Flat Coordinates and Integrable Systems

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Flat Coordinates and Hydrodynamic Type Systems

B.A. Dubrovin and S.P. Novikov introduced the concept of homogeneous differential-geometric Poisson brackets of first order in 1983.

Corresponding Hamiltonian systems can be written in flat coordinates

$$a_t^k = \eta^{km} \left(\frac{\delta \mathbf{H}}{\delta a^m} \right)_x, \quad \mathbf{H} = \int h(a, a_x, a_{xx}, ...) dx, \quad k, m = 1, 2, ..., N.$$

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Here the matrix η^{km} is symmetric, constant and non-degenerate.
The above N component system has two extra conservation laws. The conservation law of momentum

$$p_t = \left(a^m \frac{\delta \mathbf{H}}{\delta a^m} - F\right)_x,$$

where

$$p = \frac{1}{2}\eta_{km}a^k a^m$$

and the function $F(a, a_x, a_{xx}, ...)$ can be found from

$$\partial_x F = \frac{\delta \mathbf{H}}{\delta a^m} a_x^m.$$

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In a dispersionless and in a dispersive cases, the common feature of the Dubrovin–Novikov Hamiltonian structure is the momentum density

$$p = \frac{1}{2}\eta_{km}a^k a^m.$$

Flat Coordinates and Hydrodynamic Type Systems

If some hydrodynamic type system has N local conservation laws

$$a_t^k = (b^k(a))_x, \quad k = 1, 2, ..., N$$

and $(N+1)$-st local conservation law

$$p_t = f_x,$$

such that

$$p = \frac{1}{2}\eta_{km} a^k a^m.$$

Then this hydrodynamic type system has the local Hamiltonian structure

$$a_t^k = \eta^{km} \left(\frac{\delta \mathbf{H}}{\delta a^m} \right)_x, \quad \mathbf{H} = \int h(a) dx, \quad k, m = 1, 2, ..., N,$$

where the Hamiltonian density can be found in quadratures

$$dh = \eta_{ms} b^s(a) da^m.$$

Flat Coordinates and Hydrodynamic Type Systems

The KP hierarchy is determined by infinitely many linear equations (the so called Lax representation)

$$\psi_{t^k} = \hat{L}_k \psi, \quad \hat{L}_k = \frac{1}{k}(\hat{L}^k)_+$$

where

$$\hat{L} = \partial_x + a_0 \partial_x^{-1} + a_1 \partial_x^{-2} + a_2 \partial_x^{-3} + \ldots$$

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First two linear equations

$$\psi_t = \frac{1}{2}\psi_{xx} + u\psi, \quad \psi_y = \frac{1}{3}\psi_{xxx} + u\psi_x + v\psi$$

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Under the substitution $r = (\ln \psi)_x$ these linear equations become

$$r_t = \left(\frac{1}{2}(r^2 + r_x) + u\right)_x, \quad r_y = \left(\frac{1}{3}(r^3 + 3rr_x + r_{xx}) + ur + v\right)_x.$$

The substitution of the expansion ($\lambda \to \infty$)

$$r = \lambda - \frac{H_0}{\lambda} - \frac{H_1}{\lambda^2} - \frac{H_2}{\lambda^3} - \dots$$

into the first above equation

$$r_t = \left(\frac{1}{2}(r^2 + r_x) + u \right)_x$$

yields the dispersive integrable chain (where $u = H_0$)

$$H_{k,t} = \left(H_{k+1} + \frac{1}{2} H_{k,x} - \frac{1}{2} \sum_{m=0}^{k-1} H_m H_{k-1-m} \right)_x , \quad k = 0, 1, 2, \dots$$

Flat Coordinates and Hydrodynamic Type Systems

In a dispersionless limit this integrable chain

$$H_{0,t} = H_{1,x}, \quad H_{k,t} = \left(H_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H_m H_{k-1-m} \right)_x, \quad k = 1, 2, \ldots$$

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is well known as the Benney moment chain

$$A_t^k = A_x^{k+1} + kA^{k-1} A_x^0, \quad k = 0, 1, 2, ...,$$

where $H_k(A)$ are polynomials.
The Benney moment chain has plenty of finite-component reductions, where the simplest is determined by the reduction

$$H_N = 0.$$

B.A. Kupershmidt and Yu.I. Manin found the local Hamiltonian structure of the Benney moment chain in 1978

$$A_t^k = [kA^{k+m-1}\partial_x + m\partial_x A^{k+m-1}]\frac{\partial H_2}{\partial A^m}, \quad H_2 = \frac{1}{2}[A^2 + (A^0)^2].$$

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Nevertheless, the reduction of the above (Kupershmidt–Manin) Hamiltonian structure to the finite-component case $H_N = 0$ is a nontrivial problem. By this reason, corresponding local Hamiltonian structure was found independently.

Flat Coordinates and Hydrodynamic Type Systems

The reduction $H_N = 0$ determines N component hydrodynamic type system

$$H_{k,t} = \left(H_{k+1} - \frac{1}{2} \sum_{m=0}^{k-1} H_m H_{k-1-m} \right)_x, \quad k = 0, 1, 2, ..., N-1.$$

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However the next conservation law is

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Taking into account that $H_N = 0$, we obtain the momentum density

$$H_{N+1} = \frac{1}{2} \sum_{m=0}^{N-1} H_m H_{N-1-m}.$$

This means H_m are flat coordinates and the above system is Hamiltonian, i.e.

$$H_{m,t} = \left(\frac{\partial H_{N+2}}{\partial H_{N+1-m}} \right)_x .$$

Flat Coordinates and Dispersive Integrable Systems

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also has a similar reduction

$$\tilde{H}_N = 0,$$

where $\tilde{H}_N$ is a differential polynomial with respect to lower conservation law densities H_k including their higher order derivatives.