Symmetry approach to the problem of the gas expansion into a vacuum

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OUTLINE

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- Symmetries and integrals of motion
- Self-similar quasi-classical solution
- Role of quantum pressure in the quantum gas expansion
- Discussion of experimental data and comparison with obtained results
- Conclusion

Motivation

In this talk, we will consider how symmetries can be applied to the problem of expansion into vacuum of quantum and classical gases within the framework of the Gross-Pitaevskii (GP) equations and gas dynamics equations, namely, the continuity equation and the Euler equation for monatomic gases with the adiabatic exponent $\gamma = 5/3$.
For the quantum gases there will be considered the chemical potentials with power dependence on density n with the exponent $\nu = 2/d$ (d is the space dimension). Only for these values ν an additional symmetry arises in the problem. Note that in the GP approximation at a temperature $T \to 0$ for a weakly nonideal Bose gas the main interaction between atoms is connected with s scattering.

Motivation

For the scattering length $a_s > 0$ the interaction between the Bose atoms corresponds to repulsion. In this case $\mu = gn$ with $g = 4\pi\hbar^2 a_s/m$. Thus, an additional symmetry arises only for the $2D$ Bose gas .

For $a_s < 0$ attraction corresponds to the self-focusing of light in Kerr nonlinear media and respectively Bose condensates turn out to be unstable leading to collapse. For Fermi gases s-attraction provides the formation of Cooper pairs, which at $T \to 0$ form a superfluid Bose condensate. In the so-called unitary limit (Pitaevskii, 2008), $\mu(n) = 2(1+\beta)\varepsilon_F$, $\beta = -0.63$ is a universal constant, and

$$\varepsilon_F = \frac{\hbar^2}{2m}\left(6\pi^2 n\right)^{2/3} \quad .$$

Thus, in the unitary limit, $\nu = 2/3$.

Motivation

In nonlinear optics and plasma physics, the GP equation is usually called the nonlinear Schrodinger equation (NLS). Its standard form in the Hamiltonian form is written as

$$i\frac{\partial\psi}{\partial t} = \frac{\delta H}{\delta\psi^*}$$

with the Hamiltonian

$$H = \int \left[\frac{1}{2}|\nabla\psi|^2 + |\psi|^{2(\nu+1)}\right] d\mathbf{r}.$$

Applying the transformation $\psi = \sqrt{n(r,t)}\exp\left(i\varphi(r,t)\right)$ remain the Hamiltonian form for equations for n and φ,

$$\frac{\partial n}{\partial t} = \frac{\delta H}{\delta\varphi}, \; \frac{\partial\varphi}{\partial t} = -\frac{\delta H}{\delta n},$$

Motivation

or

$$\frac{\partial n}{\partial t} + (\nabla \cdot n \nabla \varphi) = 0,$$
$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} + (\nu + 1) n^{\nu} = \frac{\Delta \sqrt{n}}{2\sqrt{n}}.$$

where the rhs term in the second equations is responsible for the so-called quantum pressure and $\mathbf{v} = \nabla \varphi$ has a meaning of the velocity. Here the Hamiltonian

$$H = \int \left[\frac{n\,(\nabla \varphi)^2}{2} + \frac{(\nabla \sqrt{n})^2}{2} + n^{\nu+1} \right] d\mathbf{r}$$

$$= H_{kin} + H_{QP} + H_{internal}.$$

Quasiclassical approximation

Neglecting the quantum pressure term leads to the quasiclassical equations (Thomas-Fermi approximation) which coincide with the Euler equations at $d = 3$ for perfect monoatomic gas with $\gamma = 5/3$

$$\begin{aligned} \frac{\partial n}{\partial t} + (\nabla \cdot n \nabla \varphi) &= 0, \\ \frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} + \frac{5}{3} n^{2/3} &= 0. \end{aligned}$$

Symmetries and integrals of motion

This $\nu+1$ is remarkable for both NLSE and its quasiclassical limit. It turns out that the equations of motion have two additional symmetries. The first symmetry forms dilatation group of the scaling type: $\mathbf{r} \to \alpha \mathbf{r}$ and $t \to \alpha^2 t$. In usual quantum mechanics, this symmetry appears for the potential $V(r) \sim r^{-2}$. For the NLSE such symmetry appears as a result of the conservation of $N = \int |\psi|^2 d\mathbf{r}$ so that at $d=3$ only the nonlinear potential $\sim |\psi|^{4/3}$ has the same scaling as the Laplace operator Δ. At $d=2$ such symmetry takes place for the potential $\sim |\psi|^2$ (the stationary self-focusing of light in the Kerr nonlinear media). In the general case, $\nu_{cr} = 2/d$. The second symmetry is of the conformal type first time found by Talanov for the cubic NLSE at $d=2$.

Symmetries and integrals of motion

Under the Talanov transformations of the general form (for all $\nu = 2/d$), the NLSE remains invariant under the change of the wave function ψ, coordinates $\mathbf{r}$ and time t to the new wave function $\tilde{\psi}$ and new coordinates $\mathbf{r}'$ and time t' :

$$\psi(\mathbf{r}, t) = [\tau/(\tau + t)] \exp\left[\frac{ir^2}{4(\tau + t)}\right] \tilde{\psi}(\mathbf{r}', t'),$$

$$\mathbf{r}' = \mathbf{r}\tau/(\tau + t), \;\; t' = t\tau/(\tau + t).$$

In linear optics, these relationships are known as lens transforms.

It is important to note that the superposition of transforms with $\lambda_1 = \tau_1^{-1}$ and $\lambda_2 = \tau_2^{-1}$ represents a transform with $\lambda_3 = \lambda_1 + \lambda_2$. Thus, the transformations form the abelian group.

Symmetries and integrals of motion

In the case of gas dynamics at $\gamma = 5/3$, this symmetry was first found by L.V. Ovsyannikov (1956). It was effectively used by S.I. Anisimov and Yu.V. Lysikov in 1970 to construct an exact axisymmetric self-similar solution describing nonlinear angular deformations of a gas cloud against the background of its expansion into vacuum. Subsequently, it was found that such deformations are observed in various physical systems, e.g. for the action of the powerful laser radiation on the solid substance. As a result its original shape in the form of a disk is converted into a sigar-shape against the background of the expanding gas.

Symmetries and integrals of motion

A direct consequence of this symmetry is the virial theorem obtained by Vlasov, Petrishchev, Talanov (1971) for the 2D NLSE:

$$\frac{d^2}{dt^2}\int r^2|\psi|^2 d\mathbf{r} = 4H.$$

This theorem was first established for focusing NLS. It is easy to see that the theorem is true for any value of $\nu = 2/d$ for defocusing case also. The NLS with $\nu = 2/d$ belongs to the so-called critical models.

Simple integration gives that the mean square of the cloud size $R^2 = \int r^2|\psi|^2 d\mathbf{r}/N$ changes in time quadratically:

$$NR^2 = 2Ht^2 + C_1 t + C_2.$$

Symmetries and integrals of motion

In the case of repulsion , $H > 0$. Therefore, for $t \to \infty$, the average size of R grows linearly with time. Two constants C_1 and C_2 are new integrals of motion, they differ from H and N by the presence of an explicit dependence on the time t:

$$C_1 = \frac{d}{dt}\int r^2|\psi|^2 d\mathbf{r} - 4Ht,$$

$$C_2 = \int r^2|\psi|^2 d\mathbf{r} - 2Ht^2 - C_1 t.$$

Integrals of this kind are non-autonomous. These relations are valid for the GPE and its quasiclassical limit.

Self-similar quasi-classical solution

Let us search for a quasi-classical solution in the self-similar form at $d = 3$

$$n = \frac{1}{a_x a_y a_z} f\left(\frac{x}{a_x}, \frac{y}{a_y}, \frac{z}{a_z}\right)$$

which conserves the total number of particles, assuming scaling parameters a_x, a_y, a_z to be functions of t.

Then the continuity equation admits integration

$$\varphi = \varphi_0(t) + \sum_l \frac{\dot{a}_l a_l}{2} \xi_l^2.$$

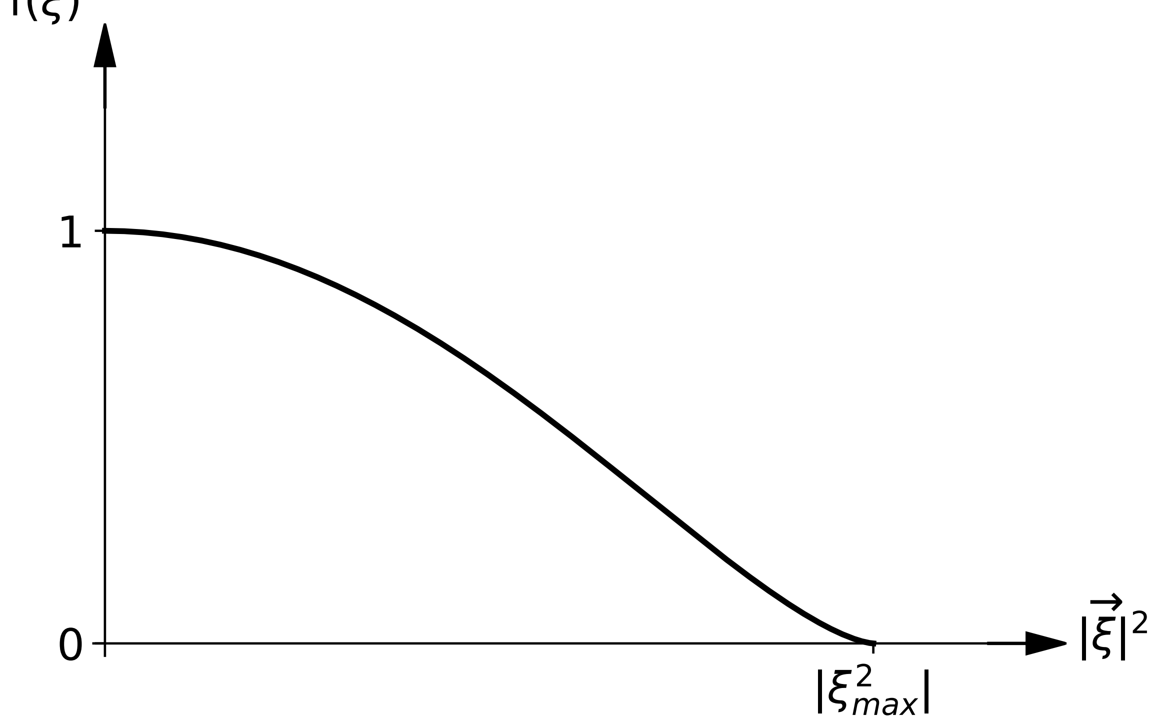
Substitution φ in the eikonal equation yields 3 Newton equations for motion of a particle

$$\ddot{a}_i = -\frac{\partial U}{\partial a_i}, \; U = \frac{3\lambda}{2\left(a_x a_y a_z\right)^{2/3}}.$$

Self-similar quasi-classical solution

Here constant $\lambda > 0$ is found from the initial condition. The density is

$$n = \frac{1}{a_x a_y a_z} \left[1 - \frac{3\lambda}{10} \xi^2 \right]^{3/2}.$$



The behavior of the density factor $f(\xi)$ (arbitrary units).

Self-similar quasi-classical solution

The Newton equations have the standard energy integral

$$E = \frac{1}{2}\sum_{i=1,2,3} \dot{a_i}^2 + \frac{3\lambda}{2\left(a_x a_y a_z\right)^{2/3}}.$$

Secondly, for these equations we have the virial identity

$$\frac{d^2}{dt^2}\sum_i a_i^2 = 4E.$$

Its twice integration gives two constants $C_1,\ C_2$. In the spherically symmetric case when $a_x = a_y = a_z \equiv a$, the equations of motion transform into one equation $\ddot{a} = \frac{\lambda}{a^3}$. Its solution shows that gas cloud expands in radial direction at $t \to \infty$ with constant velocity $v_\infty = \sqrt{2E/3}$ (ballistic regime).

Self-similar quasi-classical solution

In the cylindrically symmetric case when $a_x = a_y = a/\sqrt{2}$, $a_z = b$ we have

$$\ddot{a} = -\frac{\partial U}{\partial a}, \; \ddot{b} = -\frac{\partial U}{\partial b}.$$

where $U = 3\lambda 2 \left(a^2 b/2\right)^{-2/3}$.

This system belongs to the so called Ermakov type for two degrees of freedom. To integrate this system one needs to have two autonomous integrals of motion in involution. In our case we have three integrals of motion:

$$E = \frac{1}{2}(\dot{a}^2 + \dot{b}^2) + \frac{3\lambda}{2\left(a^2 b/2\right)^{2/3}}.$$

and two constants $C_1, \, C_2$. The integrals $C_1, \; C_2$ are not autonomous and can not provide a complete integration.

Self-similar quasi-classical solution

In terms of the polar coordinates $a = r\cos\Phi$, $b = r\sin\Phi$

$$E = \frac{1}{2}(\dot{r}^2 + r^2\dot{\Phi}^2) + \frac{3\lambda}{2^{1/3}r^2\left(\cos^2\Phi\sin\Phi\right)^{2/3}}.$$

The combination $\widetilde{E} = Er^2 - \frac{1}{2}r^2\dot{r}^2 = EC_2 - C_1^2/8$ gives the needed constant (the Ermakov integral) resulting in conservation law for new "energy"

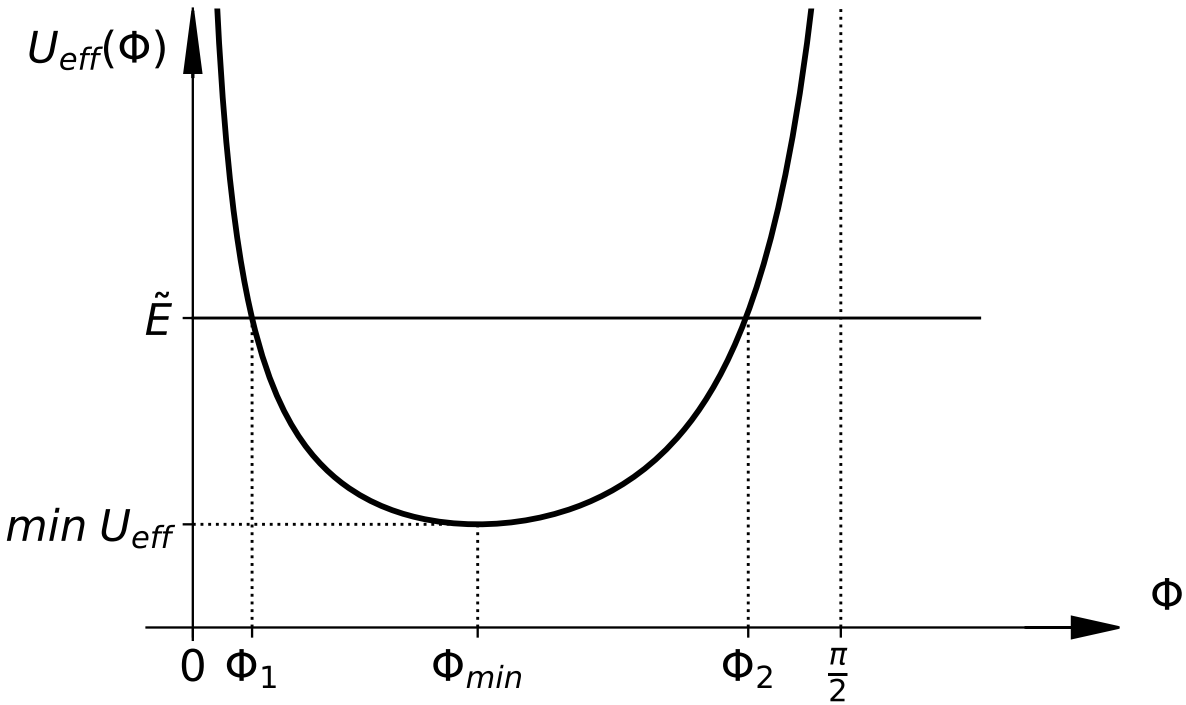
$$\widetilde{E} = \frac{1}{2}\left(\frac{d\Phi}{d\tau}\right)^2 + U_{eff}(\Phi),$$

with new time $\tau = \int_0^t \frac{dt'}{2E(t')^2 + C_1 t' + C_2}$, where

$$U_{eff}(\Phi) = 3\lambda 2^{1/3}\left(\cos^2\Phi\sin\Phi\right)^{-2/3}.$$

Self-similar quasi-classical solution

Effective potential $U_{eff}(\Phi)$:



The new time τ is expressed through t ,

$$\sqrt{2\widetilde{E}}\tau = \arctan\frac{\sqrt{2E}(t+t_0)}{\chi} - \arctan\frac{\sqrt{2E}t_0}{\chi}$$

where $\chi^2 = \widetilde{E}/E$.

Self-similar quasi-classical solution

If the initial velocity is equal zero $C_1 = 0$ and $\sqrt{2\widetilde{E}}\tau = \arctan\frac{\sqrt{2\widetilde{E}}t}{C_2}$. In this case, as $t \to \infty$ $\tau \to \tau_\infty = \frac{\pi}{2\sqrt{2\widetilde{E}}}$.
Hence the τ-period of the oscillations in the potential $U_{eff}(\Phi)$ is expressed as

$$T = 2\int_{\Phi^{(-)}}^{\Phi^{(+)}} \frac{d\Phi}{\sqrt{2\left[\widetilde{E} - U_{eff}(\Phi)\right]}},$$

where $\Phi^{(\pm)}$ are reflection points.
At large value of $\widetilde{E}$ oscillations are almost independent on the details of $U_{eff}(\Phi)$. In this case the angular velocity $\frac{d\Phi}{d\tau} \to \pm\sqrt{2\widetilde{E}}$ and the τ -period $T \to \frac{\pi}{\sqrt{2\widetilde{E}}}$.

Self-similar quasi-classical solution

Namely, in this limit T exceeds in two times τ_∞. Notice also that dependence of T with respect to $\widetilde{E}$ is monotonic for the given potential $U_{eff}(\Phi)$. This means that in the real experiment in the better case it is possible one to observe only half of such oscillation, t_{osc}. Thus, a recurrence to the initial shape is impossible in this case. The gas shape behavior will be different for cigar and disk initial conditions. In the cigar case we start from the left reflection point of the potential $U_{eff}(\Phi)$, in the disk case – from the right reflection point. Note that at fixed $\widetilde{E}$ starting from any reflection point we can not reach its opposite reflection point.

The solution presented here was obtained first time by Anisimov and Lysikov in 1970 for expansion of ideal gas with $\gamma = 5/3$.

Self-similar quasi-classical solution

In the general anisotropic case, when all the scaling parameters are different we introduce the spherical coordinates where the Ermakov reduced energy reads

$$\widetilde{E} = C_2 E - \frac{1}{8} C_1^2 = \left(\frac{d\theta}{dt}\right)^2 + \sin^2\theta \left(\frac{d\varphi}{dt}\right)^2 + U_{eff}.$$

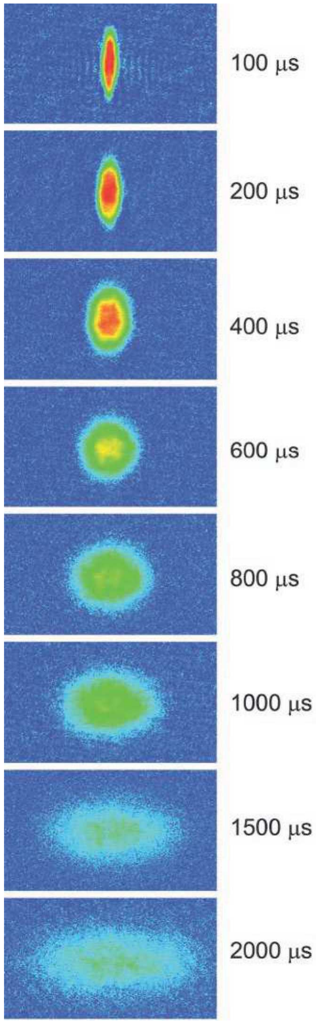
Here

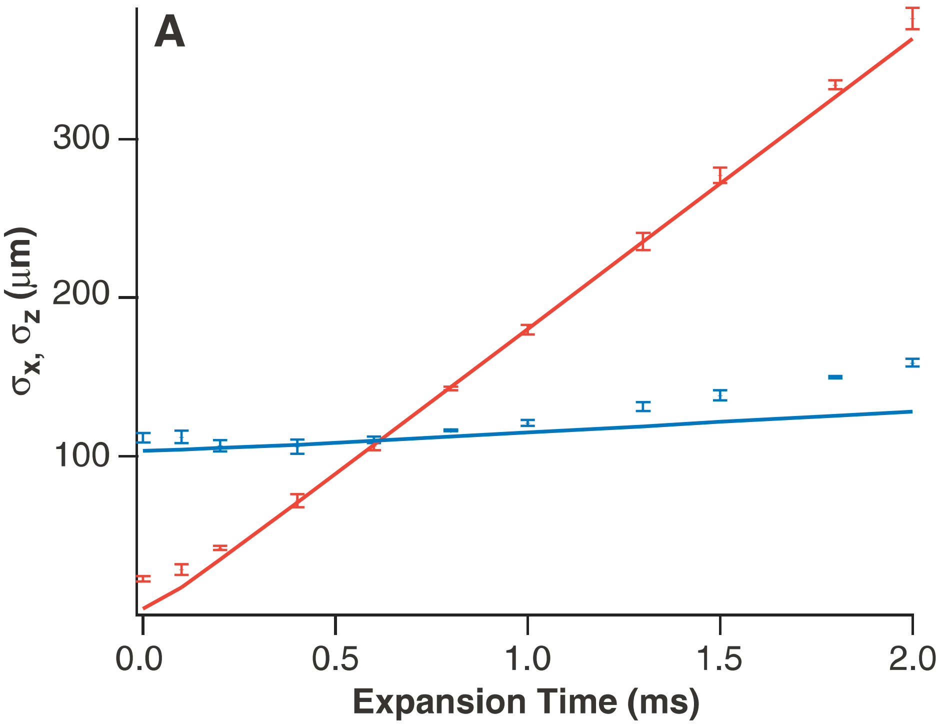
$$U_{eff} = \frac{3\lambda}{2^{1/3}\left(\sin^2\theta\cos\theta\sin 2\varphi\right)^{2/3}}.$$

As it was shown by Gaffet in 1996, this system has one additional integral which follows from the Painleve test. As in the previous limit motion in this potential remains its nonlinear quasi-oscillation character.

Comparison with experimental data

The self-similar expansion of a strongly interacting Fermi gas was observed by the Thomas group (2002).





Ellipsoid (cigar) $\rightarrow$ sphere $\rightarrow$ ellipsoid $\perp$ to the initial cigar.

Comparison with experimental data

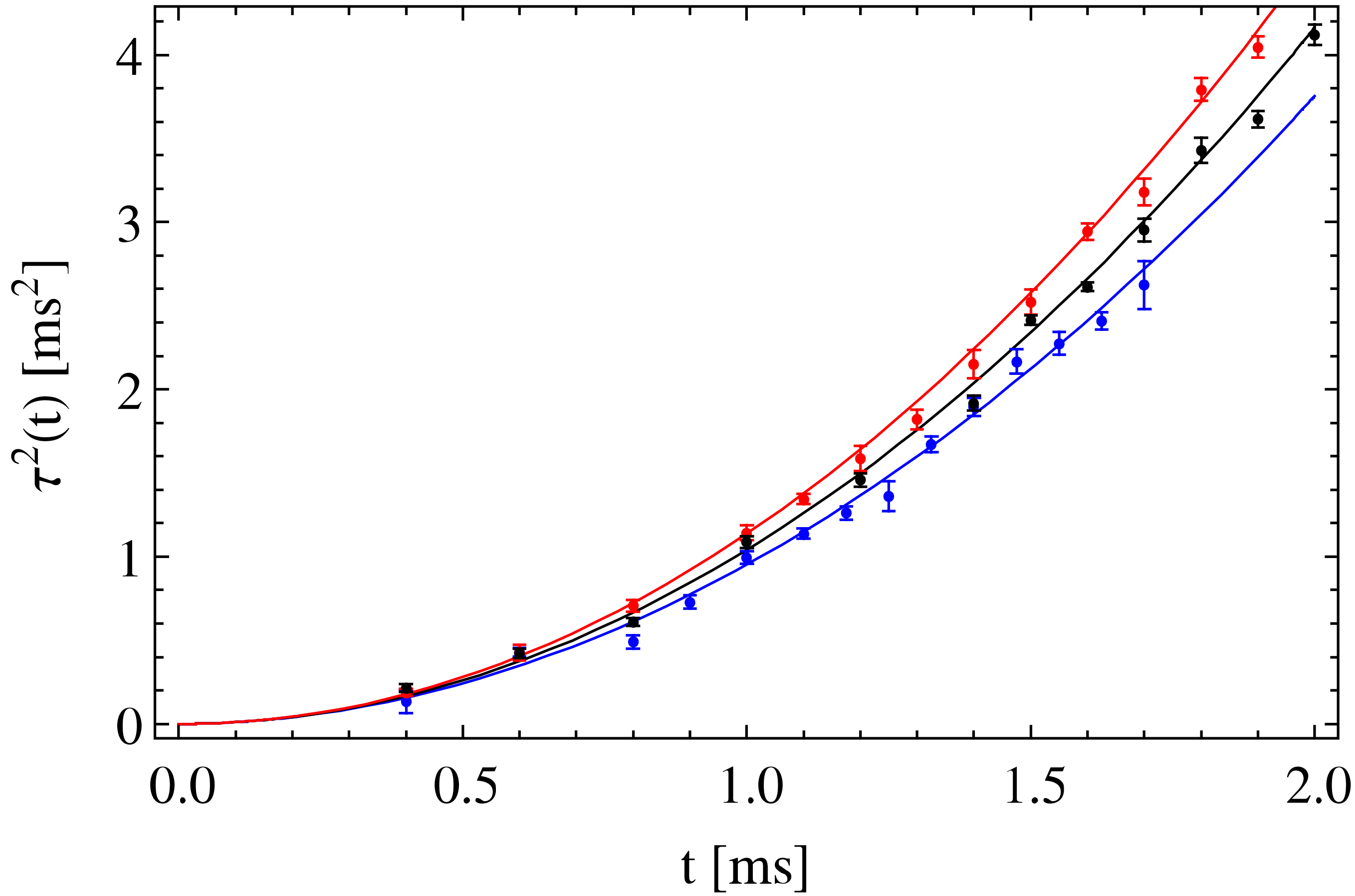
Exactly on resonance, the mean squared cloud size $\langle \mathbf{r}^2 \rangle \equiv \langle x^2 \rangle + \langle y^2 \rangle + \langle z^2 \rangle$ is found (2014) to evolve as

$$\langle \mathbf{r}^2 \rangle = \langle \mathbf{r}^2 \rangle_{t=0} + \frac{t^2}{m} \langle \mathbf{r} \cdot \nabla U(\mathbf{r}) \rangle_{t=0},$$

where $U(\mathbf{r})$ is the initial trapping potential. This expansion law coincides with the quasi-classical $\langle \mathbf{r}^2 \rangle$ in the unitarian limit. When the system is far from the unitarian point $(k_F a_s)^{-1} = 0$ experiments nevertheless give the parabolic time dependence for $\langle \mathbf{r}^2 \rangle$. Small deviation of the data from the self-similar behavior has been attributed to the contribution of quantum pressure. This difference may be explained since in experiment the interaction parameter is not tuned exactly on resonance $1/(k_F a_s) = 0$, with the estimate $1/(k_F a_s) \simeq -0.14$.

Comparison with experimental data

Experimental values $\tau^2(t) \equiv m[\langle \mathbf{r}^2 \rangle - \langle \mathbf{r}^2 \rangle_{t=0}]/\langle \mathbf{r} \cdot \nabla U \rangle_{t=0}$



Black markers correspond to the gas on resonance, $1/(k_F a_s) = 0$, red and blue markers to $1/(p_F a_s) \simeq 0.59$ and $1/(k_F a_s) \simeq -0.61$.

Conclusion

- We have demonstrated that symmetry for the GPE in the unitarian limit, describing strongly interacting superfluid Fermi gas, provides existence of the virial theorem.
- Independently on the ratio between quantum pressure and chemical potential while the Fermi superfluid gas expansion the size of the gas cloud scales linearly with time asymptotically as $t \to \infty$ with constant velocity $v_\infty = (2H/N)^{1/2}$.

Conclusion

- For description of the Fermi gas expansion in the quasiclassical limit (the Thomas-Fermi approximation) we have constructed the self-similar anisotropic solution. For large time scales the theory matches quite well with simple ballistic ansatz and also with the initial quasi-classical distribution of trapping gas.
- For the initial condition in the cigar-shape form the self-similar solution demonstrates successively all the stages of gas expansion, starting from the distribution extended along the cigar axis, bypassing the spherically symmetrical one and ending with the distribution, turned at angle $\pi/2$ with respect to the initial cigar form. Such behavior was observed first time in experiments.

THANKS FOR YOUR ATTENTION