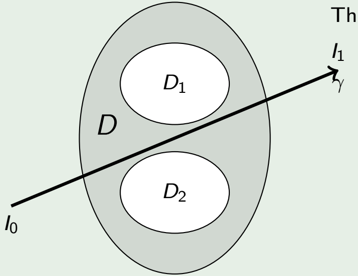

Thus, we assume that

$$a(x) = \begin{cases} 0, & x \in \mathbb{R}^2 \setminus D, \\ 0, & x \in D_1 \cup D_2, \\ 1, & x \in D \setminus (D_1 \cup D_2). \end{cases}$$

X-ray tomography

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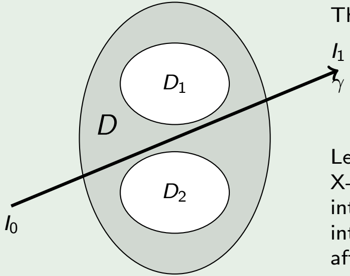

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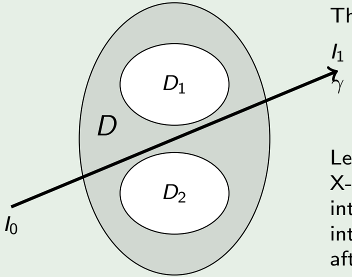

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We know the value of $\frac{I_1}{I_0}$ for a sufficiently large set of oriented lines γ. We need to reconstruct the absorption coefficient of X-ray photons $a(x)$.

X-ray tomography

One of the main formulas of X-ray tomography, which allows to formulate the mathematical problem of restoring $a(x)$, is based on the Beer–Bouguer–Lambert law and looks like this:

$$I_1(\gamma) = I_0 \exp\left(-\int_\gamma a(x)\,dx\right). \qquad (1)$$

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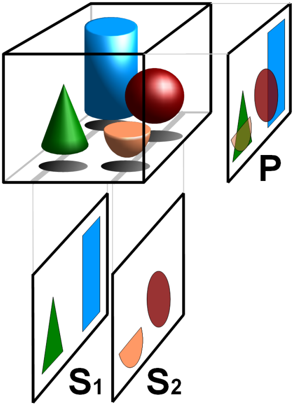
The mathematical foundations of X-ray tomography were given in the works of [Radon, 1917], [John, 1937], [Cormack, 1963], Gelfand and co-authors (in the 1960s and later), [Helgason, 1965]. In 1979, mathematician Alan Cormack and engineer Godfrey Hounsfield received the Nobel Prize in Physiology or Medicine for the synthesis of ideas that led to the creation of the first X-ray tomograph.

X-ray tomography



source: wiki.

Beer–Bouguer–Lambert law:

$$I_1(\gamma) = I_0 \exp\left(-\int_\gamma a(x)\,dx\right)$$

Main differential equation of X-ray tomography

Differential equation

$$\theta \nabla_x \psi + a(x)\psi = 0,\ \psi = \psi(x,\theta),\ x \in \mathbb{R}^3,\ \theta \in S^{d-1}, \qquad (2)$$

is called the *basic differential equation of X-ray tomography*. The parameter θ is called the *spectral parameter*.

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is called the *basic differential equation of X-ray tomography*. The parameter θ is called the *spectral parameter*. We assume that the function $a(x)$ is sufficiently regular and localized (rapidly decreasing at infinity). Now, we also assume that the function $a(x)$ is known.

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The absorption coefficient of X-ray photons $a(x)$ is defined by the formula

$$-a(x)\psi(x, \theta) := \lim_{\varepsilon \to 0} \frac{\psi(x + \varepsilon\theta, \theta) - \psi(x, \theta)}{\varepsilon}. \tag{4}$$

On the right side the derivative in the direction θ of the function $\psi(x)$ is given. Assume the function $\psi(x, \theta)$ to be sufficiently smooth. We can rewrite this definition of the absorption coefficient as an equation (2).

Main differential equation of X-ray tomography

Let us consider a special solution $\psi^+(x,\theta)$ of the equation (2), characterized by the limit condition

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$$S(x,\theta) := \lim_{s\to+\infty} \psi^+(x+s\theta,\theta), \quad x\in\mathbb{R}^d, \quad \theta\in S^{d-1}, \quad (6)$$

called *spectral data* (or *scattering data*).

Main differential equation of X-ray tomography

Note that:

$$S(x+\tau\theta,\theta)=S(x,\theta),\ S(x,\theta)=S(\pi_\theta x,\theta),$$

where π_θ is the orthogonal projection onto the plane $X_\theta=\left\{x\in\mathbb{R}^d \mid x\theta=0\right\}$.

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The function $S(x,\theta)$ is defined on the direct product $\mathbb{R}^d \times S^{d-1}$, it is natural to consider it on the space of the tangent bundle over the unit sphere

$$TS^{d-1} = \left\{(x,\theta) \in \mathbb{R}^d \times S^{d-1} \mid x\theta = 0\right\}. \tag{7}$$

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We interpret the points $\gamma=(x,\theta)\in TS^{d-1}$ as oriented lines

$$\gamma=(x,\theta)=\left\{y\in\mathbb{R}^d\mid y=x+s\theta,\ s\in\mathbb{R}\right\},\ \text{vector } \theta \text{ defines orientation.}$$

Thus, we interpret TS^{d-1} as the set of all oriented lines in $\mathbb{R}^d$.

Direct and inverse scattering problems

The direct problem for the differential equation (2): given absorption coefficient $a(x)$, find the functions $\psi^{+}(x,\theta)$, and then the spectral data $S(x,\theta)$:

$$a \mapsto \psi^{+} \mapsto S.$$

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The inverse problem for the differential equation (2) consists of finding the absorption coefficient $a(x)$ from the given spectral data $S(x,\theta)$:

$$S \mapsto a.$$

Direct and inverse scattering problems

The solution to the direct problem is given by the following formulas. The function ψ^+ is given by the formula

$$\psi^+(x, \theta) = \exp\big(-Da(x, -\theta)\big), \tag{8}$$

where $Da(x, \theta) := \int_0^{+\infty} a(x + s\theta)\, ds$ is *divergent beam transform*.

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$$S(x,\theta) = \exp\big(-Pa(x,\theta)\big),\ \ Pa(x,\theta) := \int\limits_{\mathbb{R}} a(x+s\theta)\,ds = \int\limits_{y\in\gamma=(x,\theta)} a\,dy. \tag{9}$$

The transformation P is called the *Radon transform along lines* or the *X-ray transform*.

The formula for the spectral data S follows from the definition (6) and from the formula (8) for the function ψ^+. To obtain the formula (8), we are looking for the solution ψ^+ of the equation (2) in the form $\psi^+(x,\theta) = \exp\big(\varphi^+(x,\theta)\big)$. Then for the function φ^+ we get the problem

$$\theta\nabla_x\varphi^+ + a(x) = 0, \quad \lim_{s\to-\infty} \varphi^+(x+s\theta,\theta) = 0.$$

The solution to this problem is given by the formula $\varphi^+(x,\theta) = Da(x,\theta)$.

The X-ray transform and the Fourier transform

X-ray transform of a function $f(x)$, $x \in \mathbb{R}^d$, as defined

$$Pf(x,\theta) = \int_{\mathbb{R}^d} f(x + s\theta)\, ds, \quad (x,\theta) \in TS^{d-1}. \tag{10}$$

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we also define the Fourier transform of the X-ray transform $P_\theta f$ by the formula

$$(P_\theta f)^\wedge(\xi) = (2\pi)^{-\frac{d-1}{2}} \int\limits_{X_\theta} e^{i\xi x} P_\theta f(x)\, dx, \quad \xi \in X_\theta,\ \theta \in S^{d-1}.$$

The X-ray transform and the Fourier transform

[Projection theorem] We have that

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The proof follows from the following equalities:

$$\int_{X_\theta} e^{i\xi x} P_\theta f(x) dx = \int_{X_\theta} e^{i\xi x} \int_{\mathbb{R}} f(x + s\theta)\, ds\, dx = [\xi\theta = 0] = \int_{\mathbb{R}^d} e^{i\xi y} f(y)\, dy.$$

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The projection theorem allows us to reconstruct the function f from the X-ray transform Pf using the scheme

$$Pf \mapsto \hat{f} \mapsto f.$$

The X-ray transform and the Fourier transform

Note also that in order to recover the value of the function f at the point $x \in \mathbb{R}^d$, we only need to be able to restore the values of a function defined in $\mathbb{R}^2$. Actually, in order to recover the value of the function f at the point $x \in \mathbb{R}^d$, we only need to draw a two-dimensional plane $\Xi \simeq \mathbb{R}^2$ through the point x and to apply the reconstruction scheme to the function $f|_\Xi$. Thus, we find the value of the function f at the point x. Therefore, below we will consider the case $d = 2$ in details.

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where $\theta = (\theta_1, \theta_2) \in S^1$, $\theta^{\perp} = (-\theta_2, \theta_1)$ — the vector obtained by rotating the vector θ by a right angle counterclockwise. Therefore, for the case $d = 2$

$$\hat{f}(r\theta^{\perp}) = (2\pi)^{-1} \int\limits_{\mathbb{R}} e^{irs} q_{\theta}(s)\, ds, \qquad (12)$$

where $q_{\theta}(s) := Pf(s\theta^{\perp}, \theta)$, $s \in \mathbb{R}$, $\theta \in S^1$.

Radon formula

Theorem (Radon, 1917)

The following formula holds:

$$f(x) = \frac{1}{4\pi} \int\limits_{S^1} \theta^{\perp} \nabla \tilde{q}_{\theta}(x\theta^{\perp})\, d\theta, \quad x \in \mathbb{R}^2,$$

$$\tilde{q}_{\theta}(s) = (Hq_{\theta})(s) := \frac{1}{\pi}\mathrm{p.v.} \int\limits_{\mathbb{R}} \frac{q_{\theta}(t)}{s-t}\, dt,$$

$$q_{\theta}(s) = Pf(s\theta^{\perp}, \theta), \quad s \in \mathbb{R},\ \theta = (\theta_1, \theta_2) \in S^1,\ \theta^{\perp} = (-\theta_2, \theta_1).$$

The function $\tilde{q}_{\theta}(s)$ is called the filtered backprojection.

Radon formula

Here p.v. denotes the integral in the sense of the principal value:

$$\text{p.v.}\int\limits_{\mathbb{R}} \frac{u(t)}{s-t}\,dt := \lim_{\varepsilon\to+0}\left[\int\limits_{-\infty}^{s-\varepsilon} + \int\limits_{s+\varepsilon}^{+\infty}\right]\left(\frac{u(t)\,dt}{s-t}\right)$$

$$\equiv \lim_{\varepsilon\to+0}\int\limits_{\mathbb{R}} \frac{1}{2}\left(\frac{1}{s-t+i\varepsilon}+\frac{1}{s-t-i\varepsilon}\right)u(t)\,dt.$$

For the numerical calculation of integrals in the sense of the principal value, the following formula based on the residue theorem can be used:

$$\int\limits_{\mathbb{R}} e^{-ips}ds\,\text{p.v.}\int\limits_{\mathbb{R}} \frac{u(t)}{s-t}\,dt = -i\pi p\int\limits_{\mathbb{R}} e^{-ipt}u(t)\,dt.$$

Radon's formula is now the basic one in applications. It is a basis for *filtered backprojection algorithm*).

Proof.

Using the expression for the inverse Fourier transform in polar coordinates, we obtain:

$$f(x) = (2\pi)^{-1} \int\limits_{S^1} \int\limits_0^{+\infty} e^{-ir\theta^\perp x} \hat{f}(r\theta^\perp) r \, dr \, d\theta$$

$$= \frac{1}{4\pi} \int\limits_{S^1} \int\limits_{\mathbb{R}} e^{-ir\theta^\perp x} \hat{f}(r\theta^\perp) r \operatorname{sgn} r \, dr \, d\theta = \frac{1}{8\pi^2} \int\limits_{S^1} \int\limits_{\mathbb{R}} \int\limits_{\mathbb{R}} e^{ir(s-\theta^\perp x)} q_\theta(s) \, ds \, r \operatorname{sgn} r \, dr \, d\theta.$$

Next, since

$$\int\limits_{\mathbb{R}} e^{irs} (-i) \operatorname{sgn} r \, dr = \frac{1}{s+i0} + \frac{1}{s-i0} \text{ (in the sense of distributions)},$$

we obtain the theorem. □

Cormack–Helgason theorem

Theorem (Cormack (1963–1964), $d = 2$; Helgason (1965), $d \geqslant 2$)

Let $f \in \mathcal{S}(\mathbb{R}^d)$. $R_{2,1} f|_{\Omega_{d,d-1}(B_r)}$ defines $f|_{\mathbb{R}^d \setminus B_r}$.

Cormack–Helgason theorem

Using polar coordinates, we represent the coordinates x_1 and x_2 as $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$ and expand the function $f(x)$ in a Fourier series in the variable φ for each fixed r:

$$f(x) = \sum_{l=-\infty}^{+\infty} f_l(r)e^{il\varphi}. \tag{13}$$

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It is sufficient to show that $f_l(r)$ for $r \geqslant \rho$ depends only on $q_l(s)$ $s \geqslant \rho$.

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$$f_l(r) = -\frac{1}{\pi} \int_r^{+\infty} (s^2 - r^2)^{-\frac{1}{2}}\, T_{|l|}\left(\frac{s}{r}\right) q_l'(s)\, ds, r > 0. \tag{16}$$

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1. *recurrently:*

$$T_0(x) \equiv 1,\ T_1(x) = x,\ T_{l+1}(x) + T_{l-1}(x) = 2xT_l(x), \quad l = 1, 2, \dots,$$

2. *by an explicit formula:*

$$T_l(x) = \begin{cases} \cos(l \arccos x), & |x| \leqslant 1, \\ \cosh(lx), & |x| \geqslant 1, \end{cases} \qquad x \in \mathbb{R}.$$

Thanks for your attention!