Temporal logics

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Introduction

"In logic, *temporal logic* is any system of rules and symbolism for representing, and reasoning about, propositions qualified in terms of time. It is sometimes also used to refer to *tense logic*, a modal logic-based system of temporal logic introduced by Arthur Prior in the late 1950s, with important contributions by Hans Kamp. It has been further developed by computer scientists, notably Amir Pnueli, and logicians.
Temporal logic has found an important application in formal verification, where it is used to state requirements of hardware or software systems."(Wikipedia)

Introduction

Motivation:

- sentences in natural language depending on time,
- computations depending on time, behavior of intellectual systems,
- processes depending on time in natural sciences (physics, chemistry, biology etc.),
- processes in humanitarian sciences (history, economy),
- theories of time in physics (relativity, quantum mechanics),
- philosophy of time.

These aspects were only underlying the research made by logicians. None of them has been developed in a proper way (except, maybe, computer science applications).

Priorean temporal logics

Temporal propositional formulas are constructed from a countable set of proposition letters $PL = \{p_1, p_1, \ldots\}$ and $\rightarrow, \bot, \Box, \Box^-$:

- $\bot$ ("falsity")
- p_i
- $(A \rightarrow B)$, where A, B are formulas ("if A, then B")
- $\Box A$ (" it will always be the case that A")
- $\Box^- A$ (" it was always the case that A ")

Derived connectives are $\neg, \wedge, \vee, \top, \leftrightarrow, \Diamond, \Diamond^-$:

$$\neg A := (A \rightarrow \bot),\ (A \vee B) := (\neg A \rightarrow B),\ (A \wedge B) := \neg(A \rightarrow \neg B),$$

$$\top := (\bot \rightarrow \bot),\ (A \leftrightarrow B) := ((A \rightarrow B) \wedge (B \rightarrow A)),$$

$$\Diamond A := \neg\Box\neg A \text{ (it will be the case that } A\text{)},$$

$$\Diamond^- A := \neg\Box^-\neg A \text{ (it was the case that } A\text{)}.$$

The original Prior's notation for $\Diamond, \Diamond^-, \Box, \Box^-$ was F, P, G, H.

Priorean temporal logics

A (normal) temporal logic is a set of formulas containing

- Boolean tautologies;
- $\Box(p \to q) \to (\Box p \to \Box q)$;
- $\Box^-(p \to q) \to (\Box^- p \to \Box^- q)$;
- $\Diamond\Box^- p \to p$;
- $\Diamond^-\Box p \to p$;

and closed under the rules

- (MP) $\quad \mathrm{A}, \mathrm{A} \to \mathrm{B} \,/\, \mathrm{B}$;
- (Nec_1) $\quad \mathrm{A} \,/\, \Box\mathrm{A}$;
- (Nec_2) $\quad \mathrm{A} \,/\, \Box^-\mathrm{A}$;
- (Sub) $\quad \mathrm{A} \,/\, \mathrm{SA}$, where S is a propositional substitution.

If L is a temporal logic, A a formula, $L + A$ denotes the smallest temporal logic containing L and A.

Priorean temporal logics

Examples

$\mathbf{K.t}$ is the minimal temporal logic

$\mathbf{K4.t} = \mathbf{K.t} + \Diamond\Diamond p \to \Diamond p$ (transitive time),

$\mathbf{Ver.t} = \mathbf{K.t} + \Box\bot$ (empty time),

$\mathbf{Triv.t} = \mathbf{K.t} + (p \leftrightarrow \Diamond p)$ (trivial time),

$\mathbf{K.t} + \Diamond\top$ (non-ending time),

$\mathbf{D.t} = \mathbf{K.t} + \Diamond\top + \Diamond^{-}\top$ (unbounded time).

Kripke semantics

A temporal frame is a tuple $F = (W, R)$, where
$W \neq \varnothing$ (the set of moments of time),
$R \subseteq W^2$ (the precedence relation, "earlier than").
A Kripke model over F is a pair $M = (F, \theta)$, where $\theta\colon PL \longrightarrow 2^W$ (*valuation*, a function sending proposition letters to subsets of W).

The *truth relation* between points w of a model M and temporal formulas is defined by recursion:

- $M, w \models p_i$ if $w \in \theta(p_i)$;
- $M, w \models \Box A$ if $M, w' \models A$ whenever wRw',
- $M, w \models \Box^- A$ if $M, w' \models A$ whenever $w'Rw$.

A formula A is (globally) *true in* a model M (in symbols, $M \vDash A$) if $M, w \models A$, for every $w \in W$.
A formula A is *valid on* a frame F (in symbols, $F \vDash A$) if $M \vDash A$, for every model M over F.

Kripke semantics

If Γ is a set of formulas, $\mathbf{V}(\Gamma)$ denotes the class of frames validating Γ (*temporal variety*)

Soundness theorem: $\mathbf{V}(\Gamma) = \mathbf{V}(\mathbf{K.t} + \Gamma)$.

If $\mathcal{C}$ is a class of frames, then $\mathbf{L}(\mathcal{C}) := \{A \mid \forall F \in \mathcal{C}\; F \vDash A\}$ is the logic $\mathbf{L}(\mathcal{C})$ is *determined by* $\mathcal{C}$.

A logic is Kripke complete if it is determined by some class of frames. In this case it is also determined by its own variety.

A logic has the finite model property (fmp) if it is determined by a class of finite frames.

Theorem (Harrop)

Every finitely axiomatizable logic with the fmp is decidable.

Some formulas and logics

All the logics mentioned earlier are Kripke complete and have the fmp.

$$\begin{aligned}
det &:= \Diamond p \leftrightarrow \Box p; & det^{-} &:= \Diamond^{-} p \leftrightarrow \Box^{-} p;\\
A3 &:= \Diamond^{-}\Diamond p \to p \vee \Diamond p \vee \Diamond^{-} p; & Ad &:= \Diamond p \to \Diamond\Diamond p;\\
A3^{-} &:= \Diamond\Diamond^{-} p \to p \vee \Diamond p \vee \Diamond^{-} p; \\
AL &:= \Diamond p \to \Diamond(p \wedge \Box\neg p);\\
AL^{-} &:= \Diamond^{-} p \to \Diamond^{-}(p \wedge \Box^{-}\neg p);\\
AZ &:= \Diamond p \to \Box\Diamond p \vee \Diamond(p \wedge \Box\neg p);\\
AZ^{-} &:= \Diamond^{-} p \to \Box^{-}\Diamond^{-} p \vee \Diamond^{-}(p \wedge \Box^{-}\neg p);
\end{aligned}$$

$$AR := \exists p \wedge \exists\neg p \wedge \forall(p \to \Box^{-} p) \to \exists(p \wedge \Box\neg p) \vee \exists(\neg p \wedge \Box^{-} p).$$

In the last formula $\exists A := A \vee \Diamond A \vee \Diamond^{-} A$, $\forall A := A \wedge \Box A \wedge \Box^{-} A$.

SL.t	:=	**K.t** + det + det^{-}	(the logic of Tomorrow and Yesterday)
LIN	:=	**K4.t** + $A3$ + $A3^{-}$	(the logic of linear orders)
LIND	:=	**LIN** + $\Diamond\top$ + $\Diamond^{-}\top$	(the logic of unbounded linear orders)
LQ	:=	**LIND** + Ad	(the logic of rational time)

Some formulas and logics

$$\begin{array}{llll} \mathbf{GL.t} & := & \mathbf{LIN} + AL + AL^{-} & \text{(the logic of finite linear orders);} \\ \mathbf{LZ} & := & \mathbf{LIND} + AZ + AZ^{-} & \text{(the logic of integer time);} \\ \mathbf{LR} & := & \mathbf{LQ} + AR & \text{(the logic of real time).} \end{array}$$

Theorem (Bull – Segerberg)

All the above logics are Kripke complete and have the fmp, except for **LZ**. So they are all decidable

Theorem (folklore)

LZ is decidable.

Since and Until

The language now includes to binary connectives U, S, with the following semantics in Kripke models on linear orders $(W, <)$

$$t \vDash U(A, B) \Leftrightarrow \exists u > t\,(u \vDash A \,\&\, \forall v(t < v < u \Rightarrow v \vDash B),$$

$$t \vDash S(A, B) \Leftrightarrow \exists u < t\,(u \vDash A \,\&\, \forall v(u < v < t \Rightarrow v \vDash B).$$

Then $\Diamond, \Diamond^-$ are expressible:

$$\Diamond A = U(A, \top), \; \Diamond^- A = S(A, \top).$$

There is a *standard translation* $(-)^*(t)$ from US-formulas to classical first-order formulas with the binary predicate symbol $<$, unary predicate symbols P_i and a single parameter t (briefly, $\mathcal{L}$-formulas):

$$p_i^*(t) := P_i(t), \; \bot^*(t) := \bot, \; (A \to B)^*(t) := (A^*(t) \to B^*(t)),$$

$$U(A, B)^*(t) := \exists u > t\,(A^*(u) \wedge \forall v(t < v < u \to B^*(v))),$$

$$S(A, B)^*(t) := \exists u < t\,(A^*(u) \wedge \forall v(u < v < t \to B^*(v))).$$

Since and Until

Expressive Completeness Theorem (Kamp)

On the model $(\mathbf{R}, <)$ every 1-parametric $\mathcal{L}$-formula is equivalent to the standard translation of some US-formula.

Decidability Theorem (Gurevich – Burgess)

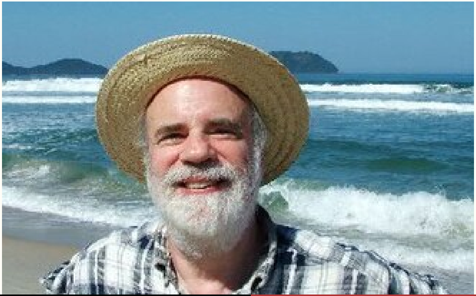
The US-logic of the real line is decidable.

Further directions

- Interval temporal logics
- Logics of branching time
- Relativistic temporal logics
- Many-dimensional temporal logics

Some persons

Saul Kripke (1940 – 2022)



Some persons

William Ockham (1287 – 1347)

Some persons

Charles Peirce (1839 – 1914)

Some persons

Arthur Prior (1914 – 1969)

Some persons

Hans Kamp (born 1940)

Some persons

Krister Segerberg (1936 – 2025)

Some persons

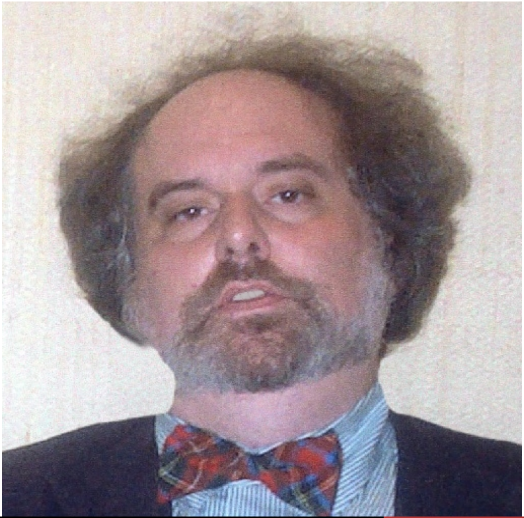
Amir Pnueli (1941 – 2009)

Some persons

Edmund Clarke (1945 – 2020)

Some persons

Ernest Allen Emerson (1954 – 2024)



THANK YOU!