An Epistemic Model with Boundedly-Rational Players*

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Abstract

I propose a general and simple framework for studying epistemic foundations of noncooperative solution concepts: A player faces a strategic situation described by an epistemic picture, which is a set of statements written in a formal language. The player takes his epistemic picture as given; constructs a state space and a set of acts; forms preferences over acts; and selects an optimal act. The player has a finite level of reasoning: Into his state space, he includes all possible states of affairs for opponents with levels below his one.

The object of study is the correspondence between epistemic pictures and behavior they generate. I provide sufficient conditions for the generated behavior to be robust with respect to (i) increasing precision of epistemic pictures; (ii) increasing players' levels of reasoning; (iii) changing extensive-form representation of a given game. Additionally, I re-evaluate epistemic foundations of several key noncooperative solution concepts.

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1.1 Die Welt is die Gesamtheit der Tatsachen, nicht der Dinge.

2.1 Wir Machen uns Bilder der Tatsachen.

Ludwig Wittgenstein[1]

1 Introduction

Over the course of several decades, game theorists have proposed a variety of noncooperative solution concepts. Yet occasionally, those solutions were proposed in a purely mathematical way, without any discussion of whether they are logically consistent, or when they should be applied. To clarify these issues, one may look at epistemic foundations of a given solution: If the solution has any foundation, then it is consistent. Also, the foundation by itself may indicate environments in which the solution is appropriate.

In this paper, I provide a formal model that allows one to study epistemic foundations of noncooperative solution concepts in a simple and transparent way. Specifically, I consider a player who is given an *epistemic picture*, E, that describes a strategic situation he faces. Epistemic picture E is a set of *epistemic statements*, which are written in a formal *epistemic language* (Section 2). Each epistemic statement has the form

the player **epistemic modal** **fact**,

where 'epistemic modal' is the verb for an epistemic modality, and 'fact' is a proposition about the strategic situation. An epistemic modality expresses the player's degree of certainty in that a fact is true. Examples of epistemic modalities are the modalities of knowledge, belief, and assumption.

The player takes epistemic picture E as given, and engages in the following three-step *process of reasoning* (Section 3.1): First, the player constructs a decision problem which consists of a state space and a set of acts. (Acts are also called 'strategies'.) Second, the player forms preferences over acts. Finally, the player selects a strategy that is optimal according to the formed preferences. The model allows for general types of preferences so that the player's behavior can have general *optimality type*. For example, the player's behavior can be maxmin optimal, cautiously optimal, sequentially optimal, and so on.

The central assumption in my model is that the player has a finite *level of reasoning*, $k \in \mathbb{Z}_+$, which is understood as follows. A level-0 player is nonstrategic and can select any available strategy. A level-$k \geqslant 1$ player is strategic: He constructs a decision problem, forms preferences, and selects an optimal act. Level k means that into his state space, the player includes all states of affairs that are possible for his opponents for all of their levels from 0 to $k-1$. The player determines those states through an inductive algorithm which computes all possible behaviors of lower-level opponents.

[1] Wittgenstein (1922): "1.1 The world is the totality of facts, not of things. 2.1 We make to ourselves pictures of facts."

The object of study is the *prediction correspondence*, $\Pi(i, opt, k, E)$ (Section 3.2). The inputs of Π are the player's name i, optimality type *opt*, level of reasoning k, and epistemic picture E. The output is the set of strategies that type-*opt* level-k player i can choose when he is given picture E. Noncooperative solutions are described by epistemic pictures that generate the required behavior through correspondence Π. Epistemic pictures are viewed as epistemic foundations for solutions they generate.

I present my main results in Section 4, where I provide sufficient conditions on inputs of correspondence Π that guarantee that the generated prediction is *epistemically robust* in one of the following three senses: The prediction is monotonically decreasing in the precision of the player's epistemic picture (Section 4.1). The prediction is monotonically decreasing in the player's level of reasoning (Section 4.2). The prediction is invariant with respect to a particular extensive-form representation of the studied dynamic game (Section 4.3).

In Section 5, I assess epistemic foundations of key noncooperative solution concepts: In Section 5.1, I analyze properties of several rationalizability procedures. In Section 5.2, I critically re-evaluate the epistemic conditions for Nash equilibrium proposed by Aumann and Brandenburger (1995). In Section 5.3, I reconsider the issues with common knowledge of admissibility reported by Samuelson (1992).

1.1 Simple Example

To get an idea about how my model works, consider the following simple example: Two players, Ann and Bob, play the following static game, G:

	b_1	b_2
a_1	$3,1$	$1,0$
a_2	$2,2$	$3,1$
a_3	$1,0$	$2,3$

I will use the following definitions of rationality and optimality: A player, whose level of reasoning is k, views her opponent as *rational* if she thinks that the opponent's level of reasoning is precisely $k-1$, the highest level whose behavior the player can still analyze. A player's behavior is *optimal* if she forms a probabilistic belief on her subjective state space, and selects a strategy that has the highest expected payoff computed under that belief.

With the above definitions, suppose that Ann holds an epistemic picture, E, which consists of (i) common knowledge that G is played; (ii) common knowledge of rationality; and (iii) common knowledge of optimality. What can one say about Ann's play provided that her level of reasoning is $k \in \mathbb{N}$ and her behavior is optimal?

To answer this question for each $k \in \mathbb{N}$, one proceeds inductively. Suppose first that Ann's level of reasoning is 1, so that she only considers possibilities in which Bob's level is 0. Her process of reasoning is the following: At the first stage, she constructs to herself the decision problem that corresponds to picture E. As Ann knows that the played game is G, her state space Ω consists of the following two states: $\omega_1 :=$ 'Bob's level is 0 and he plays b_1' and $\omega_2 :=$ 'Bob's level is 0 and he plays b_2'. Ann has three acts that give the following payoffs in states $\{\omega_1, \omega_2\}$: act a_1 pays $(3, 1)$, act a_2 pays $(2, 3)$, and act a_3 pays $(1, 2)$. At the second stage, Ann forms a belief on Ω. At the third stage, she selects an act that maximizes the expected payoff against that belief. This means that level-1 Ann will play either strategy a_1 or a_2. She will not play strategy a_3 as it is strictly dominated by strategy a_2.

Suppose now that Ann's level of reasoning is 2, so that she considers possibilities in which Bob's level is either 0 or 1. Her reasoning is the following: At the first stage, she constructs the decision problem in which the state space, Ω, is a subset of $\{\omega_1, \omega_2, \omega_3, \omega_4\}$, where ω_1 and ω_2 are the same as above, ω_3 :='Bob's level is 1, he knows G is played, his behavior is optimal, and he plays b_1' and ω_4 :='Bob's level is 1, he knows G is played, his behavior is optimal, and he plays b_2'. As Ann knows that Bob is rational, she excludes states ω_1 and ω_2. As both b_1 and b_2 can be optimally played by a level-1 Bob, level-2 Ann concludes that $\Omega = \{\omega_3, \omega_4\}$. Again, she has three acts that give the following payoffs in states $\{\omega_3, \omega_4\}$: act a_1 pays $(3, 1)$, act a_2 pays $(2, 3)$, and act a_3 pays $(1, 2)$. Thus, level-2 Ann will play either strategy a_1 or a_2.

Suppose now that Ann's level of reasoning is 3, so that she considers possibilities in which Bob's level is either 0, 1, or 2. Her reasoning is the following: At the first stage, she constructs the decision problem in which the state space, Ω, is a subset of $\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$, where ω_1, ω_2, ω_3, and ω_4 are the same as above, ω_5 :='Bob's level is 2, he knows that [G is played; Ann's level is 1, she knows G is played, and her behavior is optimal], his behavior is optimal, and he plays b_1' and ω_6 :='Bob's level is 2, he knows that [G is played; Ann's level is 1, she knows G is played, and her behavior is optimal], his behavior is optimal, and he plays b_2'. As Ann knows that Bob is rational, she excludes states ω_1, ω_2, ω_3, and ω_4. Further, Ann analyzes level-2 Bob who knows that [G is played; Ann's level is 1, she knows G is played, and her behavior is optimal]. Such Bob concludes that Ann will play either strategy a_1 or a_2, but never a_3. As strategy b_2 is strictly dominated by b_1 against strategies $\{a_1, a_2\}$, such Bob will optimally play b_1, but not b_2. Thus, Ann concludes that $\Omega = \{\omega_5\}$. Her acts pay the following payoffs in state ω_5: act a_1 pays 3, act a_2 pays 2, and act a_3 pays 1. Thus, level-3 Ann will play strategy a_1.

For levels $k > 3$, the argument is analogous: Under epistemic picture E, level-k Ann concludes that level-$(k-1)$ Bob will play strategy b_1. As a result, she will play strategy a_1.

The above argument can be used in any finite static game. The predictions that the argument generates correspond to the rationalizability procedure proposed by Bernheim (1984) and Pearce (1984). In this sense, epistemic picture E provides an epistemic foundation for the rationalizability procedure in finite static games. Yet the argument is not completely formal. For example, I did not define 'common knowledge', or did not explain what it means that 'a player knows that game G is played'. In this paper, I formalize all these and many other statements, and provide an inductive algorithm that captures the reasoning of a boundedly rational player who possesses a completely general epistemic picture.

1.2 Related Literature

Compared with the existent epistemic frameworks, my model has a clear advantage in that it *simultaneously* has the following attractive properties:

P1. Bayesian: A player constructs to himself a decision problem in the spirit of Savage (1954): In that problem, strategies are acts that are separated from the state space of uncertainty.

P2. Finite: Given a finite epistemic picture, a player constructs a finite state space.

P3. Inductive: A player reasons through an inductive recursive algorithm. Consequently, theorems can be proven by induction, rather than by fixed-point arguments.

P4. Universal: The model allows for general types of preferences, epistemic modalities, and facts.

P5. Direct: Results are expressed directly in terms of statements given to a player, rather than in terms of esoteric objects such as common priors, information partitions, belief-complete type structures, and so on.

Other epistemic frameworks fail to satisfy some of the above properties. Indeed, the literature on epistemic game theory contains a vast array of epistemic models. Most of these models, however, fall into one of the following two categories:

First, there are papers that develop *the partition model*. Some of the prominent examples are Aumann (1976, 1987, 1999a,b), Samuelson (1992), Aumann and Brandenburger (1995), and Feinberg (2000, 2005a,b). The advantage of the partition model is that it is finite and universal. The disadvantage is that it is not Bayesian, inductive, or direct. One of the reasons why the partition model is not Bayesian is that it puts implicit equilibrium restrictions on players' beliefs through the objective state space: See the discussion at the end of Section 5.2 for more details.

Second, there are papers that advance *the type-structure model*. The necessarily incomplete list includes Harsanyi (1967), Mertens and Zamir (1985), Brandenburger and Dekel (1993), Epstein and Wang (1996), and Battigalli and Siniscalchi (1999). A great and detailed exposition of the type-structure model is provided in Dekel and Sinischalchi (2014). The advantage of the type-structure model is that it is Bayesian as it separates a player's strategy from his type that encodes a belief about opponents' types and strategies. The disadvantage of the type-structure model is that it is neither finite, inductive, universal, nor direct. One of the reasons why the type-structure model is not universal is that this epistemic model does not permit the modality of knowledge.

The idea of looking at players with bounded levels of reasoning is by no means new. For instance, Battigalli (1996) talks about different rationality levels of players' strategies in dynamic games, and uses this idea to conveniently reformulate the extensive-form rationalizability procedure proposed by Pearce (1984). In the epistemic context, Kets (2017) incorporates players with bounded levels of reasoning into the type-structure model. The vast experimental literature has explored various ways to model bounded rationality. From that literature, my model is related to the cognitive-hierarchy model developed by Camerer et al. (2004).

Finally, my model combines some ideas expressed in Wittgenstein (1922) and Savage (1954). From Wittgenstein (1922), I borrow the ideas (i) that a state of affairs is a combination of truth values of non divisible facts (a state is 'a picture of facts'), and (ii) that a subjective player constructs to himself such states. From Savage (1954), I take the idea that a player's choice should be independent from the external world represented by a state space.

2 Epistemic Language

A player's epistemic picture is a set of epistemic statements written in an epistemic language. In this paper, I primarily deal with two epistemic languages: the most simple language, $\mathcal{L}_0$, and the extended language, $\mathcal{L}_1$. I first describe language $\mathcal{L}_0$. In Section 2.3, I extend it to language $\mathcal{L}_1$.

2.1 Language $\mathcal{L}_0$

Epistemic language $\mathcal{L}_0 \coloneqq \{\mathcal{T}_0, \mathcal{F}_0, \mathcal{M}_0, \mathcal{R}_0\}$ is given by a set of *symbols for things*, $\mathcal{T}_0$; a set of *atomic facts*, $\mathcal{F}_0$; a set of *epistemic modalities*, $\mathcal{M}_0$; and a set of *grammar rules*, $\mathcal{R}_0$. I now describe the constituents of language $\mathcal{L}_0$ in more detail.

(i) Symbols for Things. The set of symbols for things, $\mathcal{T}_0 \coloneqq \{\mathcal{I}, \mathcal{S}\}$, consists of infinites sets of *player symbols*, $\mathcal{I}$, and *strategy symbols*, $\mathcal{S}$. In general, sets $\mathcal{I}$ and $\mathcal{S}$ can be arbitrarily large. To describe players' behavior in finite games, it suffices to use countably-infinite sets of symbols.

(ii) Atomic Facts. Atomic facts are primitive statements about things in $\mathcal{T}_0$. Each atomic fact $f \in \mathcal{F}_0$ has the following two characteristics: (i) The *level* of f, denoted $Lev(f)$. Statements that include atomic fact f will be analyzed by a player only if his level of reasoning exceeds $Lev(f)$. (ii) The *subjects* of f, denoted $Subj(f)$. Intuitively, the subjects of an atomic fact are the players who can affect the truth value of the fact.

I now list all atomic facts in language $\mathcal{L}_0$ together with their characteristics. Atomic facts in $\mathcal{L}_0$ belong to one of the following five categories:

F1. Levels of reasoning: $\forall i \in \mathcal{I}$, $\forall k \in \mathbb{Z}_+$, statement $l_{i,k} \coloneqq$'player i's level of reasoning is k' is an atomic fact with $Lev(l_{i,k}) \coloneqq k$ and $Subj(l_{i,k}) \coloneqq \{i\}$.

F2. Rationality: $\forall i \in \mathcal{I}$, statement $r_i \coloneqq$'player i is rational' is an atomic fact with $Lev(r_i) \coloneqq 0$ and $Subj(r_i) \coloneqq \{i\}$.

F3. Optimality: $\forall i \in \mathcal{I}$, statements $o_i \coloneqq$'player i's behavior is optimal' and $mo_i \coloneqq$'player i's behavior is maxmin optimal' are atomic facts with $Lev(0_i) = Lev(mo_i) \coloneqq 0$ and $Subj(o_i) = Subj(mo_i) \coloneqq \{i\}$.

F4. Strategies: $\forall i \in \mathcal{I}$, $\forall s \in \mathcal{S}$, statement $st_{i,s} \coloneqq$'player i plays strategy s' is an atomic fact with $Lev(st_i) \coloneqq 0$ and $Subj(st_{i,s}) \coloneqq \{i\}$.

F5. Payoffs: $\forall i \in \mathcal{I}$, $\forall g \in \mathbb{R}$, statement $u_{i,g} :=$'player i receives g units of utility' is an atomic fact with $Lev(u_{i,g}) := 0$ and $Subj(u_{i,g}) := \mathcal{I}$.

(iii) Epistemic Modalities. An epistemic modality expresses a player's degree of certainty in that a fact is true. Language $\mathcal{L}_0$ includes the following epistemic modalities: (i) *knowledge*, denoted K; and (ii) for $p \in [0,1]$, *p-belief*, denoted B_p. That is, $\mathcal{M}_0 := \{K, \{B_p\}_{p\in[0,1]}\}$. Using the usual convention, for $i \in \mathcal{I}$ and $M \in \mathcal{M}_0$, I will write M^i in place of the phrase 'player i verb M that'. For example, K^1 should be read as 'player 1 knows that'.

(iv) Grammar. I now describe $\mathbb{S}_0$, the set of *statements* that can be formed in language $\mathcal{L}_0$. Statements in $\mathbb{S}_0$ are finite statements that can be written using atomic facts, logical connectives, brackets, and applications of epistemic modalities. Each statement $P \in \mathbb{S}_0$ has three characteristics: (i) The *level* of P, denoted $Lev(P)$. Statement P will be analyzed by a player only if his level of reasoning is weakly above $Lev(P)$. (ii) The *variables* of P, denoted $Var(P)$, which is the set of non divisible facts whose truth values determine the truth value of P. (iii) The *subjects* of P, denoted $Subj(P)$, which is the set of players that affect the truth value of P. The statements in $\mathbb{S}_0$ and their characteristics are completely determined by the following six grammar rules, $\mathcal{R}_0$:

Rule 1 (Atomic Facts). *Each atomic fact $f \in \mathcal{F}_0$ is a statement in $\mathbb{S}_0$ with $Var(f) := \{f\}$, and $Lev(f)$ and $Subj(f)$ being the level and the subjects of atomic fact f.*

Rule 2 (Brackets). *If $P \in \mathbb{S}_0$, then $(P) \in \mathbb{S}_0$ with $Lev\big((P)\big) := Lev(P)$, $Var\big((P)\big) := Var(P)$, and $Subj\big((P)\big) := Subj(P)$.*

Rule 3 (Simple Negation). *If $P \in \mathbb{S}_0$ and $Var(P) \subset \mathcal{F}_0$, then $\neg P \in \mathbb{S}_0$ with $Lev(\neg P) := Lev(P)$, $Var(\neg P) := Var(P)$, and $Subj(\neg P) := Subj(P)$.*

Rule 4 (Monotone Connectives). *If $P \in \mathbb{S}_0$ and $Q \in \mathbb{S}_0$, then $P \wedge Q \in \mathbb{S}_0$ and $P \vee Q \in \mathbb{S}_0$ with $Lev(P \wedge Q) = Lev(P \vee Q) := \max\{Lev(P), Lev(Q)\}$, $Var(P \wedge Q) = Var(P \vee Q) := Var(P) \cup Var(Q)$, and $Subj(P \wedge Q) = Subj(P \vee Q) := Subj(P) \cup Subj(Q)$.*

Remark 1. If a statement in $\mathbb{S}_0$ contains a chain of sub-statements connected without brackets, then one should parse it using the following priority of operations: from $\neg$, to $\wedge$, to $\vee$. For example, expression $a \vee \neg\neg b \wedge c$ should be parsed as $a \vee \Big(\big(\neg(\neg b)\big) \wedge c\Big)$.

Applying Rules 1-4, one can write all statements in $\mathbb{S}_0$ whose variables are atomic facts. In particular, one can write the rules of finite static games using statements of the following two kinds:

G1. Strategy Rules: $\forall i \in \mathcal{I}$, and for each nonempty finite $S \subset \mathcal{S}$,

$$ST_{i,S} := \bigvee_{s\in S} \big(st_{i,s} \bigwedge_{\tilde{s}\in S\setminus\{s\}} \neg st_{i,\tilde{s}}\big) \tag{1}$$

is the statement that 'player i plays exactly one strategy from set S'. The characteristics of $ST_{i,S}$ are $Lev(ST_{i,S}) = 0$, $Var(ST_{i,S}) = \{st_{i,s}\}_{s\in S}$, and $Subj(ST_{i,S}) = \{i\}$.

G2. Payoff Rules: $\forall i \in \mathcal{I}$, for each nonempty finite $I \subset \mathcal{S}$, for each map $\hat{s} : I \to \mathcal{S}$, and $\forall g \in \mathbb{R}$,

$$U_{i,I,\hat{s},g} \coloneqq \neg\big(\bigwedge_{j\in I} st_{j,\hat{s}(j)}\big) \vee u_{i,g} \tag{2}$$

is the statement that 'player i will receive g units of utility if players I play strategy profile $\{\hat{s}(j)\}_{j\in I}$'. The characteristics of $U_{i,I,\hat{s},g}$ are $Lev(U_{i,I,\hat{s},g}) = 0$, $Var(U_{i,I,\hat{s},g}) = \{st_{j,\hat{s}(j)}\}_{j\in I} \cup \{u_{i,g}\}$, and $Subj(U_{i,I,\hat{s},g}) = \mathcal{I}$.

I will denote by $\aleph_0$ the set of strategy and payoff rules. That is, $\aleph_0$, is the union of all the statements that have either form (1) or (2).

Example 1 (Rules of a Static Game)**.** Let G be the prisoner's dilemma given by the payoff matrix

	$s_3 = C$	$s_4 = D$
$s_1 = C$	$2, 2$	$0, 3$
$s_2 = D$	$3, 0$	$1, 1$

Associating the first and the second player with player symbols 1 and 2, strategies C and D of the first player with strategy symbols s_1 and s_2, and strategies C and D of the second player with strategy symbols s_3 and s_4, one writes the *rules of game* G, denoted $\langle G\rangle$, as the following list of statements from $\aleph_0$:

$$\langle G\rangle \coloneqq \{ST_{1,\{s_1,s_2\}}, ST_{2,\{s_3,s_4\}},\\ U_{1,\{1,2\},\{s_1,s_3\},2}, U_{1,\{1,2\},\{s_1,s_4\},0}, U_{1,\{1,2\},\{s_2,s_3\},3}, U_{1,\{1,2\},\{s_2,s_4\},1},\\ U_{2,\{1,2\},\{s_1,s_3\},2}, U_{2,\{1,2\},\{s_1,s_4\},3}, U_{2,\{1,2\},\{s_2,s_3\},0}, U_{2,\{1,2\},\{s_2,s_4\},1}\}.$$

That is, rules $\langle G\rangle$ is the list of ten level-0 statements. The first two statements in $\langle G\rangle$ are strategy rules: they define the sets of strategies for players 1 and 2. The remaining eight statements in $\langle G\rangle$ are payoff rules: four statements for each player that define their payoff functions.

Other statements in $\mathbb{S}_0$ are obtained through applications of epistemic modalities. Normally, epistemic modalities should be non-introspective: a player's modality should not be applied to a statement in which the player is among subjects. The only exception is the knowledge modality that can be introspectively applied to the rules of a played game. The following two grammar rules govern the application of epistemic modalities in language $\mathcal{L}_0$:

Rule 5 (Non-Introspective Modality)**.** *If $P \in \mathbb{S}_0$, $i \in \mathcal{I}\backslash Subj(P)$, and $M \in \mathcal{M}_0$, then $M^i(P) \in \mathbb{S}_0$ with $Lev\big(M^i(P)\big) \coloneqq Lev(P) + 1$, $Var\big(M^i(P)\big) \coloneqq \{M^i(P)\}$, and $Subj\big(M^i(P)\big) \coloneqq \{i\}$.*

Rule 6 (Introspective Knowledge)**.** *If $\alpha \in \aleph_0$, $i \in \mathcal{I}$, then $K^i(\alpha) \in \mathbb{S}_0$ with $Lev\big(K^i(\alpha)\big) \coloneqq 1$, $Var\big(K^i(\alpha)\big) \coloneqq \{K^i(\alpha)\}$, and $Subj\big(K^i(\alpha)\big) \coloneqq \{i\}$.*

Rules 1-6 define statements in $\mathbb{S}_0$ together with their characteristics. Rules 1-6 are exhaustive: nothing else is a statement in $\mathbb{S}_0$.

Remark 2. Characteristics of statements in $\mathbb{S}_0$ are defined in a purely syntactic way. As a result, they do not always agree with the classical semantics. For example, the characteristics of statement $P = (l_{i,k} \vee \neg l_{i,k})$ are $Lev(P) = k$, $Var(P) = \{l_{i,k}\}$, and $Subj(P) = \{i\}$ rather than 0, $\varnothing$, and $\varnothing$.

Remark 3. The definition of rules $\aleph_0$ is syntactic and strict: the order of terms and brackets matters. For example, $(st_{i,s_1} \wedge \neg st_{i,s_2}) \vee (st_{i,s_2} \wedge \neg st_{i,s_1}) \in \aleph_0$, whereas $(st_{i,s_1} \wedge \neg st_{i,s_2}) \vee (\neg st_{i,s_1} \wedge st_{i,s_2}) \notin \aleph_0$ and $\big((st_{i,s_1} \wedge \neg st_{i,s_2}) \vee (st_{i,s_2} \wedge \neg st_{i,s_1})\big) \notin \aleph_0$.

Example 2. The following statements belong to $\mathbb{S}_0$:

1. $P_1 = l_{1,2} \wedge \neg(r_1 \vee \neg r_1)$ is the statement that '(player 1's level of reasoning is 2) and $\big($it is not the case that $\big($(player 1 is rational) or (it is not the case that player 1 is rational)$\big)\big)$'. The characteristics of statement P_1 are $Lev(P_1) = 2$, $Var(P_1) = \{l_{1,2}, r_1\}$, and $Subj(P_1) = \{1\}$.

2. $P_2 = B^3_{0.75}(st_{1,s_1} \wedge \neg r_2)$ is the statement that 'player 3 believes that with probability at least 75% $\big($(player 1 plays strategy s_1) and (it is not that the case that player 2 is rational)$\big)$'. The characteristics of statement P_2 are $Lev(P_2) = 1$, $Var(P_2) = \{P_2\}$, and $Subj(P_2) = \{3\}$.

3. $P_3 = o_2 \vee K^1(ST_{1,\{s_1,s_2\}})$ is the statement that '(player 2's behavior is optimal) or $\big($player 1 knows that (player 1 must play one of the strategies s_1 or s_2)$\big)$'. The characteristics of statement P_3 are $Lev(P_3) = 1$, $Var(P_3) = \{o_2, K^1(ST_{1,\{s_1,s_2\}})\}$, and $Subj(P_3) = \{1, 2\}$.

4. $P_4 = K^3(l_{2,3}) \vee B^2_1(K^1(ST_{1,\{s_1,s_2\}}))$ is the statement that '(player 3 knows that player 2's level of reasoning is 3) or $\big($player 2 believes that with probability 1 (player 1 knows that (player 1 must play one of the strategies s_1 or s_2))$\big)$'. The characteristics of statement P_4 are $Lev(P_4) = 4$, $Var(P_4) = \{K^3(l_{2,3}), B^2_1(K^1(ST_{1,\{s_1,s_2\}}))\}$, and $Subj(P_4) = \{2, 3\}$.

The following statements *do not* belong to $\mathbb{S}_0$:

5. $P_5 = B^1_1\big(o_2 \wedge mo_1\big)$ is the statement that 'player 1 believes that with probability 1 $\big($ (player 2's behavior is optimal) and (player 1's behavior is maxmin optimal)$\big)$'. Statement P_5 is not possible in language $\mathcal{L}_0$: In P_5, the belief modality of player 1 is applied introspectively to a statement whose subjects are $\{1, 2\}$ – this violates Rule 5.

6. $P_6 = K^1\big(r_2 \wedge ST_{1,\{s_1,s_2\}}\big)$ is the statement that 'player 1 knows that $\big($(player 2 is rational) and (player 1 must play one of the strategies s_1 or s_2)$\big)$'. Statement P_6 is not possible in language $\mathcal{L}_0$: In P_6, the knowledge modality of player 1 is applied introspectively to the statement $r_2 \wedge ST_{1,\{s_1,s_2\}}$, which is not a statement from $\aleph_0$ – this violates Rule 6. Instead of P_6, one can write two separate statements, $K^1(r_2)$ and $K^1(ST_{1,\{s_1,s_2\}})$, each of which is possible in $\mathcal{L}_0$ and which together express P_6.

An important class of statements are *epistemic statements* which are produced via Rule 5:

Definition (Epistemic Statement). *A statement $e \in \mathbb{S}_0$ is an epistemic statement of player $i \in \mathcal{I}$ if $e = M^i(P)$ for some epistemic modality $M \in \mathcal{M}_0$ and statement $P \in \mathbb{S}_0$. The pre-variables of epistemic statement $e = M^i(P)$, denoted $Var_{-1}(e)$, are the variables of P:*

$$Var_{-1}(e) := Var(P).$$

The set of epistemic statements of player i in language $\mathcal{L}_0$ is denoted $\mathbb{E}_0^i$.

Each statement $P \in \mathbb{S}_0$ is a propositional formula of its variables, $Var(P)$. Each variable in $Var(P)$ is either an atomic fact or an epistemic statement. Denote by $\bar{P} : \{\top, \bot\}^{Var(P)} \to \{\top, \bot\}$ the classical predicate associated with statement P. I will say that statement P expresses predicate $\bar{P}$. The following proposition characterizes non-trivial predicates that can be expressed in language $\mathcal{L}_0$:

Proposition 1. *A non-constant predicate $\bar{P}[\bar{F}, \bar{E}]$ over truth values of atomic facts $F \subset \mathcal{F}_0$ and epistemic statements $E \subset \bigcup_{i \in \mathcal{I}} \mathbb{E}_0^i$ can be expressed in $\mathcal{L}_0$ if and only if $\bar{P}[\bar{F}, \bar{E}]$ is increasing in $\bar{E}$.*

Proof. See Appendix A.1. □

2.2 Epistemic Pictures

I now introduce the concept of an epistemic picture, one of the key concepts in my model.

Definition (Epistemic Picture). *An epistemic picture, E^i, of player $i \in \mathcal{I}$ is a set of his epistemic statements, $E^i \subseteq \mathbb{E}_0^i$. The pre-variables of epistemic picture E^i, denoted $Var_{-1}(E^i)$, is the union of pre-variables of statements in E^i:*

$$Var_{-1}(E^i) := \bigcup_{e \in E^i} Var_{-1}(e).$$

Definition (Truncation). *For $k \in \mathbb{N}$, the k-th truncation of an epistemic picture $E^i \subseteq \mathbb{E}_0^i$, denoted $E^i_{\leqslant k}$, is the set of statements in E^i whose levels do not exceed k: $E^i_{\leqslant k} := \left\{e \in E^i \,\middle|\, Lev(e) \leqslant k\right\}$.*

Truncations of epistemic pictures play an important role in the reasoning of players in my model: In his process of reasoning, a level-k player who has an epistemic picture E will analyze only the statements from the corresponding truncation, $E_{\leqslant k}$. That player can not compute the consequences of the higher-level statements in $E^i \backslash E^i_{\leqslant k}$, and so he will just dismiss them.

Definition (Finite Epistemic Picture). *An epistemic picture $E^i \subset \mathbb{E}_0^i$ is finite if all of its truncations are finite sets: $\forall k \in \mathbb{N}, |E^i_{\leqslant k}| < \infty$.*

Finite epistemic pictures are especially convenient to work with: A player with a finite epistemic picture constructs a finite state-space during his process of reasoning. In what follows, all epistemic pictures are assumed to be finite. The following epistemic pictures are particularly important:

Example 3 (Common Knowledge/p-Belief)**.** Let $I \subseteq \mathcal{I}$, $I \neq \varnothing$, and $P \in \mathbb{S}_0$. The phrase 'P is common knowledge among players I' translates into $\mathcal{L}_0$ as the following list of statements:

$$CK^I(P) \coloneqq \big(\bigcup_{n\in\mathbb{N};\, i_1,i_2,..,i_n\in I;} \{K^{i_1}(K^{i_2}(..K^{i_n}(P)..))\}\big) \cap \mathbb{S}_0.$$

The intersection with $\mathbb{S}_0$ is taken to exclude statements that violate Rules 5-6.

The phrase 'P is common p-belief among players I' translates as the following list of statements:

$$CB_p^I(P) \coloneqq \big(\bigcup_{n\in\mathbb{N};\, i_1,i_2,..,i_n\in I} \{B_p^{i_1}(B_p^{i_2}(..B_p^{i_n}(P)..))\}\big) \cap \mathbb{S}_0.$$

For $i \in I$, the phrase 'player i thinks that P is common knowledge among players I' is the following epistemic picture:

$$CK^{i,I}(P) \coloneqq \big\{e \in CK^I(P) \,\big|\, Subj(e) = \{i\}\big\}.$$

The phrase 'player i thinks that P is common p-belief among I' is the following epistemic picture:

$$CB_p^{i,I}(P) \coloneqq \big\{e \in CB_p^I(P) \,\big|\, Subj(e) = \{i\}\big\}.$$

If I is a finite set, then $CK^{i,I}(P)$ and $CB_p^{i,I}(P)$ are finite epistemic pictures.

Example 4 (Knowledge/Common Knowledge of the Game)**.** Let G be a finite static game played by a nonempty set of players $I \subseteq \mathcal{I}$. Let $\langle G\rangle \subset \aleph_0$ be the rules of G (Example 1). For $i \in I$, the phrase 'player i know that he plays game G' is the following epistemic picture:

$$K^i\langle G\rangle \coloneqq \bigcup_{\alpha\in\langle G\rangle} K^i(\alpha).$$

The phrase 'player i thinks that it is common knowledge that players I play game G' is the following epistemic picture:

$$CK^{i,I}\langle G\rangle \coloneqq \bigcup_{\alpha\in\langle G\rangle} CK^{i,I}(\alpha).$$

For a finite static game G, epistemic pictures $K^i\langle G\rangle$ and $CK^{i,I}\langle G\rangle$ are finite.

Example 5 (Common Knowledge of/p-Belief in Rationality/Optimality)**.** Let $I \subseteq \mathcal{I}$, $I \neq \varnothing$. For $i \in I$, the phrase 'player i thinks that it is common knowledge among players I that they all are rational' is the following epistemic picture:

$$CK^{i,I}[R] \coloneqq \bigcup_{j\in I} CK^{i,I}(r_j).$$

For $p \in [0,1]$, 'player i thinks that it is common p-belief among players I that they all are rational' is the following epistemic picture:

$$CB_p^{i,I}[R] := \bigcup_{j\in I} CB_p^{i,I}(r_j).$$

The corresponding epistemic pictures for optimality are analogous: $CK^{i,I}[O] := \bigcup_{j\in I} CK^{i,I}(o_j)$; $CK^{i,I}[MO] := \bigcup_{j\in I} CK^{i,I}(mo_j)$; $CB_p^{i,I}[O] := \bigcup_{j\in I} CB_p^{i,I}(o_j)$; $CB_p^{i,I}[MO] := \bigcup_{j\in I} CB_p^{i,I}(mo_j)$. If I is a finite set, then all of these epistemic pictures are finite.

2.3 Language $\mathcal{L}_1$

I now introduce epistemic language $\mathcal{L}_1 := \{\mathcal{T}_1, \mathcal{F}_1, \mathcal{M}_1, \mathcal{R}_1\}$ that allows for describing behavior in dynamic games (without chance nodes). Language $\mathcal{L}_1$ is an extension of language $\mathcal{L}_0$ obtained in the following way:

(i) Symbols for Things. Symbols for things $\mathcal{T}_1$ contain $\mathcal{T}_0$ and an additional infinite set of *information symbols*, $\mathcal{H}$.

(ii) Atomic Facts. Atomic facts $\mathcal{F}_1$ contain $\mathcal{F}_0$ and additional facts that belong to one of the following categories:

F6. Optimality: $\forall i \in \mathcal{I}$, statements co_i :='player i's behavior is cautiously optimal', so_i :='player i's behavior is sequentially optimal', and cso_i :='player i's behavior is cautiously sequentially optimal' are atomic facts with $Lev(co_i) = Lev(so_i) = Lev(cso_i) := 0$ and $Subj(co_i) = Subj(so_i) = Subj(cso_i) := \{i\}$.

F7. Information: $\forall i \in \mathcal{I}$, $\forall h \in \mathcal{H}$, statement $f_{i,h}$:='player i receives information h' is an atomic fact with $Lev(f_{i,h}) := 0$ and $Subj(f_{i,h}) := \{i\}$

(iii) Epistemic Modalities. Epistemic modalities $\mathcal{M}_1$ contain $\mathcal{M}_0$ and additional *assumption* modality, denoted A.

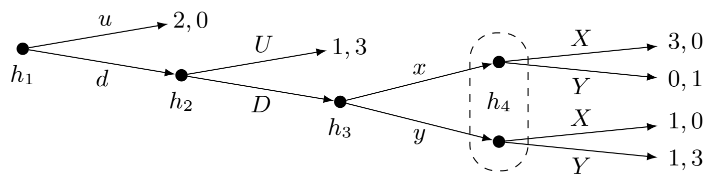
(iv) Grammar. The set of statements possible in language $\mathcal{L}_1$, denoted $\mathbb{S}_1$, is determined by the following six grammar rules, $\mathcal{R}_1$:

Rule 1′ (Atomic Facts)**.** *Each atomic fact $f \in \mathcal{F}_1$ is a statement in $\mathbb{S}_1$ with $Var(f) := \{f\}$, and $Lev(f)$ and $Subj(f)$ being the level and the subjects of atomic fact f.*

Rule 2′ (Brackets)**.** *If $P \in \mathbb{S}_1$, then $(P) \in \mathbb{S}_1$ with $Lev\big((P)\big) := Lev(P)$, $Var\big((P)\big) := Var(P)$, and $Subj\big((P)\big) := Subj(P)$.*

Rule 3′ (Negation)**.** *If $P \in \mathbb{S}_1$, then $\neg P \in \mathbb{S}_1$ with $Lev(\neg P) := Lev(P)$, $Var(\neg P) := Var(P)$, and $Subj(\neg P) := Subj(P)$.*

Rule 4′ (Connectives)**.** *If $P \in \mathbb{S}_1$, $Q \in \mathbb{S}_1$, and $c \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$, then $P\,c\,Q \in \mathbb{S}_1$ with $Lev(P\,c\,Q) := \max\{Lev(P), Lev(Q)\}$, $Var(P\,c\,Q) := Var(P) \cup Var(Q)$, and $Subj(P\,c\,Q) := Subj(P) \cup Subj(Q)$.*



(a) Dynamic game Γ

	$s_4 = U$	$s_5 = DX$	$s_6 = DY$
$s_1 = u$	$2, 0$	$2, 0$	$2, 0$
$s_2 = dx$	$1, 3$	$3, 0$	$0, 1$
$s_3 = dy$	$1, 3$	$1, 0$	$1, 3$

(b) Normal form $G(\Gamma)$

Figure 1: Example 6

Remark 4. If a statement in $\mathbb{S}_1$ contains a chain of sub-statements connected without brackets, then one should parse it using the following priority of operations: from $\neg$, to $\leftrightarrow$, to $\rightarrow$, to $\wedge$, to $\vee$. Moreover, a chain of implications, $\rightarrow$, should be parses from left to right. For example, expression $a \rightarrow b \rightarrow \neg c \leftrightarrow d \vee e \wedge g$ should be parsed as $\Big((a \rightarrow b) \rightarrow \big((\neg c) \leftrightarrow d\big)\Big) \vee (e \wedge g)$.

Rule 5′ (Non-Introspective Modality)**.** *If $P \in \mathbb{S}_1$, $i \in \mathcal{I} \backslash Subj(P)$, and $M \in \mathcal{M}_1$, then $M^i(P) \in \mathbb{S}_1$ with $Lev\big(M^i(P)\big) := Lev(P) + 1$, $Var\big(M^i(P)\big) := \{M^i(P)\}$, and $Subj\big(M^i(P)\big) := \{i\}$.*

To describe rules of dynamic games, additionally to statements $\aleph_0$, language $\mathcal{L}_1$ includes the following two types of rules:

G3. Relevance Rules: $\forall i \in \mathcal{I}$, $\forall h \in \mathcal{H}$, and for each nonempty finite $S \subset \mathcal{S}$,

$$R_{i,h,S} := f_{i,h} \rightarrow \ \big(\bigvee_{s \in S} st_{i,s}\big) \tag{3}$$

is the statement that 'if player i receives information h then he plays a strategy from S'. Alternatively, 'set S is the set of player i's strategies for which his information set h is relevant'.

G4. Information Rules: $\forall i \in \mathcal{I}$, $\forall h \in \mathcal{H}$, for each nonempty finite $J \subset \mathcal{S} \backslash \{i\}$, for each nonempty finite $N \subset \mathbb{N}$, and for each map $\hat{s} : J \times N \rightarrow \mathcal{S}$,

$$H_{i,h,J,N,\hat{s}} := f_{i,h} \rightarrow \big(\bigvee_{n \in N} \bigwedge_{j \in J} st_{j,\hat{s}(j,n)}\big) \tag{4}$$

is the statement that 'if player i receives information h then his opponents J play a strategy profile $\{\hat{s}(j,n)\}_{j \in J}$, for some $n \in N$'.

Denote by $\aleph_1$ the set of strategy, payoff, relevance, and information rules. That is, $\aleph_1$, is the union of all the statements that has either form (1), (2), (3), or (4).

Rule 6′ (Introspective Knowledge)**.** *If $\alpha \in \aleph_1$, $i \in \mathcal{I}$, then $K^i(\alpha) \in \mathbb{S}_1$ with $Lev\big(K^i(\alpha)\big) := 1$, $Var\big(K^i(\alpha)\big) := \{K^i(\alpha)\}$, and $Subj\big(K^i(\alpha)\big) := \{i\}$.*

Example 6. (Rules of a Dynamic Game.) Consider the two-player dynamic game, Γ, depicted in Figure 1a: Player 1 moves at information sets h_1 and h_3; player 2 moves at information sets h_2

and h_4. The normal form of Γ, denoted $G(\Gamma)$, is depicted in Figure 1b. The *rules of dynamic game* Γ, denoted $\langle\Gamma\rangle$, is the following list of statements from $\aleph_1$:

$$\langle\Gamma\rangle \coloneqq \{\langle G(\Gamma)\rangle, R_{1,h_1,\{s_1,s_2,s_3\}}, R_{1,h_3,\{s_2,s_3\}}, R_{2,h_2,\{s_4,s_5,s_6\}}, R_{2,h_4,\{s_5,s_6\}}$$
$$H_{1,h_1,\{2\},3,\{s_4,s_5,s_6\}}, H_{1,h_3,\{2\},2,\{s_5,s_6\}}, H_{2,h_2,\{1\},2,\{s_2,s_3\}}, H_{2,h_4,\{1\},2,\{s_2,s_3\}}\},$$

where $\langle G(\Gamma)\rangle$ is the list of rules of $G(\Gamma)$ (Example 1).

Remark 5. In the above example, a player's strategy in a dynamic game corresponds to Reny (1992)'s notion of a plan. In some contexts, one may wish to use the notion of a full strategy: For example, full strategies are needed to describe a player's behavior when he can make trembling mistakes (Selten (1975)). To allow for full strategies, one needs to extend language $\mathcal{L}_1$ by including facts about actions that players take at their information sets and by augmenting the rules of dynamic games with rules that associate full strategies with actions players take at all of their information sets.

3 Model

Having formally described epistemic pictures, I now introduce my model and define the prediction correspondence, the main object of study in this paper.

3.1 Process of Reasoning

I now describe the decision process of a player who is given an epistemic picture. For simplicity, I first focus on pictures written in language $\mathcal{L}_0$. After that, I explain how to adapt the construction to allow for pictures in language $\mathcal{L}_1$. The key element in my description is the following *possibility function*, $\pi(s, i, opt, k, E^i)$: The inputs of $\pi(s, i, opt, k, E^i)$ are a strategy symbol, $s \in \mathcal{S}$; a player symbol, $i \in \mathcal{I}$; a type of optimality, $opt \in \{o, mo\}$; and an epistemic picture, $E^i \subset \mathbb{E}_0^i$. The output is $\pi(s, i, opt, k, E^i) = \top$ if strategy s can ever be played by player i with optimality type opt, level of reasoning k, and epistemic picture E^i. Otherwise, $\pi(s, i, opt, k, E^i) = \bot$. The main assumption is that a k-level player can compute the value of $\pi(s, i, opt, \hat{k}, E^i)$ for all inputs with $\hat{k} < k$.

Function $\pi(s, i, opt, k, E^i)$ is defined inductively in k: For $k = 0$, assign $\pi(s, i, opt, 0, E^i) \coloneqq \top$ for all inputs s, i, opt, and E^i. Next, take some $k \in \mathbb{N}$ and suppose that π has been defined for all inputs with levels strictly below k. To define $\pi(s, i, opt, k, E^i)$, I assume that player i with optimality type opt, level of reasoning k, and epistemic picture E^i engages is the following three-step *process of reasoning*:

Step 1: Decision Problem. Player i starts by constructing his decision problem (Ω, S), where Ω is the subjective state-space of uncertainty and $S \subset \mathbb{R}^\Omega$ is the set of acts. Problem (Ω, S) is completely determined by the player's level of reasoning k and epistemic picture E^i, and is constructed using the following algorithm:

Step 1.1. Let $\mathcal{G}$ be the statements that describe the player's knowledge about the game. That is, $\mathcal{G} := \{e \in E^i \mid \exists \alpha \in \aleph_0,\, e = K^i(\alpha)\}$. The player must know the game he plays: If there is no static game G where i is a player and such that $\mathcal{G} = K^i\langle G\rangle$, then the algorithm stops with an error.

Step 1.2. Otherwise, let G be the game from the player's point of view: $\mathcal{G} = K^i\langle G\rangle$. Let J be the set of the player's opponents in G. Let $E = E^i_{\leqslant k}\backslash\mathcal{G}$. In game G, level-k player i will analyze only statements E: He will completely dismiss those statements from E^i that have levels higher than k. The analyzed statements, E, must not involve facts about players outside J: If $\exists t \in \mathcal{I}\backslash J, \exists v \in Var_{-1}(E)$, such that $Subj(v) = \{t\}$, then the algorithm stops with an error.

Step 1.3. Otherwise, for each $j \in J$, do the following: Let S^j be the set of player j's strategies in game G. Let $V^j = \{v \in Var_{-1}(E) \mid Subj(v) = \{j\}\}$ be the pre-variables of E that are facts about player j. Set V^j must not contain strategy facts outside S^j: If $\exists s \in \mathcal{S}\backslash S^j$ such that $st_{j,s} \in V^j$, then the algorithm stops with an error. Otherwise, let $E^j = V^j \cap \mathbb{E}^j_0$ be the set of player j's epistemic statements among variables V^j. Construct the following set of variables:

$$W^j := \bigsqcup_{s \in S^j} \{st_{j,s}\} \sqcup \{o_j, mo_j\} \sqcup \bigsqcup_{m \in \{0,..,k-1\}} \{l_{j,m}\} \sqcup \{r_j\} \sqcup E^j.$$

That is, W^j is the disjoint union of (i) player j's strategy facts; (ii) facts about her optimality; (iii) facts about her levels of reasoning below k; (iv) the fact about her rationality; (v) epistemic facts E^j. A *state of affairs* for player j, denoted ω^j, is a combination of truth values for facts in W^j. That is, $\omega^j \in \{\top, \bot\}^{W^j}$. Compute Ω^j, the set of all *possible states of affairs* for player j. Precisely, Ω^j is the union of elements $w^j \in \{\top, \bot\}^{W^j}$ that satisfy the following five restrictions:

R1. Strategies: exactly one fact in $\{st_{j,s}\}_{s \in S^j}$ is true.

R2. Optimality: exactly one fact in $\{o_j, mo_j\}$ is true.

R3. Levels: exactly one fact in $\{l_{j,m}\}_{m \in \{0,..,k-1\}}$ is true.

R4. Rationality: fact r_j is true if and only if fact $l_{j,k-1}$ is true.

R5. Possibility: $\pi(\hat{s}, j, \hat{opt}, \hat{k}, \hat{E}^j) = \top$, where $st_{j,\hat{s}}$, $\hat{opt}$, and $l_{j,\hat{k}}$, are the true strategy, optimality, and level facts, and $\hat{E}^j \subseteq E^j$ is the set of epistemic facts that are true in the state.

If $\Omega^j = \varnothing$, then the algorithm stops with an error.

Step 1.4. Having constructed Ω^j for each $j \in J$, construct the state space, Ω, as follows. State space Ω is a subset of the set of *possible states*, $\times_{j \in J} \Omega^j$. The set of variables for possible states is $W = \times_{j \in J} W^j$. Let $\mathcal{K}$ be the set of statements in E that have the form $K^i(P)$, for some $P \in \mathbb{S}_0$. For each $K^i(P) \in \mathcal{K}$, variables of P are included into the set of variable of possible states, $Var(P) \subseteq W$. Therefore, statement P can be associated with a classical predicate, $\bar{P} : \{\top, \bot\}^W \to \{\top, \bot\}$. The final restriction on the player's state space is that Ω is the set of all possible states $\omega \in \times_{j \in J} \Omega^j$ that respect the player's knowledge:

R6. Knowledge: For each $K^i(P) \in \mathcal{K}$, it must be that $\bar{P}(\omega) = \top$.

If $\Omega = \varnothing$, then the algorithm stops with an error.

Step 1.5. Otherwise, construct the set of acts, S, as follows. Acts S are associated with player i's strategies in game G in the natural way. For each strategy s of player i, the corresponding act $s : \Omega \to \mathbb{R}$ is uniquely determined by the payoff rules of G: By restriction R1, in each state $\omega \in \Omega$, there is a unique strategy profile, $s^J(\omega)$, that players J play. The payoff for act s in state ω then is the payoff that player i receives in game G under the strategy profile $\big(s, s^J(\omega)\big)$.

Step 2: Preferences. If Step 1 have not ended with an error, player i forms a subjective preference relation, $\succeq$, over the set of all acts, $\mathbb{R}^\Omega$. Relation $\succeq$ must agree with the player's optimality type, *opt*, and must satisfy the semantic restrictions imposed by the player's non-knowledge epistemic statements, $E\backslash\mathcal{K}$. If the player can not find such preferences, then the process ends with an error.

For example, if the player's optimality type is o, his preference relation must be represented by a probability measure, λ, distributed on Ω. If $E\backslash\mathcal{K}$ contains a statement $B^i_{0.75}(P)$, then λ must be such that $\lambda\{\omega \in \Omega \,|\, \bar{P}(\omega) = \top\} \geqslant 0.75$. If $E\backslash\mathcal{K}$ additionally contains statement $B^i_{0.75}\big(\neg(P)\big)$, then it must be that $\lambda\{\omega \in \Omega \,|\, \bar{P}(\omega) = \bot\} \geqslant 0.75$. In this case, the player will not be able to form preferences that will satisfy these two restrictions simultaneously. His process of reasoning will end with an error. See section 3.3, for the detailed description of players' optimality types and semantic restrictions.

Step 3: Decision. If Steps 1 and 2 have not ended with an error, player i selects any act $s \in S$ that maximizes his preferences $\succeq$ in S. This concludes the process of reasoning.

In the above process, decision problem (Ω, S) is determined uniquely. To define the value of $\pi(s, i, opt, k, E^i)$, one needs to iterate over all possible preferences the player may form in Step 2, and check if strategy s corresponds to an optimal act for any of such preferences. If the answer is 'yes', then assign $\pi(s, i, opt, k, E^i) \coloneqq \top$. If the answer is 'no' or if the process of reasoning has ended with an error at either Step 1 or Step 2, then assign $\pi(s, i, opt, k, E^i) \coloneqq \bot$.

Remark 6. To compute function π one generally needs to go through a deeply recursive procedure. The main assumption in my model is that a level-k player is capable of computing up to $(k-1)$-levels of such recursions.

Remark 7. Restriction R5 in Step 1.3 assumes the following semantic interpretation of epistemic statements: If player i thinks that statement $M^j(P)$ is false, he means merely that $M^j(P)$ is not included into player j's epistemic picture. Player i does not impose restrictions on player j's belief. For example, player i, who thinks that $B^j_{0.5}(P)$ is false, may simultaneously think that player j's belief puts probability 0.75 on the event that P is true.

Remark 8. The above process of reasoning can be readily adapted to dynamic contexts described in language $\mathcal{L}_1$. To do this, one needs to make the following changes: Player i can now have three additional types of optimality, $\{co, so, cso\}$, and his epistemic picture can include statements from $\mathbb{S}_1$. In Step 1.1, set $\mathcal{G} = \{e \in E^i \mid \exists \alpha \in \aleph_1,\ e = K^i(\alpha)\}$ must describe the player's knowledge of some dynamic game Γ; that is, $\mathcal{G} = K^i\langle\Gamma\rangle$. In Step 1.3, variables W^j must include three additional optimality facts, $\{co_j, so_j, cso_j\}$, and Restriction R2 should require that in each state $\omega^i \in \Omega^j$, exactly one of the facts in $\{o_j, mo_j, co_j, so_j, cso_j\}$ is true.

Remark 9. One may wish to model the behavior of a player who knows the rules of the game, but who also has additional knowledge that his opponents do not play certain strategies. Such restrictions can be readily incorporated into the model: One just needs to write these restrictions in a form that is different from the form of rules $\aleph_0$. For example, one can write such restrictions using an extra layer of brackets (Remark 3).

3.2 Prediction Correspondence

In the previous section, I described the process in which one defines the possibility function, $\pi(s, i, opt, k, E^i)$. I now define the closely related *prediction correspondence*, $\Pi(i, opt, k, E^i)$, which is the main object of study in this paper. The definition is for contexts described in language $\mathcal{L}_0$. The adaptation to language $\mathcal{L}_1$ is straightforward.

Definition (Prediction Correspondence). *For a player symbol $i \in \mathcal{I}$, optimality type $opt \in \{o, mo\}$, level $k \in \mathbb{N}$, and epistemic picture $E^i \subseteq \mathbb{E}_0^i$, the value of the prediction correspondence is given by*

$$\Pi(i, opt, k, E^i) \coloneqq \{s \in \mathcal{S} \mid \pi(s, i, opt, k, E^i) = \top\}.$$

That is, $\Pi(i, opt, k, E^i)$ is the set of strategies that can be predicted for a level-k player i who has optimality type opt and epistemic picture E^i. Prediction correspondence is not defined for level-0 players. If a player's process of reasoning ends with an error, then $\Pi(i, opt, k, E^i) = \varnothing$.

3.3 Semantics

I now explain the meaning of rationality, optimality, and epistemic modalities in my model.

Rationality. A player thinks that his opponent is rational if he ascribes to her the highest level of reasoning that is still comprehensible to him: From the perspective of a level-k player, the fact that 'the opponent is rational' is the same as the fact that 'the opponent's level of reasoning is $k-1$'. This is precisely Restriction R4 in Step 1.3 in his process of reasoning. This interpretation of rationality does not mean that the player dismisses the possibility that his opponent can have a higher level: On the contrary, the player thinks that the opponent's level is extremely high. His problem is that he can not comprehend the opponent's behavior for levels higher than $k-1$. The best he can do is to treat her as a player with level $k-1$. See also Remark 12.

Optimality. Savage (1954). A player's behavior is *optimal* if (i) his preferences over acts $\mathbb{R}^\Omega$ are represented by a probability measure, λ, distributed on Ω; and (ii) the player selects an act that maximizes the expected payoff with respect to that measure:

$$\max_{s \in S} \sum_{\omega \in \Omega} s(\omega)\lambda(\omega).$$

Maxmin Optimality. Gilboa and Schmeidler (1989). A player's behavior is *maxmin optimal* if (i) his preferences over acts $\mathbb{R}^\Omega$ are represented by a set of probability measures, $\{\lambda_c\}_{c \in C}$, each distributed on Ω; and (ii) the player selects an act that maximizes the minimum of expected payoffs with respect to those measures:

$$\max_{s \in S} \min_{c \in C} \sum_{\omega \in \Omega} s(\omega)\lambda_c(\omega).$$

Cautious Optimality. Blume et al. (1991). A player's behavior is *cautiously optimal* if (i) his preferences over acts $\mathbb{R}^\Omega$ are represented by an LPS, $\rho = \{\lambda_n\}_{n=1}^N$, with full support on Ω and (ii) the player selects an act that maximizes the corresponding vector of expected payoffs lexicographically:

$$\max_{s \in S} \Big(\sum_{\omega \in \Omega} s(\omega)\lambda_n(\omega) \Big)_{n=1}^N.$$

Sequential Optimality. Kreps and Wilson (1982), Reny (1992). A player whose behavior is *sequentially optimal* forms preferences over acts $\mathbb{R}^\Omega$ that are represented by an LPS, $\rho = \{\lambda_n\}_{n=1}^N$, with full support on Ω. His choice s must be sequentially optimal: For each of his information h that is relevant for s, act s must be optimal among all acts relevant for h is the sense that it has the highest expected payoff against the first measure in ρ that survives the conditioning on information h.

Cautious Sequential Optimality. A player whose behavior is *cautiously sequentially optimal* forms preferences over acts $\mathbb{R}^\Omega$ that are represented by an LPS, $\rho = \{\lambda_n\}_{n=1}^N$, with full support on Ω. His choice s must be cautiously sequentially optimal: For each of his information h that is relevant for s, act s must be optimal among all acts relevant for h in the sense that it has the highest expected payoff against LPS ρ conditioned on information h.

Knowledge. In my model, knowledge statements are used in the construction of a player's decision problem as described in Step 1 of the process of reasoning.

p-Belief. Monderer and Samet (1989). Statement $B^i_p(P)$, with $p \in [0, 1]$, puts the following restrictions on preferences that player i can form in Step 2 of his process of reasoning:

- for type-o optimality: $\lambda\{\omega \in \Omega \,|\, \bar{P}(\omega) = \top\} \geqslant p$;
- for type-mo optimality: $\forall c \in C,\ \lambda_c\{\omega \in \Omega \,|\, \bar{P}(\omega) = \top\} \geqslant p$;
- for type-$\{co, so, cso\}$ optimality: $\lambda_1\{\omega \in \Omega \,|\, \bar{P}(\omega) = \top\} \geqslant p$.

Assumption. The assumption modality was originally proposed by Brandenburger et al. (2008). I use the more appealing reformulation proposed by Dekel et al. (2016). Statement $A^i(P)$ puts the following restrictions on preferences that player i can form in Step 2 of his process of reasoning:

- for type-$\{o, mo\}$ optimality: Step 2 ends with an error;
- for type-$\{co, so, cso\}$ optimality: $supp(\rho_P) = \{\omega \in \Omega \mid \bar{P}(\omega) = \top\}$, where ρ_P is the beginning of ρ that first covers set $\{\omega \in \Omega \mid \bar{P}(\omega) = \top\}$.

4 Main Results: Three Robustness Theorems

An important characteristic of any solution concept is its robustness to changes in the description of the studied environment. In this paper, I formulate noncooperative solution concepts in terms of epistemic pictures that generate the required behavior. In such a formulation, it is natural to think about *epistemic robustness* of generated solution concepts. In this section, I address the following questions about epistemic robustness of a given solution concept: (i) If a player has more precise information than that contained in the solution's epistemic picture, will his behavior still conform with the solution? (Section 4.1.) (ii) Do predictions remain valid when a player's level of reasoning is higher than what was expected? (Section 4.2.) (iii) When are the predictions independent of a particular extensive-form representation of the played dynamic game? (Section 4.3.)

4.1 Monotonicity in Pictures

I first address the question of when the prediction about a player's behavior is monotonically decreasing in the precision of his epistemic picture. In language $\mathcal{L}_0$, statements that contain expressions $K^j(\alpha)$, for some $j \in \mathcal{I}$ and $\alpha \in \aleph_0$, play a special role during the process of reasoning: they describe players' knowledge of the game (Step 1.1) and define players' acts (Step 1.5). As a result, tempering with such statements may inadvertently cause the algorithm to break down. The role of other statements in $\mathcal{L}_0$, which I will call *regular statements*, is completely different: Regular statements impose semantic restrictions on players' state spaces (Step 1.4) and beliefs (Step 2). In this section, I study how changing these semantic restrictions may affect predictions about players' behavior.

Definition (Regular Epistemic Picture). *A statement in $\mathbb{S}_0$ is called regular if it is constructed from atomic facts using only grammar Rules 1-5, but not Rule 6. An epistemic picture is called regular if all of its statements are regular.*

Regular epistemic pictures are partially ordered by the following relation of *being more precise*:

Definition (More Precise). *Let $E^i, \tilde{E}^i \subseteq \mathbb{E}^i_0$ be two regular epistemic pictures of player $i \in \mathcal{I}$ in language $\mathcal{L}_0$. Epistemic picture $\tilde{E}^i$ is called more precise than epistemic picture E^i if $\tilde{E}^i$ can be obtained from E^i through a finite sequence of strengthening operations of the following types:*

T.1: Add any regular epistemic statement of player i.

T.2: Replace some statements by other statements using one of the following patterns:

	take statements	*replace by*	*under condition that*
T.2.1	$M^i(P)$	$M^i(Q)$	$P \sim Q$, $(M \neq K) \vee (Q \notin \aleph_0)$
T.2.2	$B^i_{p_1}(P)$	$B^i_{p_2}(P)$	$p_1 < p_2$
T.2.3	$B^i_p(P)$	$K^i(P)$	$P \notin \aleph_0$
T.2.4	$B^i_p(P), B^i_p(Q)$	$B^i_p(P \wedge Q)$	*always*
T.2.5	$B^i_p(P \wedge Q)$	$B^i_{p_1}(P), B^i_{p_2}(Q)$	$p_1 + p_2 \geqslant p + 1$
T.2.6	$K^i(P \wedge Q)$	$K^i(P), K^i(Q)$	$P \notin \aleph_0, Q \notin \aleph_0$
T.2.7	$K^i(P)$, $K^i(Q)$	$K^i(P \wedge Q)$	$P \wedge Q \notin \aleph_0$

T.3: Rewrite any proper part of any statement by another one using one of the following patterns:

	take part	*replace by*	*under condition that*
T.3.1	$\ldots M^j(P) \ldots$	$\ldots M^j(Q) \ldots$	$P \sim Q$, $(M \neq K) \vee (Q \notin \aleph_0)$
T.3.2	$\ldots B^j_{p_1}(P) \ldots$	$\ldots B^j_{p_2}(P) \ldots$	$p_1 < p_2$
T.3.3	$\ldots B^j_p(P) \ldots$	$\ldots K^j(P) \ldots$	$P \notin \aleph_0$
T.3.4	$\ldots B^j_p(P \vee Q) \ldots$	$\ldots \big(B^j_p(P) \vee B^j_p(Q)\big) \ldots$	*always*
T.3.5	$\ldots \big(B^j_p(P) \vee B^j_p(Q)\big) \ldots$	$\ldots B^j_{2p}(P \vee Q) \ldots$	$p \leqslant 1/2$
T.3.6	$\ldots \big(B^j_p(P) \wedge B^j_p(Q)\big)$	$\ldots B^j_p(P \wedge Q) \ldots$	*always*
T.3.7	$\ldots B^j_p(P \wedge Q) \ldots$	$\ldots \big(B^j_{p_1}(P) \wedge B^j_{p_2}(Q)\big) \ldots$	$p_1 + p_2 \geqslant p + 1$
T.3.8	$\ldots K^j(P \vee Q) \ldots$	$\ldots \big(K^j(P) \vee K^j(Q)\big) \ldots$	$P \notin \aleph_0, Q \notin \aleph_0$
T.3.9	$\ldots K^j(P \wedge Q) \ldots$	$\ldots \big(K^j(P) \wedge K^j(Q)\big) \ldots$	$P \notin \aleph_0, Q \notin \aleph_0$
T.3.10	$\ldots \big(K^j(P) \wedge K^j(Q)\big) \ldots$	$\ldots K^j(P \wedge Q) \ldots$	$P \wedge Q \notin \aleph_0$

In the above patterns: P and Q are generic statements in $\mathbb{S}_0$; $P \sim Q$ means that statements P and Q have the same variables and express equivalent predicates; j is a generic player symbol in $\mathcal{I}$ which may equal i; M is a generic modality in $\mathcal{M}_0$; p, p_1, p_2 are generic numbers in $[0,1]$.

The general property of patterns T_2 and T_3 is that in them, the replaced parts are implied by the replacing parts if one interprets these parts as semantic restrictions imposed on an individual player. In this sense, a more precise epistemic picture "implies" a less precise one. One may conjecture that if a behavior is possible under an epistemic picture, it should also be possible under a picture that is less precise. The following theorem shows that when the played game is common knowledge, this conjecture is true:

Theorem 1 (Monotonicity in Pictures)**.** *Let G be a finite static I-player game. Suppose a player $i \in I$ thinks that G is common knowledge. Let E^i and $\tilde{E}^i$ be finite regular epistemic pictures written in language $\mathcal{L}_0$ that player i may hold additionally. If $\tilde{E}^i$ is more precise than E^i, then so is the* *corresponding prediction about player i's behavior:*

$$\forall opt \in \{o, mo\}, \forall k \in \mathbb{N},$$
$$\Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) = \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i\}\big).$$

Proof. See Appendix A.2. □

Extension to Cautious Optimality: I now provide an extension of Theorem 1 that allows for players with cautiously-optimal behavior. Unlike optimal and maxmin-optimal behavior, cautious behavior is not monotonic with respect to restrictions put by knowledge statements: Knowledge statements are extremely *dogmatic* in that they curtail the set of possible states, rather than the set of beliefs a player can have. Adding such statements may decrease the support of the player's cautious belief. As a result, the player may find more strategies to be optimal. To restore monotonicity, one needs to exclude dogmatic statements from the consideration.

Specifically, let $\mathcal{L}_+$ be the epistemic language which includes atomic facts $\{co_i\}_{i \in \mathcal{I}}$, but is otherwise the same as language $\mathcal{L}_0$: symbols, other atomic facts, modalities, and grammar rules are exactly as in $\mathcal{L}_0$. Note that $\mathcal{L}_+$ does not include the assumption modality.

Definition (Undogmatic Epistemic Picture)**.** *A statement in language $\mathcal{L}_+$ is called undogmatic if it is constructed without applying the knowledge modalities, $\{K^i\}_{i \in \mathcal{I}}$. An epistemic picture is undogmatic if all of its statements are undogmatic.*

The role that undogmatic epistemic pictures play in my model is similar to the role played by belief-complete type spaces in the type-structure model. The following theorem is an analog of Theorem 1 for epistemic pictures written in language $\mathcal{L}_+$:

Theorem 1′. *Let G be a finite static I-player game. Let E^i and $\tilde{E}^i$ be finite, regular, and undogmatic epistemic pictures of player $i \in I$ written in language $\mathcal{L}_+$. If $\tilde{E}^i$ is more precise than E^i, then*

$$\forall opt \in \{o, mo, co\}, \forall k \in \mathbb{N},$$
$$\Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) = \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i\}\big).$$

Proof. The proof adapts the proof of Theorem 1. See Appendix A.3. □

Remark 10. It would be interesting to know if Theorem 1 can be extended to a language that also contains the assumption modality. I leave this question for future research.

4.2 Monotonicity in Levels

I now address the question of when the prediction about a player's behavior is monotonically decreasing in his level of reasoning. The following notion will be useful:

Definition (Positive Epistemic Picture)**.** *A statement in* $\mathbb{S}_0$ *is called positive if it is constructed without applying negation to propositions that have among their variables atomic facts about players' levels,* $\{l_{i,k}\}_{i\in\mathcal{I},k\in\mathbb{Z}_+}$*, or about their rationality,* $\{r_i\}_{i\in\mathcal{I}}$*. An epistemic picture is positive if all of its statements are positive.*

For example, picture $\{B^1_p(\neg st_{2,s_1} \wedge l_{2,3})\}$ is positive, while picture $\{B^1_p(\neg(st_{2,s_1} \wedge r_2))\}$ is not positive. Many rationalizability concepts can be produced from regular epistemic pictures that are positive (Section 5.1). The general property of rationalizability concepts is that they correspond to certain strategy-elimination procedures: At each step in the procedure, the set of strategies that a player can play decreases monotonically. The following theorem establishes that this property is quite general and is satisfied by predictions generated from positive regular epistemic pictures in language $\mathcal{L}_0$:

Theorem 2 (Monotonicity in Levels)**.** *Let G be a finite static I-player game. Let E^i be a finite, regular, and positive epistemic picture of player $i \in I$ written in language $\mathcal{L}_0$. The prediction about player i's behavior under picture $\{CK^{i,I}\langle G\rangle, E^i\}$ is decreasing in the player's level of reasoning:*

$$\forall opt \in \{o, mo\}, \forall k \in \mathbb{N}, \quad \Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i\}\big).$$

Proof. See Appendix A.4. □

Remark 11. Take a solution concept described by prediction $\hat{S}(k) := \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i\}\big)$, for a finite game G and a finite, regular, and positive epistemic picture E^i. By Theorem 2, the sequence $\{\hat{S}(k)\}_{k\in\mathbb{N}}$ decreases monotonically. As G is finite, the sequence must converge to some set. That set can then be interpreted as the prediction about the behavior of a perfectly rational type-*opt* player i under picture $\{CK^{i,I}\langle G\rangle, E^i\}$.

Remark 12. Recall that when a level-k player i thinks that player j is rational, he treats her as a level-$(k-1)$ player. If player i's epistemic picture satisfies the assumptions of Theorem 2, then such interpretation of rationality seems reasonable: Player i would not be able to compute player j's behavior beyond level $k-1$ anyways. Yet he may be comfortable as her actual behavior will be within the set he predicts, even if she is of a much higher level.

Remark 13. The restriction to positive pictures in Theorem 2 is important. For example, let G be the prisoner's dilemma. Let $E^1 = \{B^1_{0.5}(\neg r_2)\}$ be an epistemic picture for player i. Picture E^1 is finite and regular, but not positive. From the point of view of a level-1 player 1, the opponent is always rational. Thus, he would not be able to construct a belief that would satisfy the restriction of E^1. The prediction about his behavior is then empty. Yet, from the point of view of a level-2 player 1, the opponent can be irrational (here, level-0) with probability 1. Thus, the prediction about his behavior is that he would play 'defect', which violates the monotonicity.

Finally, one can extend Theorem 2 to allow for cautiously-optimal behavior as follows.

Theorem 2′. *Let G be a finite static I-player game. Let E^i be a finite, regular, positive, and undogmatic epistemic picture of player $i \in I$ written in language $\mathcal{L}_+$. The prediction about player i's behavior under picture $\{CK^{i,I}\langle G\rangle, E^i\}$ is decreasing in the player's level of reasoning:*

$$\forall opt \in \{o, mo, co\}, \forall k \in \mathbb{N}, \quad \Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i\}\big).$$

Proof. The proof adapts the proof of Theorem 2 in exactly the same way as the proof of Theorem 1′ adapts the proof of Theorem 1. □

4.3 Invariance

A common requirement for players' behavior in dynamic games is that it should be sequentially optimal: A player's strategy should be optimal not just at the beginning of the game, but also conditional on any information that the player may receive during the play. To formally define sequential optimality, one needs to refer to information sets, which belong to the extensive-form description of the played game. Yet, a given game may be represented by an extensive-form tree in many different ways. Solution concepts that employ sequentially optimal behavior then may fail to be invariant with respect to these extensive-form representations. In this section, I address this issue, and provide sufficient conditions on a player's epistemic picture that would ensure that the predicted behavior depends only on the normal form of the played game. Solution concepts that are generated by epistemic pictures that satisfy these sufficient conditions are then guaranteed to be invariant. Specifically, I study epistemic pictures that are written in language $\mathcal{L}_1$. The key definition is the following:

Definition (Invariant Epistemic Picture)**.** *A statement in language $\mathcal{L}_1$ is called invariant if it is constructed only from atomic facts that describe players' strategies, levels of reasoning, rationality, type of optimality, and payoffs. An epistemic picture is invariant if all of its statements are invariant.*

It turns out that even though one needs to refer to a particular extensive-form tree to define sequential optimality, the notion of *cautious* sequential optimality is not problematic for invariance: If a player's behavior is cautiously sequentially optimal and if he has an epistemic picture that consists of (i) common knowledge of the game; (ii) common knowledge of cautious sequential optimality; and (iii) any additional invariant statements, then the prediction about his behavior does not depend on a particular extensive-form representation of the played game:

Theorem 3 (Invariance of Cautious Sequential Optimality)**.** *Let Γ_1 and Γ_2 be finite dynamic I-player games with perfect recall and without chance nodes that have the same normal form, $G(\Gamma_1) = G(\Gamma_2)$. Let E^i be a finite invariant epistemic picture of player $i \in I$ written in language $\mathcal{L}_1$. Then*

$$\forall k \in \mathbb{N}, \quad \Pi\big(i, cso, k, \{CK^{i,I}\langle\Gamma_1\rangle, CK^{i,I}[CSO], E^i\}\big) = \Pi\big(i, cso, k, \{CK^{i,I}\langle\Gamma_2\rangle, CK^{i,I}[CSO], E^i\}\big).$$

Proof. See Appendix A.5. □

Remark 14. That an acceptable solution concept should be invariant is one of the main desiderata put forward by Kohlberg and Mertens (1986), who addressed the problem of finding invariant equilibria in a purely mathematical way. Theorem 3 suggests an epistemic method for identifying invariant concepts, which can complement the mathematical approach.

Remark 15. The restriction to games without chance nodes is for expositional convenience only: Theorem 3 can be established for general finite dynamic games with perfect recall. To do this formally, one needs to extend language $\mathcal{L}_1$ so as to allow for statements about random moves by Chance and a modality that imposes the independence restriction on players' beliefs. See Section 6.

Remark 16. In Theorem 3, picture E^i may contain dogmatic statements. For example, E^i may express knowledge that player i's opponents play strategy profiles from some set $\tilde{S}^j$, which is a proper subset of the set of their strategy profiles in $G(\Gamma)$ (Remark 9). In such a case, cautious sequential optimality of a player i's strategy s is checked only at those information sets of player i which are relevant for s and $\tilde{S}^J$.

5 Applications

Throughout this section, (i) G is a finite static game; (ii) Γ is a finite dynamic game with perfect recall; (iii) I is the set of players; (iv) for $i \in I$, S^i is the set of player i's strategies.

5.1 Rationalizability

I now discuss several strategy-elimination procedures that were introduced in the previous literature. Specifically, I provide foundations for these procedures in terms of epistemic pictures, and discuss their robustness properties using the results from Section 4.

Rationalizability. Bernheim (1984), Pearce (1984). For $i \in I$, $k \in \mathbb{N}$, let R^i_k be the set of player i's strategies obtained after k-rounds of the rationalizability procedure in game G. That is, R^i_k are the strategies that survive k rounds of elimination of strictly dominated strategies in G. This procedure corresponds to optimal behavior under epistemic picture $E^i(CKR) := \{CK^{i,I}\langle G\rangle, CK^{i,I}[R], CK^{i,I}[O]\}$ written in language $\mathcal{L}_0$. That is,

$$\forall k \in \mathbb{N}, \quad \Pi\big(i, o, k, E^i(CKR)\big) = R^i_k.$$

Epistemic picture $E^i(CKR)\backslash CK^{i,I}\langle G\rangle$ is regular and positive. Thus, the rationalizability procedure is robust in the sense of Theorems 1 and 2.

One can also consider weaker versions of the rationalizability procedure generated by the following regular and positive epistemic pictures written in language $\mathcal{L}_0$:

$$\forall p \in [0,1], \quad E^i(CB_pR) := \{CK^{i,I}\langle G\rangle, CB^{i,I}_p[R], CB^{i,I}_p[O]\};$$
$$\forall p \in [0,1], \quad E^i(CB_pMR) := \{CK^{i,I}\langle G\rangle, CB^{i,I}_p[R], CB^{i,I}_p[MO]\}.$$

Theorem 2 implies that these pictures indeed generate strategy-elimination procedures. Moreover, by Theorem 1, the following inclusions hold:

$$\forall k \in \mathbb{N}, \forall p \in [0,1], \forall q \in [0,1], p < q, \;\; \Pi\big(i, o, k, E^i(CB_qR)\big) \subseteq \Pi\big(i, o, k, E^i(CB_pR)\big);$$
$$\forall k \in \mathbb{N}, \forall p \in [0,1], \forall q \in [0,1], p < q, \;\; \Pi\big(i, mo, k, E^i(CB_qMR)\big) \subseteq \Pi\big(i, mo, k, E^i(CB_pMR)\big).$$

Robust Rationalizability. Dekel and Fudenberg (1990). For $i \in I$, $k \in \mathbb{N}$, let WS^i_{k-1} be the set of player i's strategies obtained after k-rounds of the robust rationalizability (RR) procedure in game G. That is, WS^i_{k-1} are the strategies that survive one round of elimination of weakly dominated strategies followed by $k-1$ rounds of elimination of strictly dominated strategies. This procedure corresponds to cautiously optimal behavior under epistemic picture $E^i(RR) := \{CK^{i,I}\langle G\rangle, CB^{i,I}_1[R], CB^{i,I}_1[CO]\}$ written in language $\mathcal{L}_+$. That is,

$$\forall k \in \mathbb{N}, \;\; \Pi\big(i, o, k, E^i(RR)\big) = WS^i_{k-1}.$$

Epistemic picture $E^i(RR)\backslash CK^{i,I}\langle G\rangle$ is regular, positive and undogmatic. Thus, the RR procedure is robust in the sense of Theorems $1'$ and $2'$. One can also generate this procedure for cautiously sequentially optimal behavior in dynamic games using epistemic picture $E^i(\overline{RR}) := \{CK^{i,I}\langle\Gamma\rangle, CB^{i,I}_1[R], CK^{i,I}[CSO]\}$. As $E^i(\overline{RR})$ satisfies the assumptions of Theorem 3, the RR procedure is invariant.

One can also consider weaker versions of the RR procedure generated by the following regular, positive, and undogmatic epistemic pictures written in language $\mathcal{L}_+$:

$$\forall p \in [0,1], \quad E^i(CB_pCR) := \{CK^{i,I}\langle G\rangle, CB^{i,I}_p[R], CB^{i,I}_p[CO]\}.$$

Theorem $2'$ implies that these pictures indeed generate convergent strategy-elimination procedures. Moreover, by Theorem $1'$, the following inclusions hold:

$$\forall k \in \mathbb{N}, \forall p \in [0,1], \forall q \in [0,1], p < q, \;\; \Pi\big(i, co, k, E^i(CB_qCR)\big) \subseteq \Pi\big(i, co, k, E^i(CB_pCR)\big).$$

In dynamic games, all of these procedures are invariant when one generates them by epistemic pictures $E^i(\overline{CB_pCR}) := \{CK^{i,I}\langle\Gamma\rangle, CB^{i,I}_p[R], CK^{i,I}[CSO]\}$, for $p \in [0,1]$, written in language $\mathcal{L}_1$.

Iterative Admissibility. Let Γ be a static two-player game. For $i \in I = \{1,2\}$, $k \in \mathbb{N}$, let IA^i_k be the set of player i's strategies obtained after k-rounds of the iterative admissibility (IA) procedure in game Γ. That is, IA^i_k are the strategies that survive k rounds of elimination of weakly dominated strategies in Γ. Brandenburger et al. (2008) propose common assumption of rationality as a possible epistemic foundation for the IA procedure. This notion can be written in language $\mathcal{L}_1$ as follows.

Definition (Common Assumption of Rationality)**.** *The statement 'player i thinks there is common assumption of rationality between him and player j' is the following epistemic picture of player i:*

$$CA^{i,I}[R] := \Big\{A^i(r_j); A^i\big(r_j \wedge A^j(r_i)\big); A^i\Big(r_j \wedge A^j(r_i) \wedge A^j\big(r_i \wedge A^i(r_j)\big)\Big); \ldots\Big\}.$$

The IA procedure corresponds to cautiously sequentially optimal behavior under epistemic picture $E^i(IA) := \{CK^{i,I}\langle G\rangle, CA^{i,I}[R], CK^{i,I}[CSO]\}$ written in language $\mathcal{L}_1$. That is,

$$\forall k \in \mathbb{N}, \quad \Pi\big(i, cso, k, E^i(IA)\big) = IA^i_k.$$

As $E^i(IA)$ satisfies the assumptions of Theorem 3, the IA procedure is invariant.

Extensive-Form Rationalizability. Pearce (1984). For $i \in I = \{1,2\}$, $k \in \mathbb{N}$, let EFR^i_k be the set of player i's strategies obtained after k-rounds of the extensive-form rationalizability (EFR) procedure in game Γ. This procedure corresponds to sequentially optimal behavior under epistemic picture $E^i(EFR) := \{CK^{i,I}\langle G\rangle, CA^{i,I}[R], CK^{i,I}[SO]\}$ written in language $\mathcal{L}_1$. That is,

$$\forall k \in \mathbb{N}, \quad \Pi\big(i, so, k, E^i(EFR)\big) = EFR^i_k.$$

Epistemic picture $E^i(EFR)$ does not satisfy the assumptions of Theorem 3. This indicates that the EFR procedure may fail to be invariant, which is indeed the case. Note also that pictures $E^i(EFR)$ and $E^i(IA)$ are different only in one aspect: the notion of sequential optimality that is used. In this sense, the IA and the EFR procedures are epistemically close.

5.2 Nash Equilibrium

I now discuss the epistemic conditions for Nash equilibrium provided by Aumann and Brandenburger (1995)(AB95) in the context of the partition model. I will focus on the case of two-player games. For a player $i \in \{1,2\}$, a conjecture, ϕ^i, is a probability measure on the set of strategies of the other player, $\phi^i \in \Delta(S^j)$. For Nash equilibrium in two-player games, AB95 propose the following epistemic conditions:

AB95: Theorem A. *With $n = 2$ (two players), let g be a game, ϕ a pair of conjectures. Suppose that at some state, it is mutually known that $\boldsymbol{g} = g$, that the players are rational, and that $\boldsymbol{\phi} = \phi$. Then (ϕ^2, ϕ^1) is a Nash equilibrium of g.*

I now translate these conditions into my model. The notion of a conjecture translates as follows.

Definition (Conjecture)**.** *The statement 'player i's conjecture is $\phi^i \in \Delta(S^j)$' is the following epistemic picture of player i written in language $\mathcal{L}_0$:*

$$C^i(\phi^i) := \bigsqcup_{s \in S^j} \Big\{B^i_{\phi^i(s)}\big(st_{j,s}\big); B^i_{1-\phi^i(s)}\big(\neg st_{j,s}\big)\Big\},$$

where $\phi^i(s)$ is the probability that conjecture ϕ^i assigns to strategy s.

The following proposition provides an analog of AB95's Theorem A in my framework:

Proposition 2 (Conditions for Nash Equilibrium)**.** *Let G be a finite static two-player game. Suppose that a player 1, who has optimality type-o and level of reasoning $k \geqslant 3$, has the following epistemic picture:*

$$\Big\{K^1\langle G\rangle, K^1(K^2\langle G\rangle), K^1\big(K^2(K^1\langle G\rangle)\big); \quad K^1(r_2),\ K^1\big(K^2(r_1)\big), K^1(o_2),\ K^1\big(K^2(o_1)\big); \quad K^1\big(C^2(\phi^2)\big),\ K^1\Big(K^2\big(C^1(\phi^1)\big)\Big)\Big\}.$$

If Step 1 of the player's process of reasoning does not end with an error, then ϕ^2 is a best response against ϕ^1. If additionally ϕ^1 is possible in Step 2, then (ϕ^2, ϕ^1) is a Nash equilibrium of game G.

Proof. See Appendix A.6. □

Propostion 2 provides an interesting perspective on the AB95's conditions: To have an equilibrium conjecture, a player should know that his opponent knows the conjecture. This is problematic as during the process of reasoning, the player forms his conjecture *after* he takes into account all the information about the opponent. It seems that in order to ensure that players consistently have equilibrium conjectures in a game, one needs to presuppose that players are not free to form their conjectures. That is, equilibrium behavior is not natural for *unrestricted* Bayesian players.

An attractive feature of my model is that it allows one to distinguish between the belief of a player and thoughts of others' about what that belief might be. For example, taken by itself, expression $C^2(\phi^2)$ denotes a property of the belief that player 2 forms during his process of reasoning. Inside of statement $K^1\big(C^2(\phi^2)\big)$, expression $C^2(\phi^2)$ denotes a predicate over variables in the subjective state space of player 1. The partition model does not permit such nuances. In the objective state space, beliefs of player 2 are conflated with player 1's thoughts about player 2's beliefs, as if these two different concepts were the same. Already in its construction, the partition model contains an implicit equilibrium restriction. On a related note, Aumann (1987, p. 2) claimed that "far from being inconsistent with the Bayesian view of the world, the notion of equilibrium is an unavoidable consequence of that view." Perhaps, it would be more appropriate to say that the notion of equilibrium is an unavoidable consequence of implicit assumptions of the partition model.

5.3 Common Knowledge of Admissibility

A player's behavior is called *admissible* if the player is rational and behaves cautiously optimally. Samuelson (1992)(Sam92) notes the following five problems that pertain to common knowledge of admissibility (CKA) in the partition model:

S.1: The prediction of CKA does not coincide with the outcome of the IA procedure.

S.2: The prediction of CKA may be not unique.

S.3: The prediction of CKA can be empty.

S.4: Under CKA, a player may fail to know the opponent's choice set implied by CKA.

S.5: CKA is problematic even in games in which each player has a unique dominant strategy.

I now examine whether the above problems pertain to the prediction generated by common knowledge of admissibility in my model. Specifically, I consider the CKA procedure defined by $\Pi\big(i, co, k, E^i(CKA)\big)$, where $E^i(CKA)$ is the following epistemic picture written in language $\mathcal{L}_+$:

$$E^i(CKA) := \{CK^{i,I}\langle G\rangle, CK^{i,I}[R], CK^{i,I}[CO]\}.$$

Let $G = (S^1, S^2, u^1, u^2)$ be a finite static two-player game. For each $i \in \{1, 2\}$, let D^i be the operator that for each nonempty subset $\tilde{S}^j \subseteq S^j$ produces nonempty subset $D^i(\tilde{S}^j) \subseteq S^i$, where $D^i(\tilde{S}^j)$ is the set of player i's strategies that are not weakly dominated in game $\tilde{G} = (S^i, \tilde{S}^j, u^i, u^j)$. The CKA procedure corresponds to the inductive application of operators (D^1, D^2) starting from sets (S^2, S^1):

k	$\Pi\big(1, co, k, E^1(CKA)\big)$	$\Pi\big(2, co, k, E^2(CKA)\big)$
1	$D^1(S^2)$	$D^2(S^1)$
2	$D^1\big(D^2(S^1)\big)$	$D^2\big(D^1(S^2)\big)$
3	$D^1\Big(D^2\big(D^1(S^2)\big)\Big)$	$D^2\Big(D^1\big(D^2(S^1)\big)\Big)$
...	...	...

If at some step, the CKA procedure produces sets that are the same as the sets produced on the previous step, then the procedure converges: It will keep producing the same sets ever after. In this case, it is sensible to call these sets the predicted behavior for rational players under common knowledge of admissibility. Sam92 proposes the following notion of a consistent pair: A pair of nonempty sets $\tilde{S}^1 \subseteq S^1$ and $\tilde{S}^2 \subseteq S^2$ is called a *consistent pair* if $\tilde{S}^1 = D^1(\tilde{S}^2)$ and $\tilde{S}^2 = D^1(\tilde{S}^1)$. Clearly, if the CKA procedure converges, then it converges to a consistent pair. The problem with the CKA procedure is that it may fail to converge. (Convergence would have been guaranteed by Theorem 2′, but it does not apply here because $E^i(CKA)$ contains dogmatic statements $CK^{i,I}[R]$.) If the CKA procedure does not converge, then it cycles: As G is finite, there will be two steps k and $\tilde{k}$ at which the procedure produces the same pair of sets. The first time this happens, the procedure starts to cycle indefinitely.

The following five examples illustrate the working of the CKA procedure. Sam92 uses these five examples to illustrate the five problems with CKA in the partition model:

Example 7 (S.1; Sam92, Example 4)**.** Consider the following game:

	Y_1	Y_2	Y_3
X_1	$2,4$	$5,4$	$-1,0$
X_2	$3,4$	$2,4$	$-2,0$
X_3	$1,2$	$0,0$	$2,2$
X_4	$0,2$	$2,0$	$0,4$

The outcome of the IA procedure is $\big(\{X_2\},\{Y_1\}\big)$. The game has a unique consistent pair, $\big(\{X_1,X_2\},\{Y_1,Y_2\}\big)$. The CKA procedure converges to that pair:

k	$\Pi\big(1, co, k, E^1(CKA)\big)$	$\Pi\big(2, co, k, E^2(CKA)\big)$
1	X_1, X_2, X_3, X_4	Y_1, Y_3
2	X_2, X_3	Y_1, Y_3
3	X_2, X_3	Y_1
4	X_2	Y_1
5	X_2	Y_1, Y_2
$\geqslant 6$	X_1, X_2	Y_1, Y_2

Thus, the CKA procedure indeed is not equivalent to the IA procedure.

Example 8 (S.2; Sam92, Example 7)**.** Consider the following game:

	L	C	R
T	$1,2$	$2,1$	$0,0$
M	$2,1$	$1,2$	$0,0$
B	$0,0$	$0,0$	$1,1$

This game has three consistent pairs, $\big(\{T,M,B\},\{L,C,R\}\big)$, $\big(\{T,M\},\{L,C\}\big)$, and $\big(\{B\},\{R\}\big)$. Sam92 views each of these pairs as a distinct solution due to the fixed-point nature of the partition model. In my inductive model, this problem does not arise: The CKA procedure is determinate and converges at the very first step:

k	$\Pi\big(1, co, k, E^1(CKA)\big)$	$\Pi\big(2, co, k, E^2(CKA)\big)$
$\geqslant 1$	T, M, B	L, C, R

Example 9 (S.3; Sam92, Example 8)**.** Consider the following game:

	L	R
T	$1,1$	$1,0$
B	$1,0$	$0,1$

Sam92 argues that in the partition model, CKA is not possible in this game. In my model, the

CKA procedure does not converge. Instead, it cycles immediately with period 4:

k	$\Pi\big(1, co, k, E^1(CKA)\big)$	$\Pi\big(2, co, k, E^2(CKA)\big)$
$4n+1, n \geqslant 0$	T	L, R
$4n+2, n \geqslant 0$	T	L
$4n+3, n \geqslant 0$	T, B	L
$4n+4, n \geqslant 0$	T, B	L, R

This example illustrates the following difference between my model and the partition model: When common knowledge of admissibility is applied inductively in my model, it generates nonempty sets of strategies at each step. Yet these sets may cycle indefinitely, without converging. When this happens, the fixed-point logic of common knowledge in the partition model leads one to conclude that common knowledge of admissibility is impossible to attain.

Example 10 (S.4; Sam92, Example 10)**.** Consider the following game:

	L	C	R
T	$1,1$	$1,1$	$2,1$
M	$1,1$	$0,0$	$3,1$
B	$1,2$	$1,3$	$1,1$

Sam92 uses this game to illustrate how in the partition model, it is possible that under CKA, players' do not know the choice sets of their opponents. In my model, this game is problematic because the CKA procedure does not converge. Instead, it cycles with period 2 starting from step 2:

k	$\Pi\big(1, co, k, E^1(CKA)\big)$	$\Pi\big(2, co, k, E^2(CKA)\big)$
1	T, M	L, C
$2n, n \geqslant 1$	T, B	L, R
$2n+1, n \geqslant 1$	M	C

Perhaps, one can interpret cycles in the CKA procedure as the lack of common knowledge about players' choice sets.

Example 11 (S.5; Sam92, Example 12)**.** Consider the following game:

	L	C	R
T	$1,1$	$0,1$	$0,0$
B	$0,1$	$0,0$	$0,1$

Sam92 uses this game to show that even if each player has a unique dominant strategy in a game, players may still play dominated strategies under CKA. This problem pertains to my model as well:

The CKA procedure converges in step 2 to the unique consistent pair, $\big(\{T\}, \{L, C\}\big)$:

k	$\Pi\big(1, co, k, E^1(CKA)\big)$	$\Pi\big(2, co, k, E^2(CKA)\big)$
1	T	L
$\geqslant 2$	T	L, C

6 Extensions

One can extend the proposed model by enriching the epistemic language so as to allow for (i) other kinds of preferences and types of optimality; (ii) other kinds of modalities: independence, infinitely more likely, respect, strong belief, Δ; (iii) chance nodes; (iv) incomplete information; (v) trembling mistakes.

References

Aumann, R. and A. Brandenburger (1995). Epistemic conditions for Nash equilibrium. *Econometrica 63*(5), 1161–1180.

Aumann, R. J. (1976). Agreeing to disagree. *Annals of Statistics 4*(6), 1236–1239.

Aumann, R. J. (1987). Correlated equilibrium as an expression of bayesian rationality. *Econometrica 55*(1), 1–18.

Aumann, R. J. (1999a). Interactive epistemology I: knowledge. *International Journal of Game Theory 28*, 263–300.

Aumann, R. J. (1999b). Interactive epistemology II: probability. *International Journal of Game Theory 28*, 301–314.

Battigalli, P. (1996). Strategic rationality orderings and the best rationalization principle. *Games and Economic Behavior 13*, 178–200.

Battigalli, P. and M. Siniscalchi (1999). Hierarchies of conditional beliefs and interactive epistemology in dynamic games. *Journal of Economic Theory 88*, 188–230.

Bernheim, D. B. (1984). Rationalizable strategic behavior. *Econometrica 52*(4), 1007–1028.

Blume, L., A. Brandenburger, and D. Dekel (1991). Lexicographic probabilities and choice under uncertainty. *Econometrica 59*(1), 61–79.

Brandenburger, A. and E. Dekel (1993). Hierarchies of beliefs and common knowledge. *Journal of economic theory 59*(1), 189–198.

Brandenburger, A., A. Friedenberg, and J. Keisler (2008). Admissibility in games. *Econometrica 76*(2), 307–352.

Camerer, C. F., T.-H. Ho, and J.-K. Chong (2004). A cognitive hierarchy model of games. *The Quarterly Journal of Economics 119*(3), 861–898.

Dekel, E., A. Friedenberg, and M. Siniscalchi (2016). Lexicographic beliefs and assumption. *Journal of Economic Theory 163*, 955–985.

Dekel, E. and D. Fudenberg (1990). Rational behavior with payoff uncertainty. *Journal of Economic Theory 52*(2), 243–267.

Dekel, E. and M. Sinischalchi (2014). Epistemic game theory. *The Handbook of Game Theory 4*.

Epstein, L. G. and T. Wang (1996). " beliefs about beliefs" without probabilities. *Econometrica: Journal of the Econometric Society*, 1343–1373.

Feinberg, Y. (2000). Characterizing common priors in the form of posteriors. *Journal of Economic Theory 91*(2), 127–179.

Feinberg, Y. (2005a). Subjective reasoning—dynamic games. *Games and Economic Behavior 52*(1), 54–93.

Feinberg, Y. (2005b). Subjective reasoning—solutions. *Games and Economic Behavior 52*(1), 94–132.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of mathematical economics 18*(2), 141–153.

Harsanyi, J. (1967). Games with incomplete information played by "Bayesian" players, 1-3. *Management Science 14*(3).

Kets, W. (2017). Bounded reasoning and higher-order uncertainty. *Available at SSRN 2116626*.

Kohlberg, E. and J.-F. Mertens (1986). On the strategic stability of equilibria. *Econometrica 54*(5), 1003–1037.

Kreps, D. M. and R. Wilson (1982). Sequential equilibria. *Econometrica 50*(4), 863–894.

Mertens, J.-F. and S. Zamir (1985). Formulation of bayesian analysis for games with incomplete information. *International Journal of Game Theory 14*(1), 1–29.

Monderer, D. and D. Samet (1989). Approximating common knowledge with common beliefs. *Games and Economic Behavior 1*(2), 170–190.

Pearce, D. G. (1984). Rationalizable strategic behavior and the problem of perfection. *Econometrica 52*(4), 1029–1050.

Reny, P. J. (1992). Backward induction, normal form perfection and explicable equilibria. *Econometrica: Journal of the Econometric Society*, 627–649.

Samuelson, L. (1992). Dominated strategies and common knowledge. *Games and Economic Behavior 4*(2), 284–313.

Savage, L. J. (1954). *The foundations of statistics.* New York: Wiley.

Selten, R. (1975). Re-examination of the perfectness concept for equilibrium points in extensive games. *International Journal of Game Theory 4*, 25–55.

Wittgenstein, L. (1922). *Tractatus Logico-Philosophicus.* Longon, England: Kegan Paul, Trench, Trubner & Co. Ltd.

A Proofs

A.1 Proof of Propositon 1

One the one hand, if $\bar{P}[\bar{F},\bar{E}]$ is expressed by a statement $P \in \mathbb{S}_0$, then Rules 3-4 inductively imply that $\bar{P}$ must be increasing in $\bar{E}$.

On the other hand, suppose that $\bar{P}[\bar{F},\bar{E}]$ is a non-constant predicate that is increasing in $\bar{E}$. Suppose first that $F = \varnothing$. Predicate $\bar{P}$ can be expressed in language $\mathcal{L}_0$ by the statement

$$(\bigvee_{N \in \mathcal{N}(M)} \bigwedge_{e \in N} e),$$

where $\mathcal{N}(M) \subseteq 2^E$ is the set of subsets $N \subseteq E$ such that $\bar{P}(\omega) = \top$ at the valuation ω in which all the variables in N are true and all the variables in $E \backslash M$ are false. (As $\bar{P}$ is non-constant, set $\mathcal{N}(M)$ is nonempty.)

Suppose now that $F \neq \varnothing$. Fix some $\hat{f} \in F$. Predicate $\bar{P}$ can be expressed in language $\mathcal{L}_0$ by the statement

$$\bigvee_{M \subseteq F} \Big(\big(\bigwedge_{f \in M} f \wedge \bigwedge_{f \in F \backslash M} \neg f \big) \wedge T(M) \Big),$$

where for each $M \subseteq F$, either (i) $T(M) = (\hat{f} \wedge \neg \hat{f})$ if $\bar{P}(\omega) = \bot$ at the valuation ω in which all the variables in $M \sqcup E$ are true and all the variables in $F \backslash M$ are false; or (ii) $T(M) = (\hat{f} \vee \neg \hat{f})$ if $\bar{P}(\omega) = \top$ at the valuation ω in which all the variables in M are true and all the variables in $(F \backslash M) \sqcup E$ are false; or otherwise, (iii) $T(M) = \big(\bigvee_{N \in \mathcal{N}(M)} \bigwedge_{e \in N} e \big)$ where $\mathcal{N}(M) \subseteq 2^E$ is the nonempty set of subsets $N \subseteq E$ such that $\bar{P}(\omega) = \top$ at the valuation ω in which all the variables in $M \sqcup N$ are true and all the variables in $(F \sqcup E) \backslash (M \sqcup N)$ are false.

A.2 Proof of Theorem 1

The proof is by induction in level k: That is, throughout the proof below, I assume that either $k = 1$, or $k > 1$ and the theorem has been proven for all levels below k.

Consider a level-k player i. I will first prove the statement of the theorem for pictures E^i and $\tilde{E}^i$ such that $\tilde{E}^i$ is obtained from E^i by doing exactly one strengthening operation, T. I will focus on type-o players. The proof for the type-mo players is similar: For type-mo players, one just needs to construct a set of measures instead of a single measure. Each measure in the set is constructed exactly as in the case of type-o players.

The proof considers several cases. The general argument in all the cases is the following: Suppose $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, \tilde{E}^i\}\big)$. Let $\tilde{\Omega}$ be the state space of k-level player i under picture $\{CK^{i,I}\langle G \rangle, \tilde{E}^i\}$. Let Ω be his state space under picture $\{CK^{i,I}\langle G \rangle, E^i\}$. Let $\tilde{W}$ and W be the variables of states in $\tilde{\Omega}$ and Ω. Let $\tilde{\lambda}$ be a probability measure on $\tilde{\Omega}$ such that (i) $\tilde{\lambda}$ satisfies the semantic restrictions of $\tilde{E}^i$; and (ii) s is an optimal choice from S under belief $\tilde{\lambda}$. Then, one can transform measure $\tilde{\lambda}$ into another measure, λ, such that (i) λ is a measure on Ω; (ii) λ satisfies the restrictions of E^i; and (iii) under λ, the distribution over strategy profiles of player i's opponents is the same as under $\tilde{\lambda}$. This implies that $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, E^i\}\big)$.

Case 1: Operation T is of type $T.1$. Then, $\tilde{E}^i = \{\tilde{E}^i, E^i\}$, and so we only need to prove the inclusion $\Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, \tilde{E}^i\}\big) \subseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, E^i\}\big)$. If $\Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, \tilde{E}^i\}\big) = \varnothing$, we are done. Otherwise, take any $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, \tilde{E}^i\}\big)$. We need to show that $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G \rangle, E^i\}\big)$.

As $E^i \subset \tilde{E}^i$, $W \subseteq \tilde{W}$. The following observation will be useful:

Lemma 1. *If $\tilde{\omega} \in \tilde{\Omega}$, then $\tilde{\omega}_W \in \Omega$, where $\tilde{\omega}_W$ denotes the vector of W-coordinates of $\tilde{\omega}$.*

Proof. Suppose $k = 1$. Then, states in Ω and $\tilde{\Omega}$ have the same set of variables, and all these variables are atomic facts: $W = \tilde{W} \subset \mathcal{F}_0$. Then, $\tilde{\omega}_W = \tilde{\omega} \in \Omega$ as in Step 1.4, picture E^i imposes fewer restrictions on Ω than picture $\tilde{E}^i$.

Suppose $k > 1$. Then, $\tilde{W} \backslash W$ consists only of epistemic statements. Also, all epistemic statements in $\tilde{W}$ are regular. By the induction hypothesis, state $\tilde{\omega}_W$ is a possible state of affairs. Restrictions that E^i imposes on states in Step 1.4 are all expressed in terms of W-coordinates. As $\tilde{\omega}$ is included into $\tilde{\Omega}$, it satisfies these restrictions. Therefore, $\tilde{\omega}_W$ also satisfies these restrictions. Thus, $\tilde{\omega}_W \in \Omega$. □

Let $\tilde{\lambda}$ be a probability measure on $\tilde{\Omega}$ that is possible under picture $\tilde{E}^i$ and that justifies the choice of s. Let λ be the marginal of $\tilde{\lambda}$ on W-coordinates. By Lemma 1, $supp(\lambda) \subseteq \Omega$. Restrictions that statements E^i impose on $\tilde{\lambda}$ are all expressed in terms of λ. Hence, they are satisfied by λ. As λ and $\tilde{\lambda}$ correspond to the same marginal distribution over strategy profiles of player i's opponents, measure λ justifies the choice of s under picture $\{CK^{i,I}\langle G\rangle, E^i\}$. The case is proven.

Case 2: Operation T is one of operations of type $T.2$. In this case, states in $\tilde{\Omega}$ and Ω have the same variables: $\tilde{W} = W$.

Prove first that $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) \subseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E^i\}\big)$. Let $\tilde{\lambda}$ be a probability measure on $\tilde{\Omega}$ that is possible under picture $\tilde{E}^i$ and that justifies the choice of s. Picture $\tilde{E}^i$ imposes stricter restrictions on the state space and the belief of player i than picture E^i. Therefore, belief $\tilde{\lambda}$ is possible under $\{CK^{i,I}\langle G\rangle, E^i\}$. Thus, player i can play s under E^i. The inclusion is proven.

Prove now the equality $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big)$. The $\supseteq$-inclusion follows directly from Case 1. The $\subseteq$-inclusion follows from the fact that picture E^i does not impose any additional semantic restrictions above those that are already imposed by $\tilde{E}^i$. The case is proven.

Case 3: Operation T is one of operations of type $T.3$. I will prove the statement for operation $T.3.5$: In a statement, take a part $\big(B^j_p(P) \vee B^j_p(Q)\big)$ and replace it by $B^j_{2p}(P \vee Q)$ provided that $p \leqslant 1/2$. The proof for the other nine types of operations is similar.

Let $M^i(R)$ be the statement in E^i that needs to be modified by operation $T.3.5$ to obtain $\tilde{E}^i$. Let $M^i(\tilde{R})$ be the statement obtained by the modification. If $k = 1$, then statements $M^i(R)$ and $M^i(\tilde{R})$ are dismissed by player i. And so the case is trivially true. Suppose now that $k > 1$. Consider two sub-cases:

Case 3.1: Operation T changes R on the upper level. Let $v_1 = B^j_p(P)$ and $v_2 = B^j_p(Q)$ be the variables of R in the replaced part. Let $v_3 = B^j_{2p}(P \vee Q)$ be the variable of $\tilde{R}$ in the replacing part. In this case, $W \backslash \tilde{W} \subseteq \{v_1, v_2\}$.

Prove first that $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) \subseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E^i\}\big)$. Indeed, take any $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big)$. Let $\tilde{\lambda}$ be a probability measure on $\tilde{\Omega}$ that is possible for $\tilde{E}^i$ and that justifies the choice of s. Construct a measure, $\hat{\lambda}$, on the extended space $\hat{\Omega} = \{\top, \bot\}^{\tilde{W} \cup W}$ as follows. For each state $\tilde{\omega} \in \tilde{\Omega}$, measure $\hat{\lambda}$ takes probability mass $\tilde{\lambda}(\tilde{\omega})$ and puts it to the state $\hat{\omega}$ in which $\hat{\omega}_{\tilde{W}} = \tilde{\omega}$ and for each $v \in W \backslash \tilde{W}$, $\hat{\omega}_v = \bot$. As $\tilde{\lambda}$ is supported on possible states of affairs, all states in $supp(\hat{\lambda})$ are also possible. Consider any unmodified statement $e \in \tilde{E}^i \backslash \{M^i(\tilde{R})\}$. The restriction that e puts on $\hat{\lambda}$ is expressed in terms of $\tilde{\lambda}$, and thus, it is satisfied by $\hat{\lambda}$. The only restriction that $\hat{\lambda}$ may fail to satisfy is the restriction imposed by the old statement, $M^i(R)$.

From measure $\hat{\lambda}$, construct another measure, $\bar{\lambda}$, on $\hat{\Omega}$ as follows. For each state $\hat{\omega} \in supp(\hat{\lambda})$, do the following: If $\hat{\omega}_{v_3} = \bot$, then keep probability mass $\hat{\lambda}(\hat{\omega})$ in state $\hat{\omega}$. Otherwise, if either $\hat{\omega}_{v_1} = \top$

or $\hat{\omega}_{v_1} = \top$, then keep probability mass $\hat{\lambda}(\hat{\omega})$ in state $\hat{\omega}$. Finally, if $\hat{\omega}_{v_3} = \top$, but $\hat{\omega}_{v_1} = \bot$ and $\hat{\omega}_{v_1} = \bot$, then shift probability mass $\hat{\lambda}(\hat{\omega})$ to another state according to the following rule: Let $\hat{\omega}[v_1 = \top]$ be the state that coincides with $\hat{\omega}$ in all variables except for v_1. Similarly, let $\hat{\omega}[v_2 = \top]$ be the state that coincides with $\hat{\omega}$ in all variables except for v_2. Note that any belief of player j that satisfies restriction $B^j_{2p}(P \vee Q)$, must also satisfy at least one of restrictions $B^j_p(P)$ or $B^j_p(Q)$. As $\hat{\omega} \in supp(\hat{\lambda})$, state $\hat{\omega}$ describes a possible state of affairs. Therefore, at least one state $\hat{\omega}[v_2 = \top]$ or $\hat{\omega}[v_2 = \top]$ must also be possible. Shift probability mass $\hat{\lambda}(\hat{\omega})$ to that of states $\hat{\omega}[v_2 = \top]$ and $\hat{\omega}[v_2 = \top]$ which is possible. (If both states are possible, shift mass $\hat{\lambda}(\hat{\omega})/2$ to each of them.)

By construction, $\bar{\lambda}$ is supported in possible states of affairs. Also, the shift of probability mass from $\hat{\lambda}$ to $\bar{\lambda}$ is monotone in epistemic variables: When a probability is shifted from a state, it always goes to a state with higher truth values of variables v_1 and v_2, and the same truth values of other variables. Proposition 1 then implies that $\bar{\lambda}$ satisfies each semantic restriction that is satisfied by $\hat{\lambda}$. Finally, $\bar{\lambda}$ satisfies restriction $M^i(R)$ because (i) $\bar{\lambda}$ satisfies restriction $M^i(\tilde{R})$ (as it is satisfied by $\hat{\lambda}$), and (ii) in each state in $supp(\bar{\lambda})$, if $v_3 = \top$, then either $v_1 = \top$ or $v_2 = \top$, so that if $\tilde{R}$ is true, then P is also true.

Finally, let λ be the marginal of $\bar{\lambda}$ on W-coordinates. By construction, λ is supported on possible states in Ω, satisfies all the restrictions of E^i, and has the same marginal distribution as $\tilde{\lambda}$ over strategy profiles of player i's opponents. Hence, λ justifies the choice of s under picture $\{CK^{i,I}\langle G\rangle, E^i\}$. The implication is proven.

Prove now the equality $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big)$. The $\supseteq$-inclusion follows directly from Case 1. The $\subseteq$-inclusion follows from the fact that measure λ constructed above satisfies restrictions of $\tilde{E}^i$.

Case 3.2: Operation T changes R on a deeper level: That is, it changes some variable of R. In this case, T makes one of the variables of R more precise. Let v_1 be the variable of P that is changed to variable v_2 in $\tilde{R}$ (v_1 may still be a variable of $\tilde{R}$, as v_1 may enter statement R in several places, whereas operation T acts only in one place).

Take any $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big)$. We need to show that $s \in \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E^i\}\big)$. Let $\tilde{\lambda}$ be a probability measure on $\tilde{\Omega}$ that is possible for $\tilde{E}^i$ and that justifies the choice of s. To construct measure λ on Ω that satisfies restrictions of E^i one can follow a procedure similar to that in Case 3.1: Expand the set $\tilde{W}$ to include variable v_1 (if it is not already in $\tilde{W}$). Then, shift the measure increasing the truth values of epistemic variables in such a way that the shifted measure is supported only in states in which if v_2 is true then v_1 is also true. The possibility of such states is guaranteed by the induction hypothesis. The equality $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big)$ is also proven analogously to Case 3.1.

To finish the proof of the theorem, suppose that $\tilde{E}^i$ is obtained from E^i in a finite sequence of strengthening operations: $E(0) = E^i$, $E(1)$,..., $E(n) = \tilde{E}^i$, where for each $m \in \{1, .., n\}$, picture $E(m)$ is obtained from picture $E(m-1)$ by doing one strengthening operation. The inclusion $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) \subseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E^i\}\big)$ follows immediately from the chain of inclusions $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E(m)\}\big) \subseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E(m-1)\}\big)$, for $m \in \{1, .., n\}$.

Let us prove the equality $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big)$. For each $m \in \{1, .., n\}$, picture $\{E(m), E(m+1), .., E(n)\}$ can be obtained from $\{E(m-1), E(m+1), .., E(n)\}$ by doing at most one strengthening operation. This implies that $\forall m \in \{1, .., n\}$,

$$\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E(m), E(m+1), .., E(n)\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E(m-1), E(m).., E(n)\}\big).$$

Applying this equality recursively, we obtain that

$$\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, E(0), E(1), .., E(n)\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big).$$

As $\{E(0), E(1), .., E(n)\}$ is more precise than $\{\tilde{E}^i, E^i\}$, we then have

$$\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big) \supseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big).$$

Yet as $\{\tilde{E}^i, E^i\}$ is more precise than $\tilde{E}^i$, we have

$$\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big) \subseteq \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big).$$

Thus, $\Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i, E^i\}\big) = \Pi\big(i, o, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big)$, and the theorem is proven.

A.3 Proof of Theorem 1′

The proof for a type-*co* player i adapts the proof of Theorem 1 in the following way: Take any $s \in \Pi\big(i, co, k, \{CK^{i,I}\langle G\rangle, \tilde{E}^i\}\big)$. Let $\tilde{\rho} = \{\tilde{\lambda}_n\}_{n=1}^N$ be an LPS with full support of $\tilde{\Omega}$ that justifies the choice of s. As $\tilde{E}^i$ is undogmatic, for each strategy profile $s^J \in S^J$ of player i's opponents, there is a state in $\tilde{\Omega}$ at which s^J is played (Any profile in S^J is possible for level-0 opponents, and so can not be dismissed by an undogmatic player i.)

Take the first measure in $\tilde{\rho}$, $\tilde{\lambda}_1$, and construct measure λ_1 on Ω exactly as in the proof of Theorem 1. As language $\mathcal{L}_+$ contains only the knowledge and p-belief modalities, all the semantic restrictions that E^i puts on player i's belief are the restrictions on the first measure in his LPS. As E^i is undogmatic, it only puts restrictions on beliefs. Thus, any LPS with full support on Ω whose first measure is λ_1 will satisfy all of the restrictions of E^i

For each measure λ_n, $n \in N$, in LPS $\tilde{\rho}$, construct a corresponding measure, $\hat{\lambda}_n$, on Ω as follows. For each state $\tilde{w} \in \tilde{\Omega}$, measure $\hat{\lambda}_n$ takes probability mass $\tilde{\lambda}_n(\tilde{\omega})$ and distributes it uniformly over states in set $T(\tilde{\omega}) \subseteq \Omega$, which is the set of states in which player i's opponents play the same strategy profile that they play is state $\tilde{w}$. (As E^i is undogmatic, it does not exclude level-0 opponents who can play any strategy profile in the game. Therefore, $T(\tilde{\omega}) \neq \varnothing$ for all $\tilde{w} \in \tilde{\Omega}$.) Naturally, for each $n \in N$, measure $\hat{\lambda}_n$ induces the same marginal distribution on strategy profiles of player i's opponents as measure $\tilde{\lambda}_n$. As $\tilde{\rho}$ has full support on $\tilde{\Omega}$, LPS $\hat{\rho} = \{\tilde{\lambda}_n\}_{n=1}^N$ has full support on Ω.

Finally, LPS $\rho = (\lambda_1, \hat{\rho})$ is such that it (i) has full support on Ω; (ii) satisfies restrictions of E^i; and (iii) has the same marginal on strategy profiles of player i's opponents as $\tilde{\rho}$. Therefore, ρ justifies $s \in \Pi\big(i, co, k, \{CK^{i,I}\langle G\rangle\}\big)$.

A.4 Proof of Theorem 2

I will focus on type-*o* players. The proof for type-*mo* players is similar: One just needs to construct a set of measures instead of a single measure. Each measure in the set is constructed exactly as in the case of type-*o* players.

Extend function Π to level-0 players: For each $i \in I$, assign $Pi\big(i, opt, 0, \{CK^{i,I}\langle G\rangle, E^i\}\big) := S^i$, where S^i is the set of player i's strategies in game G. Prove now that

$$\forall k \in \mathbb{Z}_+, \Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i\}\big).$$

The proof is by induction in k. For $k = 0$, the statement is true as $\Pi\big(i, opt, 0, \{CK^{i,I}\langle G\rangle, E^i\}\big)$ equals the whole set of player i's strategies in game G. Suppose the statement is true for all levels strictly below $k \in \mathbb{N}$. Prove then that the statement is true for k.

As in their process of reasoning, players dismiss statements with levels higher than their own level of reasoning, it suffice to show that

$$\Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k+1}\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}\big).$$

Picture $E^i_{\leqslant k+1}$ can be obtained from $E^i_{\leqslant k}$ by adding a finite set of regular statements, $E^i_{\leqslant (k+1)} \backslash E^i_{\leqslant k}$. Thus, $E^i_{\leqslant k+1}$ is more precise than $E^i_{\leqslant k}$. By Theorem 1,

$$\Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k+1}\}\big) \subseteq \Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}\big).$$

To finish the proof, it suffices to show that

$$\Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}\big) \subseteq \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}\big).$$

Let $s \in \Pi\big(i, opt, (k+1), \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}\big)$. Let $\tilde{\Omega}$ be the state space constructed under $\{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}$ by a level-$(k+1)$ player i. Let Ω be the space constructed under $\{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}$ a level-k player i. Let $\tilde{W}$ and W be the variables in states of $\tilde{\Omega}$ and Ω. We have $\tilde{W} = W \sqcup \{l_{j,k}\}_{j\in J}$, where J is the set of player i's opponents in game G.

Let $\tilde{\lambda}$ be a measure on $\tilde{\Omega}$ that satisfies the restrictions of $E^i_{\leqslant k}$ and justifies the choice of s. Using measure $\tilde{\lambda}$, construct measure λ on Ω by downgrading level-k opponents of player i to level-$(k-1)$ as follows. For each $\tilde{\omega} \in \tilde{\Omega}$, measure λ sends the probability mass $\tilde{\lambda}(\tilde{\omega})$ to the state $\omega \in \Omega$ in which: (i) $\forall j \in J$, $w_{l_{j,k-1}} = w_{r_j} = \top$ if either $\tilde{w}_{l_{j,k}} = \top$ or $\tilde{w}_{l_{j,k-1}} = \top$; (ii) the values of all other variables are the same as in $\tilde{\omega}$.

By the induction hypothesis, all states in $supp(\lambda)$ are possible. As $E^i_{\leqslant k}$ is positive, restrictions in $E^i_{\leqslant k}$ are formulated in terms of predicates that are increasing in the level and the rationality variables. Relative to $\tilde{\lambda}$, measure λ increases the probability that these variables are true. Therefore, λ also satisfies the semantic restrictions of $E^i_{\leqslant k}$. As λ does not change the strategy variables, it has the same marginal distribution as $\tilde{\lambda}$ over strategy profiles of player i's opponents. Therefore, λ justifies the choice of s for a level-k player i under picture $\{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}$. Hence, $s \in \Pi\big(i, opt, k, \{CK^{i,I}\langle G\rangle, E^i_{\leqslant k}\}\big)$.

A.5 Proof of Theorem 3

The proof is by induction in level k. Suppose that either $k = 1$, or that $k > 1$ and the theorem has been proven for all levels below k. Prove the statement for level k.

Take a level-k player i who has picture $\{CK^{i,I}\langle\Gamma_n\rangle, CK^{i,I}[CSO], E^i\}$, $n \in \{1,2\}$. Let J be the set of player i's opponents in game $G = G(\Gamma_n)$. Let S^j be the set of strategies in game Γ_n for player $j \in J$. (As $G(\Gamma_1) = G(\Gamma_2)$, S^j does not depend on index n.) Let $E^i_{\leqslant k}$ be the set of statements from E^i that level-k player i will analyze in his process of reasoning. Let Ω_n be the state space constructed by that player i. Let W_n be its variables. Let

$$V = \Big(\bigsqcup_{j\in J,\, s\in S^j}\{st_{j,s}\} \sqcup \bigsqcup_{j\in J}\{cso_j\} \sqcup \bigsqcup_{j\in J,\, 0\leqslant m\leqslant k-1}\{l_{j,m}\} \sqcup \bigsqcup_{j\in J}\{r_j\}\Big) \cup Var_{-1}(E^i_{\leqslant k}).$$

Naturally, $V \subseteq W_n$. Let $proj_V(\Omega_n)$ be the projection of Ω_n on the V-coordinates. As $E^i_{\leqslant k}$ is invariant, all the variables in V are invariant and all the variables in $W_n \backslash V$ are not invariant. As $E^i_{\leqslant k}$ is invariant, the semantic restrictions that $E^i_{\leqslant k}$ puts on player i's LPS belief are formulated in terms of the marginal distribution on space $proj_V(\Omega_n)$. By the induction hypothesis, $proj_V(\Omega_1) = proj_V(\Omega_2)$. Therefore, the set of possible marginal distributions on S^J that are possible after Step 2 of the player i's process of reasoning does not depend on index n. This implies the equality for cautiously optimal behavior:

$$\Pi\big(i, co, k, \{CK^{i,I}\langle\Gamma_1\rangle, CK^{i,I}[CSO], E^i\}\big) = \Pi\big(i, co, k, \{CK^{i,I}\langle\Gamma_2\rangle, CK^{i,I}[CSO], E^i\}\big).$$

The equality for cautiously sequentially optimal behavior immediately follows from the following:

Lemma 2. *Let Γ be a finite dynamic game with perfect recall played by players $I = \{i, J\}$. Let $s \in S^i$ be a strategy of player i. Let $\hat{S}^J \subseteq S^J$ be a subset of strategy profiles of players J. Let ρ be an LPS with full support on $\hat{S}^J$. If s is cautiously optimal against belief ρ, then s is also cautiously sequentially optimal against ρ at all player i's information sets relevant for s and S^J.*

Proof. Take any information set h of player i that is relevant for s and $\hat{S}^J$. Let $\tilde{S}^J$ be the set of profiles in $\hat{S}^J$ that are relevant for h. As ρ has full support on $\hat{S}^J$, the conditional LPS $\rho_{\tilde{S}^J} \neq 0$. Let $\tilde{s} \in S^i$ be a strategy relevant for h. If $\tilde{s}$ is an improvement over s conditionally on $\tilde{S}^J$, then $\tilde{s}$ is also an unconditional improvement over s. As s is optimal unconditionally, it must then be optimal conditionally on $\tilde{S}^J$. Therefore, s is cautiously sequentially optimal. □

A.6 Proof of Proposition 2

Let Ω be the state space that type-o level-k player 1 constructs in his process of reasoning. Suppose that Step 1 is completed without an error. Then, $\Omega \neq \varnothing$. In each state $\omega \in \Omega$, the following facts are true: $l_{2,k-1}$; o_2; $K^2(K^1\langle G\rangle)$; $K^2(r_1)$; $K^2(o_1)$; $K^2\big(C^1(\phi^1)\big)$. Consider the type-o level-$(k-1)$ player 2 described in state ω: As ω is possible, her state space $\tilde{\Omega} \neq \varnothing$. In each $\tilde{\omega} \in \tilde{\Omega}$, the following facts are true: $l_{1,k-2}$; o_1; $K^1\langle G\rangle$; $C^1(\phi^1)$. As $k \geqslant 3$, strategy $s \in S^1$ that level-$(k-2)$ player 1 plays in state $\tilde{w}$ must be a best response against his conjecture, which is ϕ^1. Thus, the conjecture that player 2 holds is ω must assign positive weights only to those strategies in S^1 that are best against ϕ^1. But in each $\omega \in \Omega$, fact $C^2(\phi^2)$ is true. Therefore, ϕ^2 is a best response against ϕ^1.

Suppose now that additionally, conjecture ϕ^1 is possible for the type-o level-k player 1 in Step 2 of his process of reasoning: That is, suppose that $supp(\phi^1) \subseteq proj_{S^2}(\Omega)$, where $proj_{S^2}(\Omega)$ is the projection of Ω on the S^2-coordinates. In each $\omega \in \Omega$, the following facts are true: $l_{2,k-1}$; o_2; $K^2\langle G\rangle$; $C^2(\phi^2)$. Therefore, in each $\omega \in \Omega$, player 2 plays a strategy $s \in S^2$ that is a best response against ϕ^2. Thus, ϕ^1 is a best response against ϕ^2, and (ϕ^2, ϕ^1) is a Nash equilibrium of G.