Theoretical modelling of superradiant lasers and other open quantum resonant systems

I.E.Protsenko and A.V.Uskov

Quantum radio-physic division of Lebedev Physical institute of RAS

protsenk@gmail.com procenkoie@lebedev.ru

Presentation summary

Introduction. Laser model. Semiclassical approach.

Quantum model. Neglecting population fluctuations

Population fluctuation effect on spontaneous emission

Solving Heisenberg nonlinear equations for LED. Results

Notes about the method

Working in the frequency domain. Calculating quantum fluctuation power spectra. Preserving commutation relations for operators.

What systems can be investigated?

Superradiant micro-lasers, with the cavity size ~ $\lambda/2$

Nonlinear optical devices

Plasmonic nanoparticles and lasers

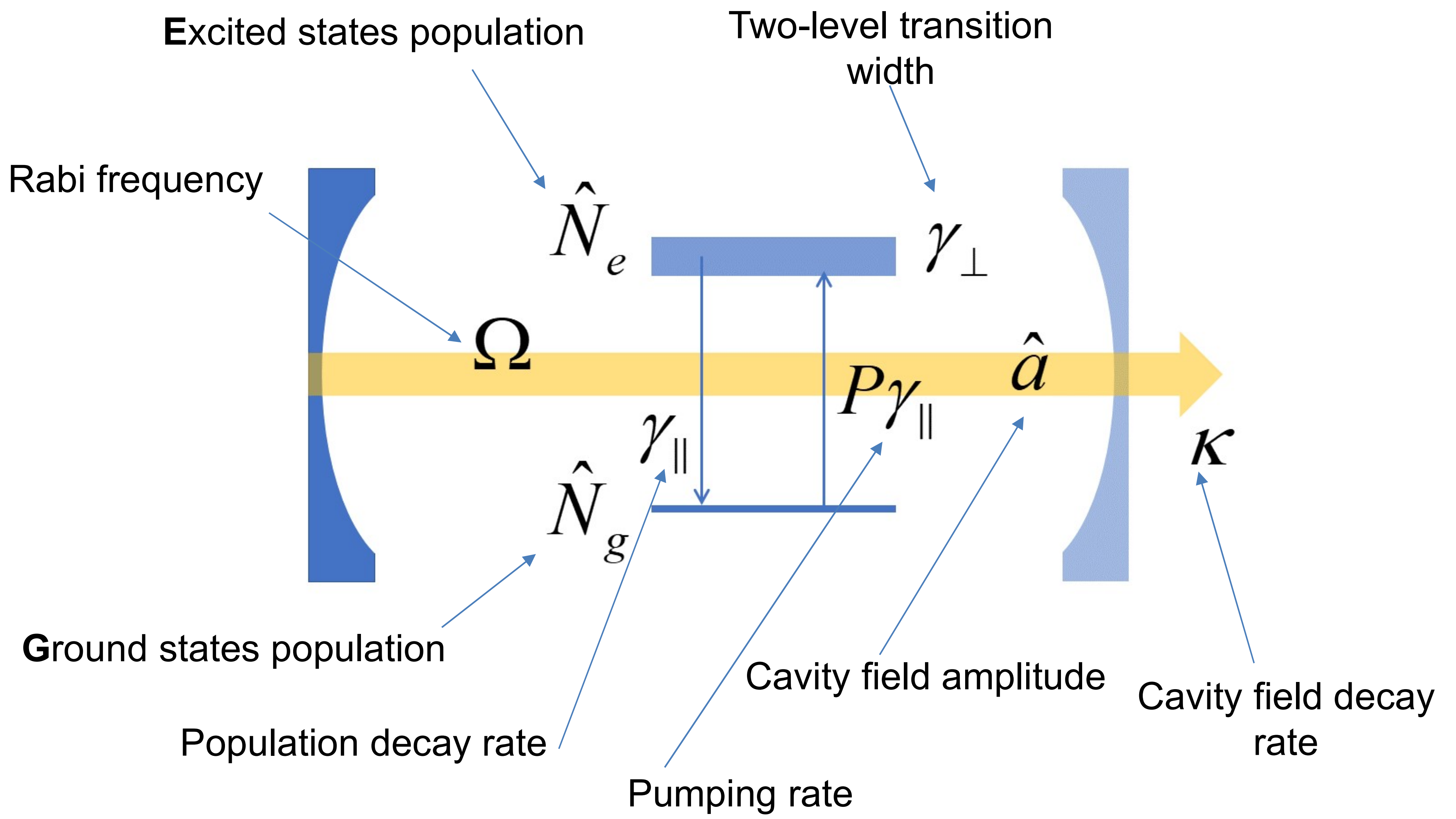
Photonic time crystal devices

Resonant fluorescence

Other resonant quantum optic systems

The system we consider here

Micro-laser with the cavity size ~ λ/2



Hamiltonian

Interaction picture, resonant approximation

$$H = i\hbar\Omega\sum_{i=1}^{N_0} f_i\left(\hat{a}^+\hat{\sigma}_i - \hat{\sigma}_i^+\hat{a}\right) + \hat{\Gamma}$$

Interaction with environment

Cavity field amplitude
Bose operators

$$\left[\hat{a},\hat{a}^+\right] = 1$$

Different coupling of
different emitters

$$f_i = \rho(x_i, y_i)\sin\left(k_0 z_i\right) \qquad \sum_{i=1}^{N_0} f_i^2 = 1$$

What is $\hat{\sigma}_i$?

i-th two-level atom

$$|e\rangle_i$$

$$|g\rangle_i$$

$$\hat{\sigma}_i = |g\rangle_i \langle e|_i$$ Transition operator

$$\hat{\sigma}_i |e\rangle_i = |g\rangle_i \langle e|e\rangle_i = |g\rangle_i$$

population operators

$$\hat{n}_{ei} = |e\rangle_i \langle e|_i$$

$$\hat{n}_{gi} = |g\rangle_i \langle g|_i$$

orthogonality

$$_i\langle e|g\rangle_i = 0$$

$$_i\langle e|e\rangle_i = 1$$

Commutation relations can be easily found

Total populations of the **e**xcited (e) and **g**round (g) medium states

$$\hat{N}_{e,g} = \sum_{i=1}^{N_0} \hat{n}_{e,gi}$$

Polarisation of the medium

$$\hat{v} = \sum_{i=1}^{N_0} f_i \hat{\sigma}_i$$

What equations we solve?

Langevin force

Field amplitude

$$\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{v} + \hat{F}_a$$

polarisation

$$\dot{\hat{v}} = -(\gamma_\perp / 2)\hat{v} + \Omega f \left(\hat{a} N + 2 \hat{a} \delta \hat{N}_e \right) + \hat{F}_v$$

Population fluctuations

$$\delta \dot{\hat{N}}_e = -\Omega \delta \hat{\Sigma} - \gamma_\parallel (P+1) \delta \hat{N}_e + \hat{F}_{N_e}$$

Fluctuations of field-medium dipole interaction

$$\delta \hat{\Sigma} = \hat{a}^+ \hat{v} + \hat{v}^+ \hat{a} - \left\langle \hat{a}^+ \hat{v} + \hat{v}^+ \hat{a} \right\rangle$$

Mean values and fluctuations of populations

$$\hat{N}_e = N_e + \delta \hat{N}_e$$

Mean values and coefficients are without hats

Energy conservation law $2\kappa n = \gamma_\parallel \left(P N_g - N_e \right)$

Population inversion $N = N_e - N_g$ Number of particles $N_0 = N_g + N_e$

Some conditions of the approach:

The stationary case $\hat{a}(t-t')$

The field propagates along axis Z

The number of particles is large

No polarization-population correlations!

Close to resonance

«Semiclassical» approach

Operators are replaced by c-numbers, fluctuations are neglected

$$\dot{a} = -\kappa a + \Omega v$$

$$\dot{v} = -(\gamma_\perp / 2) v + \Omega f a N$$

$$\dot{N}_e = -\Omega\left(a^* v + v^* a\right) + \gamma_\parallel \left(P N_g - N_e\right)$$

The stationary solution can be found

Semiclassical theory results

$$N_0 = 100$$

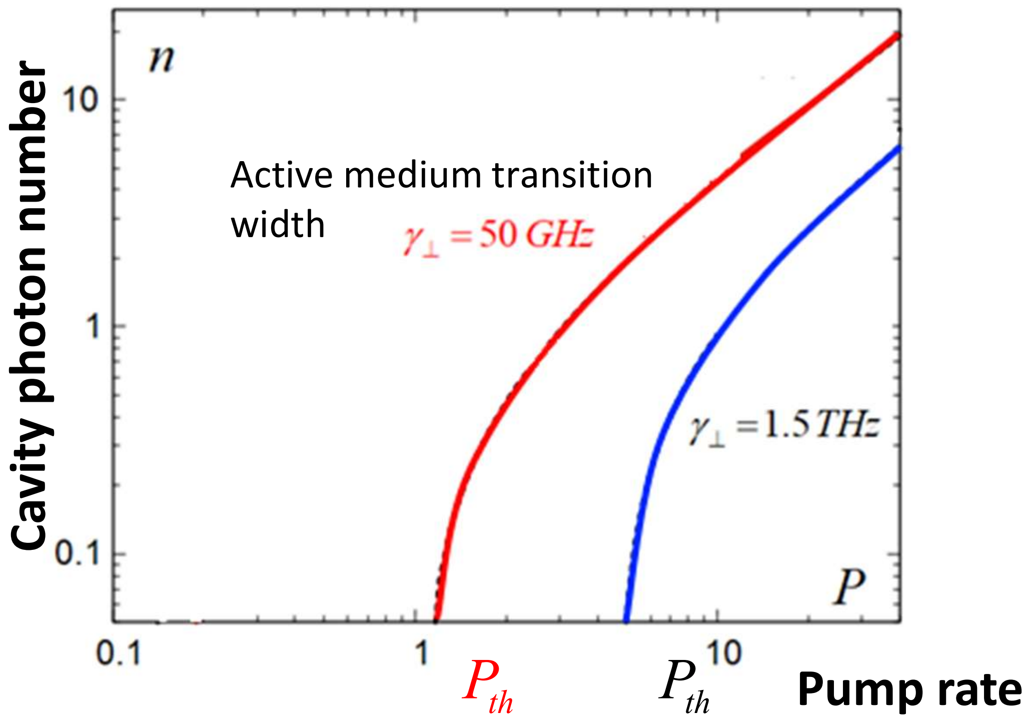
$$\lambda_0 = 1.55\ \mu m$$

Cavity quality factor $Q = 1.2 \cdot 10^4$

Cavity volume $V_m = 10\lambda_0^3$ (micro-laser)

Excited states decay rate $\gamma_{\parallel} = 1$ GHz

(free space spontaneous emission)



Quantum «zero-order» approximation

$$\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{v} + \hat{F}_a$$

$$\dot{\hat{v}} = -(\gamma_\perp / 2)\hat{v} + \Omega f \left(\hat{a} N + 2\hat{a}\delta \hat{N}_e \right) + \hat{F}_v$$

Population fluctuations are neglected

$$\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{v} + \hat{F}_a$$

$$\dot{\hat{v}} = -(\gamma_\perp / 2)\hat{v} + \Omega f \hat{a} N + \hat{F}_v$$

The mean population inversion must be found

$$N = ?$$

Equations are linear in operators and can be solved by Fourier-transform

[similar with equations for coupled oscillators]

Operator Fourier-expansion

$$\hat{a}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{a}_\omega e^{-i\omega t} d\omega$$ [The same for all other operators]

$$-i\omega \hat{a}_\omega = i\Omega \hat{v}_\omega - \kappa \hat{a}_\omega + \hat{F}_{a\omega}$$

$$-i\omega \hat{v}_\omega = -\left(\gamma_\perp / 2\right) \hat{v}_\omega - i\Omega f \hat{a}_\omega N + \hat{F}_{v\omega}$$

We solve the system of linear equations and express Fourier-component operators through Langevin force Fourier-components

$$\hat{a}_\omega = \frac{\Omega \hat{F}_{v\omega} + (\gamma_\perp / 2 - i\omega)\hat{F}_{a\omega}}{(\kappa - i\omega)(\gamma_\perp / 2 - i\omega) - \kappa\gamma_\perp N / 2N_{th}}$$

$$N_{th} = \kappa\gamma_\perp / 2\Omega^2 f$$ semiclassical threshold population inversion

$$\left\langle F_{v^+\omega} F_{v\omega'} \right\rangle = f\gamma_\perp N_e \delta(\omega - \omega')$$

Spectral density of polarisation Langevin noise

$$\hat{F}_{a\omega} = \sqrt{2\kappa}\,\hat{a}_\omega^{(in)}$$

Vacuum field

spectrum n_ω of the cavity field

$$\left\langle (\hat{a}^+)_\omega \hat{a}_{\omega'} \right\rangle = n_\omega \delta(\omega - \omega')$$

$$n_\omega = \frac{(\kappa\gamma_\perp^2 / 2) N_e / N_{th}}{\left|(\kappa - i\omega)(\gamma_\perp / 2 - i\omega) - \kappa\gamma_\perp N / 2N_{th}\right|^2}$$

We must find

$$N_e = ?$$

Mean cavity photon number

$$n = \frac{1}{2\pi}\int_{-\infty}^{\infty} n_\omega d\omega = \frac{N_e}{(1+2\kappa/\gamma_\perp)(N_{th}-N)}$$

N_e Determined from the energy conservation law $2\kappa n = \gamma_\parallel (PN_g - N_e)$

$N_g + N_e = N_0$

Population inversion $N = N_e - N_g$

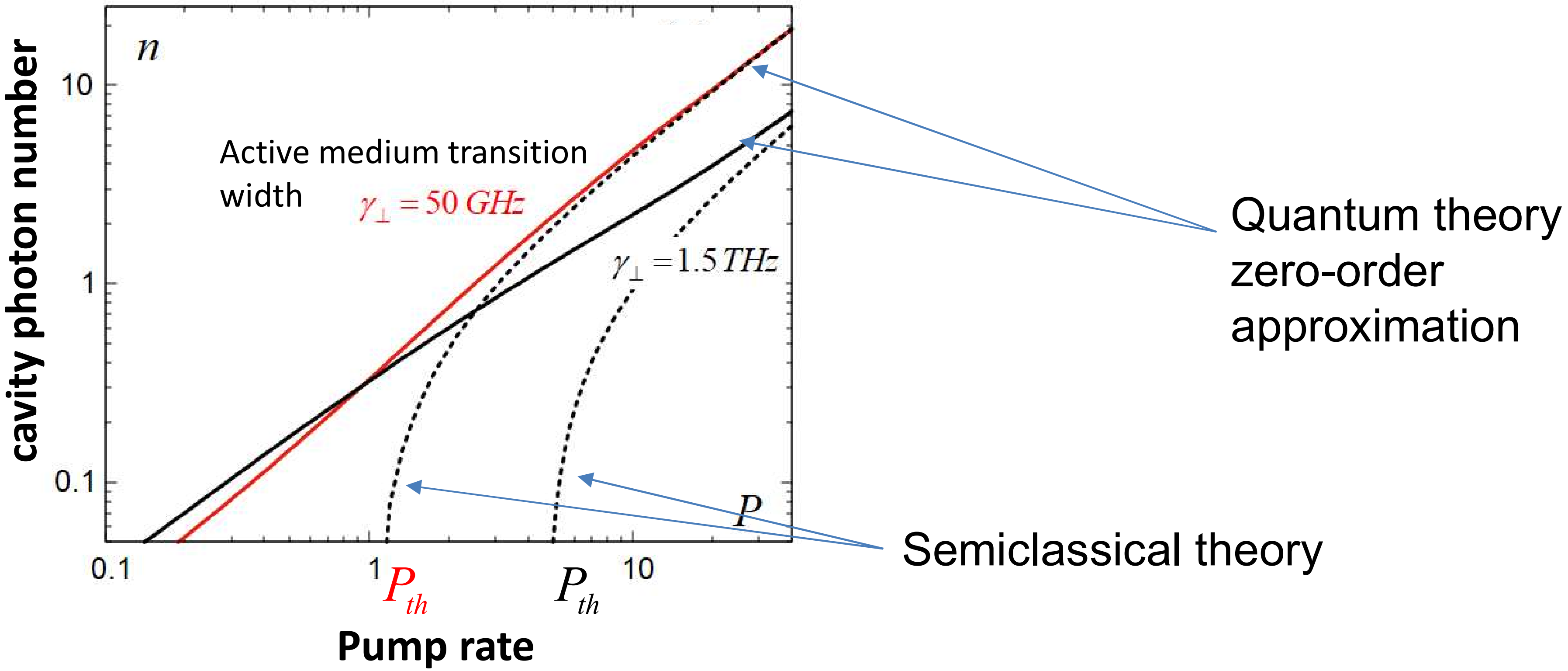
Well-known Quantum rate equations $2\kappa/\gamma_\perp \to 0$

Superradiant laser $2\kappa/\gamma_\perp > 1$

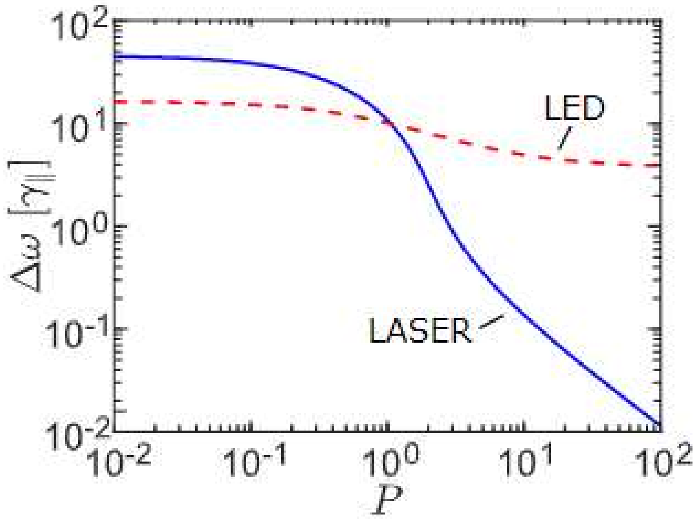
Zero-order approximation results

Thresholdless micro-laser with a large spontaneous emission to the cavity mode

Cavity photon number

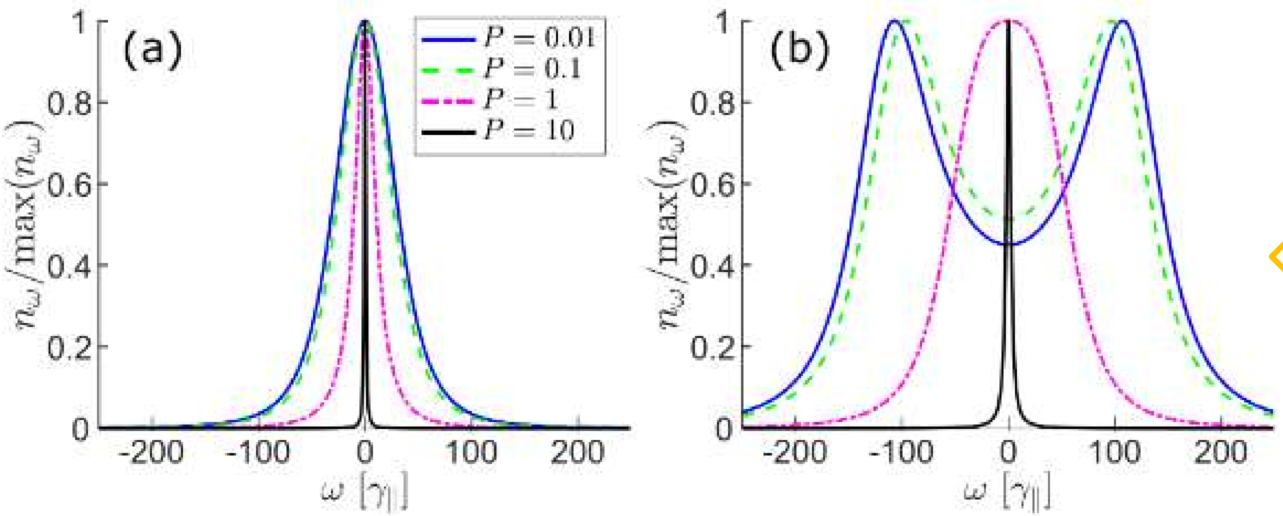


Laser linewidth



Pump rate

Laser field spectrum



Collective Rabi splitting

Zero-order approximation incompleteness

Laser linewidth

$$\gamma_{las} = \left(\frac{2\kappa\gamma_\perp}{2\kappa + \gamma_\perp} \right)^2 \frac{N_e}{N_{th}} \frac{\hbar\omega_0}{W_{out}}$$

It is good below the threshold

It must be above the threshold $\frac{1}{2}$

Output field power

Second-order autocorrelation function

$$g_2 = \frac{\langle \hat{a}^+ \hat{a}^+ \hat{a} \hat{a} \rangle}{n^2} = 2$$

It must be $g_2 > 2$ at small photon number $n \to 0$ in the superradiant laser $2\kappa > \gamma_\perp$

Therefore, population fluctuations must be considered for superradiant laser

We come back to nonlinear laser equations

$$\dot{\hat{a}} = -\kappa \hat{a} + \Omega \hat{v} + \hat{F}_a$$

$$\dot{\hat{v}} = -(\gamma_\perp / 2)\hat{v} + \Omega f\left(\hat{a}N + 2\hat{a}\delta\hat{N}_e\right) + \hat{F}_v$$

$$\delta\dot{\hat{N}}_e = -\Omega\delta\hat{\Sigma} - \gamma_\parallel (P+1)\delta\hat{N}_e + \hat{F}_{N_e}$$

We add a new term $\hat{a}\delta\hat{N}_e$, so we must change the Langevin force $\hat{F}_v$ in order to preserve $\left[\hat{a}, \hat{a}^+\right] = 1$

Population fluctuation effect on the cavity mode spontaneous emission

If the mean cavity photon number is small, we <u>neglect</u> the stimulated emission, preserving the population fluctuation effect on the spontaneous emission

This mean that we neglect nonlinear terms in equations, but take the <u>population fluctuation depended</u> Langevin force

$$\dot{\hat{a}} = -\kappa\hat{a} + \Omega\hat{v} + \hat{F}_a$$

neglected

$$\dot{\hat{v}} = -(\gamma_\perp / 2)\hat{v} + \Omega f\left(\hat{a}N + 2\hat{a}\delta\hat{N}_e\right) + \hat{F}_v$$

Depends on $\delta\hat{N}_e$

$$\delta\dot{\hat{N}}_e = -\Omega\delta\hat{\Sigma} - \gamma_\parallel(P+1)\delta\hat{N}_e + \hat{F}_{N_e}$$

neglected

Why only $\hat{F}_v$ depends on $\delta\hat{N}_e$?

Langevin force with population fluctuations

$$\left\langle \hat{F}_{v^+\omega'} \hat{F}_{v\omega} \right\rangle = 2D_{v^+v}(\omega)\delta(\omega' - \omega)$$

Requiring commutation relations $\left[\hat{a}, \hat{a}^+\right] = 1$ we find

$$2D_{v^+v}(\omega) = f\gamma_\perp N_e + 2f^2\Omega^2\left(c * \delta^2 N_e\right)_\omega$$

← convolution

$c = ?$ $\left[\hat{a}_\omega, \hat{a}^+_{\omega'}\right] = c_\omega \delta(\omega - \omega')$ Bose-commutator for the cavity mode

$$c_\omega = \frac{2\kappa\omega^2 + (\kappa\gamma_\perp^2/2)\left(1 - N/N_{th}\right)}{\left|(\kappa - i\omega)(\gamma_\perp/2 - i\omega) - \kappa\gamma_\perp N/2N_{th}\right|^2}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} c_\omega d\omega = 1 \quad \Rightarrow \quad \left[\hat{a}, \hat{a}^+\right] = 1$$

convolution $$\left(c*\delta^2 N_e\right)_\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} c_{\omega-\omega'}\delta^2 N_{e\omega'} d\omega'$$

Population fluctuation power spectrum $\delta^2 N_{e\omega} = ?$

If $n \to 0$, then $\delta\dot{\hat{N}}_e \approx -\gamma_\parallel (P+1)\delta\hat{N}_e + \hat{F}_{N_e}$

For a large number of emitters → $$\left\langle \hat{F}_{N_e\omega'}\hat{F}_{N_e\omega} \right\rangle = \gamma_\parallel \left(PN_g + N_e\right)\delta(\omega-\omega')$$

$$\delta^2 N_{e\omega} = \frac{2\gamma_\parallel N_e}{\omega^2 + \gamma_\parallel^2 (P+1)^2}$$

Now we have everything necessary for calculations

Results

A **small** mean cavity photon number

$$n = n_0\left(1+\Delta_n\right) \qquad \Delta_n = \frac{\delta^2 N_e}{N_e N_{th}}\left[3\left(\frac{2\kappa}{\gamma_\perp + 2\kappa}\right)^2 + \frac{2\kappa/\gamma_\perp}{1-N/N_{th}}\right]$$

Zero-order approximation result

Second-order autocorrelation function at $n \to 0$

$$g_2 = 2\left[1+2\left(\frac{\Delta_n}{1+\Delta_n}\right)^2\right] > 2$$

Conclusion: population fluctuations increase polarization fluctuations (through Langevin force) and therefore increase the mean cavity photon number n.

Population fluctuations lead to the super-thermal photon statistics at small n.

Solving nonlinear equations

Coming to Fourier-components

$$0=\left(i\omega-\kappa\right)\hat{a}_{\omega}+\Omega\hat{v}_{\omega}+\hat{F}_{a\omega}$$

$$0=\left(i\omega-\gamma_{\perp}/2\right)\hat{v}_{\omega}+\Omega f\left[\hat{a}_{\omega}N+2\left(\hat{a}\delta\hat{N}_{e}\right)_{\omega}\right]+\hat{F}_{v\omega}$$

$$0=\left[i\omega-\gamma_{\parallel}\left(P+1\right)\right]\delta\hat{N}_{e\omega}-\Omega\left(\hat{a}^{+}\hat{v}+\hat{v}^{+}\hat{a}-\left\langle\hat{a}^{+}\hat{v}+\hat{v}^{+}\hat{a}\right\rangle\right)_{\omega}+\hat{F}_{N_{e}\omega}$$

The field spectrum $n_{\omega}=\dfrac{\left(2\Omega^{2}f\right)^{2}S_{a\delta N_{e}}(\omega)+\gamma_{\perp}f\Omega^{2}N_{e}}{\left|\left(\kappa-i\omega\right)\left(\gamma_{\perp}/2-i\omega\right)-\kappa\gamma_{\perp}N/2N_{th}\right|^{2}}$

$$S_{a\delta N_{e}}(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\left(n_{\omega-\omega'}+c_{\omega-\omega'}/2\right)\delta^{2}N_{e\omega'}d\omega'$$

Perturbation approach: $\delta^{2}N_{e\omega}$ and $S_{a\delta N_{e}}(\omega)$ can be found from zero-order approximation

Narrow population fluctuation spectrum

Below the threshold $\gamma_{\|} << \kappa, \gamma_{\perp}$

Wide spectrum — Narrow spectrum

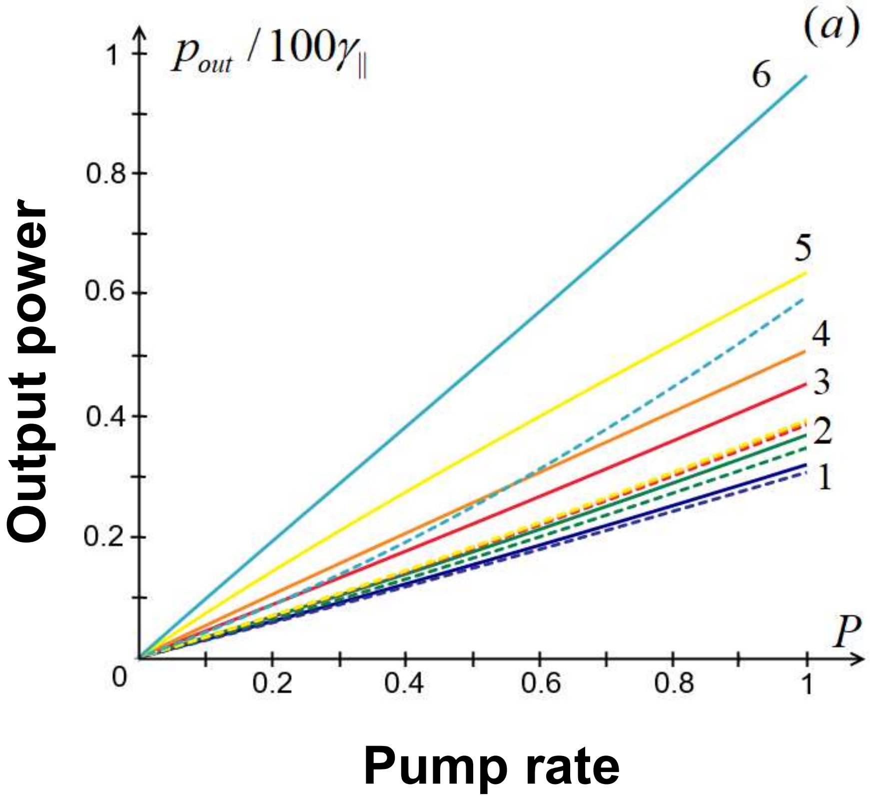
then $S_{a\delta N_e}(\omega) \approx \left(n_\omega + c_\omega / 2\right)\delta^2 N_e$

Explicit expression for the field spectrum

$$n_\omega = \Omega^2 f \frac{2\Omega^2 f c_\omega \delta^2 N_e + \gamma_\perp N_e}{\left|(\kappa - i\omega)(\gamma_\perp / 2 - i\omega) - \kappa\gamma_\perp N / 2N_{th}\right|^2 - \left(2\Omega^2 f\right)^2 \delta^2 N_e}$$

Population fluctuation dispersion $\delta^2 N_e$ can be found from generalized laser rate equations (equations for the energy operators)

Population fluctuation effect on the nonlinear LED power



Dashed curves are without population fluctuations

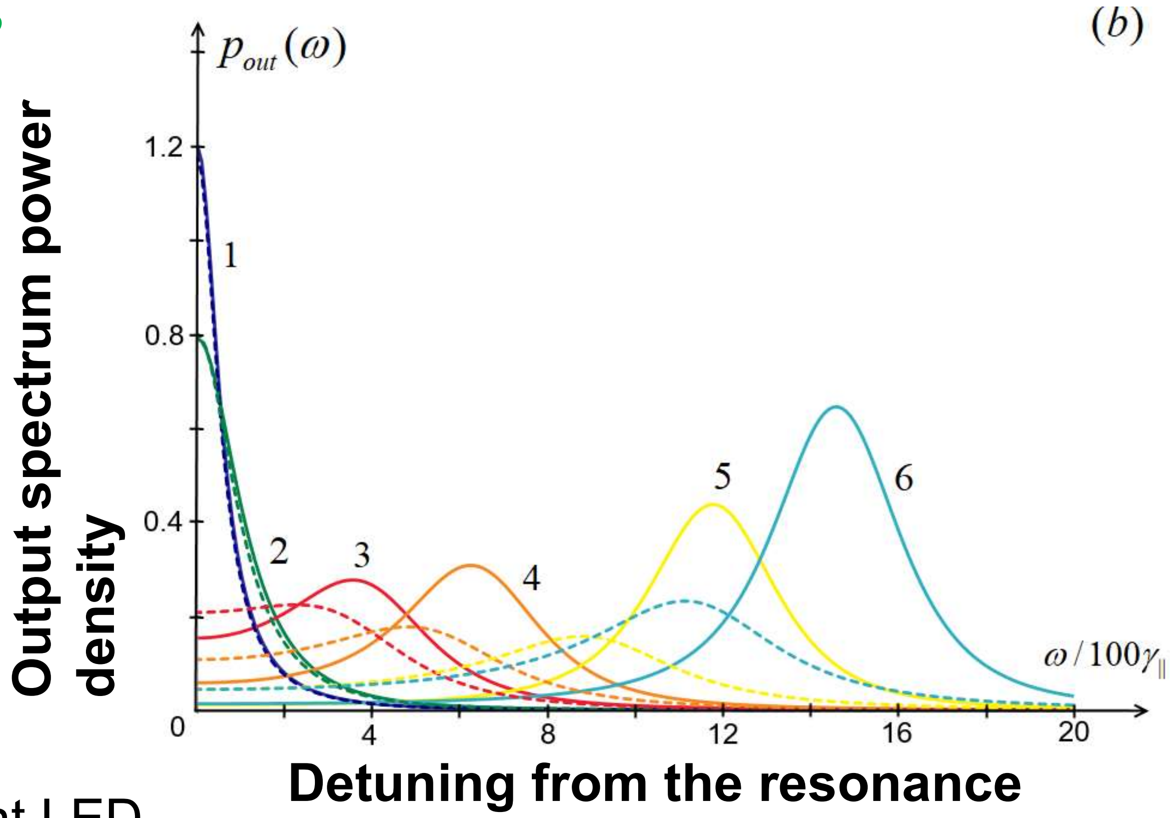
Super-radiant LED

$$2\kappa / \gamma_{\perp} = 20 >> 1$$

Cavity length decreases from 50λ (curve 1) to λ (curve 6)

Result: The mean power photon number grows 1.5-2.5 times in micro-LED because of the population fluctuations

Population fluctuation effect on Collective Rabi splitting



Dashed curves are without population fluctuations

Superradiant LED

$$2\kappa / \gamma_{\perp} = 20 >> 1$$

Cavity length decreases from 50λ (curve 1) to λ (curve 6)

Result: Population fluctuations **increase** the collective Rabi splitting

Conclusion

An approach based on the spectrum analysis of Heisenberg operator equations is developed for solving nonlinear problems of quantum optics and laser physics

The approach is applied for analysis of super-radiant micro-lasers, where active medium polarization cannot be eliminated adiabatically

New features as collective Rabi splitting, super-thermal photon statistics and the radiation power increase due to population fluctuations have been found.

Several perturbation methods based on the operator Fourier-representation lead to analytical results and forms a background for numerical studies.

The method can be applied to variety of quantum systems as nonlinear optical devices, plasmonic emitters, resonant fluorescence, photonic time crystals.

References

I.E.Protsenko, A.V.Uskov Population fluctuations increase the output of superradiant LEDs (2025) arXiv:2502.04841v1

I.E.Protsenko, A.V.Uskov Population Fluctuation Mechanism of the Super-Thermal Photon Statistic of Quantum LEDs with Collective Effects Annalen der Physik 2400121 (2024) arXiv:2311.15324v3

I.E.Protsenko, A.V.Uskov Oscillator laser model Annalen der Physik **535**, (1) 2200298 (2022) arXiv:2206.05452

I.E.Protsenko, A.V.Uskov Perturbation approach in Heisenberg equations for lasers Phys. Rev. A **105**, 053713 (2022) arXiv:2201.02872v2

I. E. Protsenko, A. V. Uskov, E. C. Andr´e, J. Mørk, and M. Wubs, Quantum langevin approach for superradiant nanolasers, New Journal of Physics **23**, 063010 (2021). arXiv: 2012.02533

E. C. Andre, I. E. Protsenko, A. V. Uskov, J. Mørk, and M. Wubs, On collective Rabi splitting in nanolasers and nano-LEDs, Opt. Lett. **44**, 1415 (2019).

I. Protsenko, P. Domokos, V. Lefevre-Seguin, J. Hare, J. M. Raimond, and L. Davidovich, Quantum theory of a thresholdless laser, Phys. Rev. A **59**, 1667 (1999).