Geometric representatives in SU-bordism classes

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1. Unitary bordism

The unitary bordism ring Ω^U consists of complex bordism classes of stably complex manifolds.

A stably complex manifold is a pair $(M, c_\mathcal{T})$ consisting of a smooth manifold M and a stably complex structure $c_\mathcal{T}$, determined by a choice of an isomorphism

$$c_\mathcal{T} \colon \mathcal{T}M \oplus \underline{\mathbb{R}}^N \xrightarrow{\cong} \xi$$

between the stable tangent bundle of M and a complex vector bundle ξ.

Theorem (Milnor–Novikov)

- *Two stably complex manifolds M and N represent the same bordism classes in Ω^U iff their sets of Chern characteristic numbers coincide.*
- *Ω^U is a polynomial ring on generators in every even degree:*

$$\Omega^U \cong \mathbb{Z}[a_1, a_2, \ldots, a_i, \ldots], \quad \deg a_i = 2i.$$

Polynomial generators of Ω^U are detected using a special characteristic class s_n. It is the polynomial in the universal Chern classes $c_1, \ldots, c_n$ obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $\sigma_i(x_1, \ldots, x_n)$ and replacing each σ_i by c_i.

$s_n[M] = s_n(\mathcal{T}M)\langle M\rangle$: the corresponding characteristic number.

Theorem

The bordism class of a stably complex manifold M^{2i} may be taken to be the polynomial generator $a_i \in \Omega^U_{2i}$ iff

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if} \quad i+1 \neq p^s \quad \text{for any prime } p, \\ \pm p & \text{if} \quad i+1 = p^s \quad \text{for some prime } p \text{ and integer } s > 0. \end{cases}$$

Problem

Find geometric representatives in (unitary) bordism classes; e.g., smooth algebraic varieties or manifolds with large symmetry.

2. Special unitary bordism

A stably complex manifold $(M, c_{\mathcal{T}})$ is special unitary (an SU-manifold) if $c_1(M) = 0$. Bordism classes of SU-manifolds form the special unitary bordism ring Ω^{SU}.

The ring structure of Ω^{SU} is more subtle than that of Ω^U. Novikov described $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ (it is a polynomial ring). The 2-torsion was described by Conner and Floyd. We shall need the following facts.

Theorem

- *The kernel of the forgetful map $\Omega^{SU} \to \Omega^U$ consists of torsion.*
- *Every torsion element in Ω^{SU} has order 2.*
- *$\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra on generators in every even degree > 2:*

$$\Omega^{SU} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][y_i \colon i > 1], \quad \deg y_i = 2i.$$

3. U- and SU-theory

$\Omega^U = U_*(pt) = \pi_*(MU)$ is the coefficient ring of the complex bordism theory, defined by the Thom spectrum $MU = \{MU(n)\}$, where $MU(n)$ is the Thom space of the universal $U(n)$-bundle $EU(n) \to BU(n)$:

$$U_n(X,A) = \lim_{k\to\infty} \pi_{2k+n}\bigl((X/A)\wedge MU(k)\bigr),$$
$$U^n(X,A) = \lim_{k\to\infty} \bigl[\Sigma^{2k-n}(X/A), MU(k)\bigr]$$

for a CW-pair (X,A).

Similarly, $\Omega^{SU} = SU_*(pt) = \pi_*(MSU)$ is the coefficient ring of the SU-theory, defined by the Thom spectrum $MSU = \{MSU(n)\}$:

$$SU_n(X,A) = \lim_{k\to\infty} \pi_{2k+n}\bigl((X/A)\wedge MSU(k)\bigr),$$
$$SU^n(X,A) = \lim_{k\to\infty} \bigl[\Sigma^{2k-n}(X/A), MSU(k)\bigr].$$

4. c_1-spherical bordism W

Consider closed manifolds M with a c_1-spherical structure, which consists of

- a stably complex structure on the tangent bundle $\mathcal{T}M$;
- a $\mathbb{C}P^1$-reduction of the determinant bundle, that is, a map $f\colon M \to \mathbb{C}P^1$ and an equivalence $f^*(\eta) \cong \det \mathcal{T}M$, where η is the tautological bundle over $\mathbb{C}P^1$.

This is a natural generalisation of an SU-structure, which can be thought of as a "$\mathbb{C}P^0$-reduction", that is, a trivialisation of the determinant bundle.

The corresponding bordism theory is called c_1-spherical bordism and is denoted W_*. It is instrumental in describing the SU-bordism ring and other calculations in the SU-theory.

As in the case of stable complex structures, a c_1-spherical complex structure on the stable tangent bundle is equivalent to such a structure on the stable normal bundle. There are forgetful transformations $MSU_* \to W_* \to MU_*$.

Homotopically, a c_1-spherical structure on a stable complex bundle $\xi\colon M \to BU$ is defined by a choice of lifting to a map $M \to X$, where X is the (homotopy) pullback:

$$\begin{array}{ccccc} & & X & \longrightarrow & \mathbb{C}P^1 \\ & \nearrow & \downarrow & & \downarrow i \\ M & \xrightarrow{\ \xi\ } & BU & \xrightarrow{\ \det\ } & \mathbb{C}P^\infty \end{array}$$

The Thom spectrum corresponding to the map $X \to BU$ defines the bordism theory of manifolds with a $\mathbb{C}P^1$-reduction of the stable normal bundle, that is, the theory W_*. We denote this spectrum by W.

Proposition (Conner–Floyd)

There is an equivalence of MSU-modules

$$W \simeq MSU \wedge \Sigma^{-2}\mathbb{C}P^2.$$

Under this equivalence, the forgetful map $W \to MU$ *is identified with the free MSU-module map* $MSU \wedge \Sigma^{-2}\mathbb{C}P^2 \to MSU \wedge \Sigma^{-2}\mathbb{C}P^\infty$.

Theorem (Conner–Floyd, Stong)

(a) *The image of the forgetful homomorphism* $\pi_*(W) \to \pi_*(MU)$ *coincides with* $\ker \Delta$.

(b) *The spectrum* W *is the fibre of* $MU \xrightarrow{\Delta} \Sigma^4 MU$.

5. Multiplications and projections

$\Omega^W_{2n} = \pi_{2n}(W)$ can be identified with the subgroup of Ω^U_{2n} consisting of bordism classes $[M^{2n}]$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes.

However, $\Omega^W = \bigoplus_{i\geqslant 0} \Omega^W_{2i}$ is *not* a subring of Ω^U: one has $[\mathbb{C}P^1] \in \Omega^W_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \notin \Omega^W_4$.

Let $\pi\colon MU \to W$ be an SU-linear projection (an idempotent operation with image W). It defines an SU-bilinear multiplication on W by the formula

$$W \wedge W \to MU \wedge MU \xrightarrow{m_{MU}} MU \xrightarrow{\pi} W.$$

This multiplication has a unit, obtained from the unit of MSU by the forgetful morphism.

Theorem (Chernykh-P)

Any SU-linear multiplication on W with the standard unit has the form

$$a * b = ab + (2[V] - w)\partial a \partial b,$$

where $[V] = [\mathbb{C}P^1]^2 - [\mathbb{C}P^2]$ *and* $w \in \Omega_4^W$. *Any such multiplication is associative and commutative. Furthermore, the multiplications obtained from SU-linear projections are those with* $w = 2\widetilde{w}$, $\widetilde{w} \in \Omega_4^W$.

In this way, W becomes a complex oriented multiplicative cohomology theory.

The standard (Stong's) multiplication corresponds to $w = 0$.

Let $m_i = \gcd\left\{\binom{i+1}{k}, 1 \leqslant k \leqslant i\right\}$

$$= \begin{cases} 1 & \text{if} \quad i+1 \neq p^\ell \quad \text{for any prime } p, \\ p & \text{if} \quad i+1 = p^\ell \quad \text{for some prime } p \text{ and integer } \ell > 0. \end{cases}$$

Then $[M^{2i}] \in \Omega^U_{2i}$ represents a polynomial generator iff $s_i[M^{2i}] = \pm m_i$.

Theorem (Stong)

Ω^W is a polynomial ring on generators in every even degree except 4:

$$\Omega^W \cong \mathbb{Z}[x_1, x_i \colon i \geqslant 3], \quad x_1 = [\mathbb{C}P^1], \quad x_i \in \pi_{2i}(W).$$

The polynomial generators x_i are specified by the condition $s_i(x_i) = \pm m_i m_{i-1}$ for $i \geqslant 3$. The boundary operator $\partial \colon \Omega^W \to \Omega^W$, $\partial^2 = 0$, is given by $\partial x_1 = 2$, $\partial x_{2i} = x_{2i-1}$, and satisfies the identity

$$\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b.$$

We have

$$\Omega^W \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][x_1, x_{2k-1}, 2x_{2k} - x_1x_{2k-1} \colon k > 1],$$

where $x_1^2 = x_1 * x_1$ is a ∂-cycle, and each x_{2k-1}, $2x_{2k} - x_1x_{2k-1}$ is a ∂-cycle.

Theorem

There exist elements $y_i \in \Omega^{SU}_{2i}$, $i > 1$, such that $s_2(y_2) = -48$ and

$$s_i(y_i) = \begin{cases} m_i m_{i-1} & \text{if } i \text{ is odd,} \\ 2m_i m_{i-1} & \text{if } i \text{ is even and } i > 2. \end{cases}$$

These elements are mapped as follows under the forgetful homomorphism $\Omega^{SU} \to \Omega^W$:

$$y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1x_{2k-1}, \quad k > 1.$$

In particular, $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ embeds in $\Omega^W \otimes \mathbb{Z}[\frac{1}{2}]$ as the polynomial subring generated by x_1^2, x_{2k-1} and $2x_{2k} - x_1x_{2k-1}$.

6. (Quasi)toric representatives in bordism classes

A toric variety is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V.

Toric varieties are classified by convex-geometrical objects called rational fans, and projective toric varieties correspond to convex lattice polytopes P.

A toric manifold is a complete (compact) nonsingular toric variety.

A quasitoric manifold is a smooth $2n$-dimensional closed manifold M with a locally standard action of a (compact) torus T^n whose quotient M/T^n is a simple polytope P. An omniorientation of a quasitoric manifold provides it with an intrinsic stably complex structure.

Theorem (Danilov–Jurkiewicz, Davis–Januszkiewicz)

Let V be a (quasi)toric manifold of real dimension $2n$. The cohomology ring $H^*(V;\mathbb{Z})$ is generated by the degree-two classes v_i dual to the torus-invariant codimension-two submanifolds V_i, and is given by

$$H^*(V;\mathbb{Z}) \cong \mathbb{Z}[v_1,\ldots,v_m]/\mathcal{I}, \qquad \deg v_i = 2,$$

where $\mathcal{I}$ is the ideal generated by elements of the following two types:

- $v_{i_1}\cdots v_{i_k}$ *such that the facets $i_1,\ldots,i_k$ do not intersect in P;*
- $\displaystyle\sum_{i=1}^{m}\langle \boldsymbol{a}_i,\boldsymbol{x}\rangle v_i$, *for any vector $\boldsymbol{x}\in \mathrm{Hom}(T^n,S^1)\cong\mathbb{Z}^n$.*

Here $\boldsymbol{a}_i\in\mathrm{Hom}(S^1,T^n)\cong\mathbb{Z}^n$ is the primitive vector defining the one-parameter subgroup fixing V_i.

It is convenient to consider the integer $n \times m$ characteristic matrix

$$\Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

whose columns are the vectors $\boldsymbol{a}_i$ written in the standard basis of $\mathbb{Z}^n$.
Then the n linear forms $a_{j1}v_1 + \cdots + a_{jm}v_m$ corresponding to the rows of Λ vanish in $H^*(V;\mathbb{Z})$.

Theorem

There an isomorphism of complex vector bundles:

$$\mathcal{T}V \oplus \underline{\mathbb{C}}^{m-n} \cong \rho_1 \oplus \cdots \oplus \rho_m,$$

where $\mathcal{T}V$ is the tangent bundle, $\underline{\mathbb{C}}^{m-n}$ is the trivial $(m-n)$-plane bundle, and ρ_i is the line bundle corresponding to V_i, with $c_1(\rho_i) = v_i$.
In particular, the total Chern class of V is given by

$$c(V) = (1+v_1)\cdots(1+v_m).$$

Proposition

An omnioriented quasitoric manifold M has $c_1(M) = 0$ if and only if there exists a linear function $\varphi\colon \mathbb{Z}^n \to \mathbb{Z}$ such that $\varphi(\boldsymbol{a}_i) = 1$ for $i = 1, \ldots, m$. Here the $\boldsymbol{a}_i$ are the columns of characteristic matrix.
In particular, if some n vectors of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ form the standard basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$, then M is SU iff the column sums of Λ are all equal to 1.

Corollary

A toric manifold V cannot be SU.

Proof. If $\varphi(\boldsymbol{a}_i) = 1$ for all i, then the vectors $\boldsymbol{a}_i$ lie in the positive halfspace of φ, so they cannot span a complete fan.

Theorem (Buchstaber-P.-Ray)

A quasitoric SU-manifold M^{2n} represents 0 in Ω^U_{2n} whenever $n < 5$.

Theorem (Lu-P.)

There exist quasitoric SU-manifolds M^{2i}, $i \geqslant 5$, with $s_i(M^{2i}) = m_i m_{i-1}$ if i is odd and $s_i(M^{2i}) = 2m_i m_{i-1}$ if i is even. These quasitoric manifolds represent polynomial generators of $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$.

7. Calabi–Yau hypersurfaces and SU-bordism

A Calabi–Yau manifold is a compact Kähler manifold M with $c_1(M) = 0$. By definition, a Calabi–Yau manifold is an SU-manifold.

A toric manifold V is Fano if its anticanonical class $V_1 + \cdots + V_m$ (representing $c_1(V)$) is ample. In geometric terms, the projective embedding $V \hookrightarrow \mathbb{C}P^s$ corresponding to $V_1 + \cdots + V_m$ comes from a lattice polytope P in which the lattice distance from 0 to each hyperplane containing a facet is 1. Such a lattice polytope P is called reflexive; its polar polytope P^* is also a lattice polytope.

The submanifold N dual to $c_1(V)$ is given by the hyperplane section of the embedding $V \hookrightarrow \mathbb{C}P^s$ defined by $V_1 + \cdots + V_m$. Therefore, $N \subset V$ is a smooth algebraic hypersurface in V, so N is a Calabi–Yau manifold of complex dimension $n - 1$.

Lemma

The s-number of the Calabi–Yau manifold N is given by

$$s_{n-1}(N) = \big\langle (v_1^{n-1} + \cdots + v_m^{n-1})(v_1 + \cdots + v_m) - (v_1 + \cdots + v_m)^n, [V] \big\rangle.$$

Example

Consider the Calabi–Yau hypersurface N_3 in $V = \mathbb{C}P^3$.

We have $c_1(\mathcal{T}\mathbb{C}P^3) = 4u$, where $u \in H^2(\mathbb{C}P^3; \mathbb{Z})$ is the canonical generator dual to a hyperplane section.

Therefore, N_3 can be given by a generic quartic equation in homogeneous coordinates on $\mathbb{C}P^3$.

The standard example is the quartic given by $z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0$, which is a $K3$-surface. Lemma above gives

$$s_3(N_3) = \langle 4u^2 \cdot 4u - (4u)^3, [\mathbb{C}P^3] \rangle = -48,$$

so N_3 represents the generator $-y_2 \in \Omega_4^{SU}$.

$\sigma = (\sigma_1, \ldots, \sigma_k)$ an unordered partition of n, $\quad \sigma_1 + \cdots + \sigma_k = n$
Δ^{σ_i} the standard reflexive simplex of dimension σ_i.
$P_\sigma = \Delta^{\sigma_1} \times \cdots \times \Delta^{\sigma_k}$ is a reflexive polytope with the corresponding toric Fano manifold $V_\sigma = \mathbb{C}P^{\sigma_1} \times \cdots \times \mathbb{C}P^{\sigma_k}$.
N_σ the canonical Calabi–Yau hypersurface in V_σ.

Theorem (Limonchenko-Lu-P.)

The SU-bordism classes of the canonical Calabi–Yau hypersurfaces N_σ in $\mathbb{C}P^{\sigma_1} \times \cdots \times \mathbb{C}P^{\sigma_k}$ multiplicatively generate the SU-bordism ring $\Omega^{SU}[\frac{1}{2}]$.

Idea of proof.

Denote by $\widehat{P}(n)$ the set of all partitions σ with parts of size at most $n-2$:

$$\widehat{P}(n) := \{\sigma = (\sigma_1, \ldots, \sigma_k)\colon \sigma_1 + \cdots + \sigma_k = n, \quad \sigma \neq (n), (1, n-1).\}$$

For each σ we have the multinomial coefficient $\binom{n}{\sigma} = \frac{n!}{\sigma_1! \cdots \sigma_k!}$ and define

$$\alpha(\sigma) := \binom{n}{\sigma}(\sigma_1 + 1)^{\sigma_1} \cdots (\sigma_k + 1)^{\sigma_k}.$$

Then for for any $\sigma \in \widehat{P}(n)$ we have

$$s_{n-1}(N_\sigma) = -\alpha(\sigma).$$

Then we prove that

$$\gcd_{\sigma \in \widehat{P}(n)} \alpha(\sigma) = \begin{cases} 2m_{n-1}m_{n-2} & \text{if } n > 3 \text{ is odd;} \\ m_{n-1}m_{n-2} & \text{if } n > 3 \text{ is even;} \\ 48 & \text{if } n = 3. \end{cases}$$

Therefore, there is a linear combination of the bordism classes $[N_\sigma] \in \Omega^{SU}_{2n-2}$ whose s-number satisfies the condition for a polynomial generator y_{n-1} of $\Omega^{SU}[\frac{1}{2}]$. □

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