КП интегрируемость непертурбативной тау функции

(по совместной статье с А.Александровым, Б.Бычковым, П.Дунин-Барковским и С.Шадриным)

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Main Theorem

Theorem ([ABDKS2024])

Non-perturbative tau function is a KP tau function

(Conjecture Eynard-Mariño 2011, Eynard-Borot 2012, Eynard-Oukassi 2024)

The np tau function is defined by a kind of convolution of two constructions:

- Krichever construction of algebraic-geometrical solutions of KP hierarchy
- CEO topological recursion

KP integrability

Viewpoints to KP integrable hierarchy of PDE's

- ΨDO's and wave functions, commuting time flows, Lax representation
- tau function, Hirota bilinear equations, Fay identities
- Sato Grassmannian, boson-fermion correspondence, KP symmetries
- determinantal equalities

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- **determinantal equalities**

Determinantal equalities

$$\log \tau(t_1, t_2, \dots) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{k_1,\dots,k_n=1}^{\infty} f_{(k_1,\dots,k_n)} t_{k_1} \dots t_{k_n}$$

$$\omega_n(z_1, \dots, z_n) = \sum_{k_1,\dots,k_n=1}^{\infty} f_{(k_1,\dots,k_n)} \prod_{i=1}^{n} z_i^{k_i - 1}$$

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Theorem

τ is a KP tau function $\Leftrightarrow$ there exists $K(z_1, z_2) = \sqrt{dz_1}\sqrt{dz_2}\left(\frac{1}{z_1 - z_2} + (regular)\right)$ such that

$$\omega_n = \det{}^{\circ} \|K(z_i, z_j)\|_{i,j=1,\dots,n} = (-1)^{n-1} \sum_{s \in \mathrm{cycl(n)}} \prod_{i=1}^{,} K(z_i, z_{s(j)}), \quad n \geq 2$$

$$\omega_1 = \left(K(z_1, z_2) - \frac{\sqrt{dz_1}\sqrt{dz_2}}{z_1 - z_2}\right)\Big|_{z_2 = z_1}$$

Corollary

The whole information on a solution of KP hierarchy is encoded in a single function $K(z_1, z_2)$

In fact,

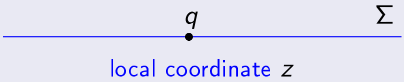
$$K(z_1,z_2) = \frac{\sqrt{dz_1}\sqrt{dz_2}}{z_1 - z_2}\, \tau \Big|_{t_k = \frac{z_1^k - z_2^k}{k}}$$

$$= \frac{\sqrt{dz_1}\sqrt{dz_2}}{z_1 - z_2}\, e^{\sum\limits_{n=1}^{\infty} \frac{1}{n!} \left(\int\limits_{z_2}^{z_1}\right)^n \left(\omega_n - \delta_{n,2} \frac{d\tilde{z}_1 d\tilde{z}_2}{(\tilde{z}_1 - \tilde{z}_2)^2}\right)}$$

Global KP integrability

Definition

We say that a tau function *possess a spectral curve* if the corresponding differentials ω_n extend as global meromorphic differentials on Σ^n for some complex curve Σ, one and the same for all $n \geq 1$.



If $\{\omega_n\}$ are defined globally, then a tau function can be associated to any choice of a point $q \in \Sigma$ regular for all ω_n's and any other choice of local coordinate at q

Theorem

KP integrability is an internal property of a system of differentials: a tau function associated to some choice of (q, z) is a KP tau function iff it is a KP tau function for any other choice. Moreover, K is invariantly defined and determinantal equalities hold as equalities of global meromorphic differentials

Krichever construction

Example

Σ is a genus g Riemann surface
$(\eta_1,\dots,\eta_g)$ a basis of holomorphic differentials, $\int_{A_i}\eta_j=\delta_{i,j}$.

$$\phi_n(p_1,\dots,p_n)=\sum_{k_1,\dots,k_n=1}^{g} c_{k_1,\dots,k_n}\eta_{k_1}(p_1)\dots\eta_{k_n}(p_n)+\delta_{n,2}B(p_1,p_2),$$

$$c_{k_1,\dots,k_n}=\frac{\partial\log(\theta(w_1,\dots,w_g))}{\partial w_{k_1}\dots\partial w_{k_n}},\qquad (p_1,\dots,p_n)\in\Sigma^n.$$

θ is the Riemann theta functin,
B is the *Bergman kernel*, a symmetric bidifferential with a pole on the diagonal with biresidue 1 fixed by condition of vanishing of $\mathfrak{A}$-periods

Theorem (A reformulation of Krichever theorem)

The system of differentials $\{\phi_n\}$ is KP integrable

Topological recursion

Chekhov-Eynard-Orantin '06
Eynard-Orantin '07

Initial data $\overset{\text{CEO TR}}{\rightsquigarrow}$ $\{\omega_n\}$,

Topological recursion provides a universal procedure leading to a solution of huge amount of enumerative problems in combinatorics and mathematical physics icluding

- enumeration of Hurwitz numbers and their variations
- enumeration of maps and hypermaps (embedded graphs)
- computation of matrix integrals
- intersection theory of moduli spaces and Gromov-Witten theory
- ...

Topological recursion

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Initial data $\overset{\text{CEO TR}}{\rightsquigarrow}$ $\{\omega_n\}$,

Initial data of TR: (Σ, x, y, B)

- Σ a smooth algebraic complex curve with fixed system of $(\mathfrak{A}, \mathfrak{B})$-cycles;
- x, y meromorphic functions on Σ satisfying the nondegeneracy condition: *all zeroes of dx are simple and dy is holomorphic and non-vanishing at each of them*;
- B the *Bergman kernel*, a symmetric 2-differential on $\Sigma \times \Sigma$ with a pole on the diagonal of the kind
$$B(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2} + (\text{holomorphic})$$
and no other singularities. It is fixed uniquely by the requirement of vanishing $\mathfrak{A}$-periods.

Topological recursion: basic properties of TR differentials

Result of the recursion: the *n*-differentials

$$\omega_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n} \omega_n^{(g)}$$

- Initial differentials: $\omega_1^{(0)} = y\,dx, \quad \omega_2^{(0)} = B$
- All other differentials $\omega_n^{(g)}$ have the following analytic properties:
  - they are globally defined on Σ^n and meromorphic;
  - all possible poles are at the zeroes of dx;
  - have vanishing $\mathfrak{A}$-periods.

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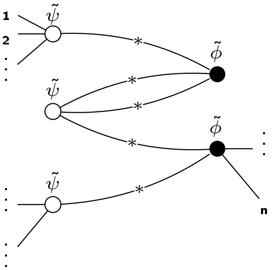
The differentials $\omega_n^{(g)}$, $2g - 2 + n > 0$, are computed recursively from the analysis of their behaviour near the poles

Non-perturbative differentials

$$\{\psi_n\},\ \{\phi_n\} \quad \rightsquigarrow \quad \{\omega_n\}$$

$\{\psi_n\}$ TR differenials for the initial data (Σ, x, y, B)
$\{\phi_n\}$ differentials of Krichever construction
the nonperurbative differentials ω_n are defined as a sum over graphs,

$$\omega_n = \sum_{\gamma\in\Gamma_n} \frac{w(\gamma)}{|\mathrm{Aut}(\gamma)|} + \delta_{n,2}B$$



- leaves are numbered
- white vertices are decorated with $\tilde\psi_m = \psi_m - \delta_{m,2}B$
- black vertices are decorated with $\tilde\phi_m = \phi_m - \delta_{m,2}B$
- internal edges are decorated with the convolution operation $*$; $\psi * \phi = \sum_{q\in\mathcal{P}} \operatorname*{res}_{p=q} \psi(p) \int^p \phi$
- graphs are connected

Non-perturbative differentials: general properties

	Krichever differentials ϕ_n	TR differentials ψ_n	np differentials ω_n
symmetric	✓	✓	✓
$\hbar$-dependency	✗	topological	non-topological
singularity on the diagonal for $n = 2$ only	✓	✓	✓
other singularities	✗	$\{dx = 0\}$	$\{dx = 0\}$
KP integrability	✓	If $\Sigma = \mathbb{C}P^1$ only	✓ (Main Theorem)

$$\psi_n = \sum_{g \geq 0} \hbar^{2g-2+n} \psi_n^{(g)}, \qquad \omega_n = \sum_{d \geq 0} \hbar^d \omega_n^{\langle d \rangle}$$

Generalization: blobbed topological recursion

Initial data of TR: (Σ, x, y, B).
Initial data of blobbed TR: $(\Sigma, x, y, \{\phi_n\})$.

- Restrict initial data to the local setting
- Take any B; we assume that $\tilde{\phi}_n = \phi_n - \delta_{2,n} B$ is holomorphic at zeroes of dx
- Compute TR differentials $\{\psi_n\}$ for the initial data (Σ, x, y, B)
- Convolve them with the blobs $\{\phi_n\}$ via summation over graphs

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Theorem

If the system of blobs $\{\phi_n\}$ is KP integrable, then the system of blobbed TR differentials $\{\omega_n\}$ is also KP integrable

KP integrability of TR differentials with a rational spectral curve

Example

TR is a special case of blobbed TR with $\phi_n = \delta_{n,2} B$. Such system of blobs is KP integrable iff B is of the form

$$B(p_1, p_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}, \quad z_i = z(p_i),$$

for some meromorphic function z.

Corollary

The system of TR differentials is KP integrable iff the spectral curve is rational

Conclusion

- The nonperturbative differentials provide a new large family of algebraic-geometrical solutions of KP hierarchy
- They generalize both Krichever construction and the KP integrability of topological recursion for the rational spectral curve observed earlier
- These solutions are non-trivial and reach even for the case of a rational spectral curve
- They are defined in an asymptotic expansion in an additional parameter $\hbar$. The restriction $\hbar = 0$ recovers the original construction of Krichever

Thanks for your attention!