Non-Markovian entropy production and its connection to global entanglement in quantum chaos

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Polyakov, Arefyeva, Probing quantum chaos with the entropy of decoherent histories, Phys. Rev. A 109, 062204 (2024)

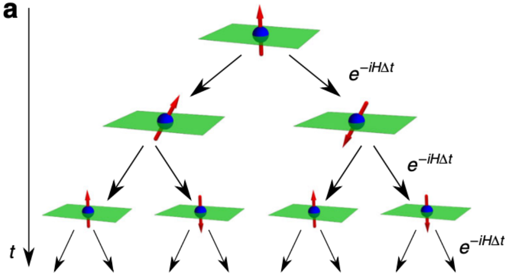
Structure of talk

- Non-Markovian open quantum system evolution
  - An efficient procedure of constructing the stream of emerging integrals of motion of the environment — records of information about OQS, containing decoherent histories
  - The full quantum many-body quench dynamics is reduced to an efficient Monte Carlo sampling of the decoherent histories
- Quantum chaos
  - The problem and approaches
  - Considered model
  - Non-Markovian entropy production as marker
  - Its connection to the Meyer-Wallach measure

Idea

- **Markov limit:** Fermi's golden rule → quantum jumps
- **Non-Markov case:** ?

Environment → emerging integrals of motion → recording device



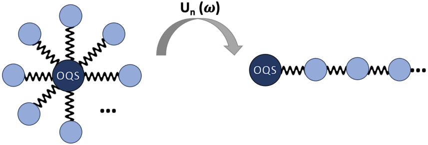
An illustration of the branching of histories for k time steps from Arrasmith, Cincio, et all, Nat. Commun. **10**, 3438 (2019)

Polyakov, "Beyond The Fermi's Golden Rule: Discrete-Time Decoherence Of Quantum Mesoscopic Devices Due To Bandlimited Quantum Noise", (2022) The considered model: $\hat{H} = \hat{H}_s + g\,\hat{O}_s^\dagger\;\hat{a} + g\,\hat{a}^\dagger\;\hat{O}_s + \hat{H}_{env}$

$$\hat{a} = \int_0^\infty c(\omega)\hat{a}(\omega)d\omega\,, \quad \hat{H}_{env} = \int_0^\infty \omega\hat{a}^\dagger(\omega)\hat{a}(\omega)d\omega \;\text{ with }\; \left[\hat{a}(\omega), \hat{a}^\dagger(\tilde{\omega})\right] = \delta(\omega - \tilde{\omega})$$

The spectral density of states of the environment $J(w) = |c(w)|^2$

Equivalent chain representation of bath: $a_n^\dagger = \int_0^\infty U_n(\omega)\hat{a}^\dagger(\omega)d\omega\,, \; [\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$



The total Hamiltonian:

$$\hat{H} = \hat{H}_S + h\,\hat{O}_s^\dagger\hat{a}_0 + h\,\hat{O}_s\hat{a}_0{}^\dagger + \sum_{n=0}^{\infty}\left(\epsilon_n\hat{a}_n^\dagger\hat{a}_n + h_n\hat{a}_{n+1}^\dagger\hat{a}_n + h_n\hat{a}_n^\dagger\hat{a}_{n+1}\right)$$

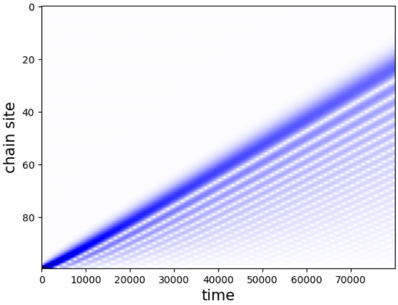
Chin, Rivas, et all., "Exact mapping between system-reservoir quantum models and semi-infinite discrete chains using orthogonal polynomials", J. Math. Phys. 51, 092109 (2010)

In the interaction picture w.r.t. free environment:

$$\hat{H}(t) = \hat{H}_s + \underbrace{h\,\hat{O}_s^\dagger \hat{a}_0(t) + h\,\hat{a}_0^\dagger(t)\,\hat{O}_s}_{H_{int}(t)}$$

Light-cone operator spread:

$$\hat{a}_0^\dagger(t) = \sum_{n=1}^{\infty} \phi_n(t)\,\hat{a}_n^\dagger$$



Satisfying:

$$\begin{cases} \partial_t \phi_n(t) = i\epsilon_n \phi_n(t) + ih_n \phi_{n+1}(t) + ih_{n-1}\phi_{n-1}(t) \\ \phi_n(0) = \delta_{n0} \end{cases}$$

Polyakov, "Beyond The Fermi's Golden Rule: Discrete-Time Decoherence Of Quantum Mesoscopic Devices Due To Bandlimited Quantum Noise", (2022)

Environment integrals of motion and records information about OQS

Which environmental degrees of freedom (EDoFs) contain information about the evolution of the open quantum system?

Records should:

- contain information about OQS and become stable
- remain unchanged at future times

As an integral of the motion of the environment, we can consider the occupation number of some quasiparticle wavepacket $\hat{n}(\kappa_j) = \hat{\kappa}_j^\dagger \hat{\kappa}_j$ or the projections onto its subspace $\hat{P}_{t_k,\kappa_j}$:

$$|\kappa_j\rangle = \hat{\kappa}_j^\dagger \, |vac\rangle \; = \sum_{k=0}^{\infty} U_{jk} \, \hat{a}_k^\dagger |vac\rangle$$

This occupation number is conserved under the free motion inside the environment. Then EDoF enter light cone, these integrals of motion became broken.
If after t_a a wavepacket leaves the interaction region:

$$\left[\hat{H}_{int}\left(\tau\right), \hat{P}_{t_a;\kappa}\right] \approx 0 \;\text{ for } \tau \geq t_a$$

and it is conserved under the future evolution.

Lieb-Robinson light cones: constructing emerging integrals of motion

In our case of the linear environment:

$$[\hat{a}_0(t), \hat{a}_j^\dagger] = C_{a_j}(t)\hat{\mathbb{1}}$$

The measure characterizing the instant interaction intensity with the OQS at time t:

$$|C_{a_j}(t)|^2 = \langle 0|[\hat{a}_0(t), \hat{a}_j^\dagger][\hat{a}_0(t), \hat{a}_j^\dagger]^\dagger|0\rangle$$

1) The EDoF $\hat{\kappa}_j^\dagger = \sum_{k=0}^{\infty} U_{jk}\, \hat{a}_k^\dagger$ is inside the light cone (interaction region), if:

$$\int_0^t |C_{\kappa_j}(\tau)|^2 d\tau - a_{cut} > 0\,,$$ forward light cone — coupled EDoFs $\kappa_1^{in}, \ldots, \kappa_{m_{in}(T)}^{in}$

hence we find broken integrals of motion

2) $$\int_t^T |C_{\kappa_j^{in}}(\tau)|^2 d\tau - a_{cut} < 0\,,$$ backward light cone — decoupled EDoFs $\kappa_1^{out}, \ldots, \kappa_{m_{out}(T)}^{out}$

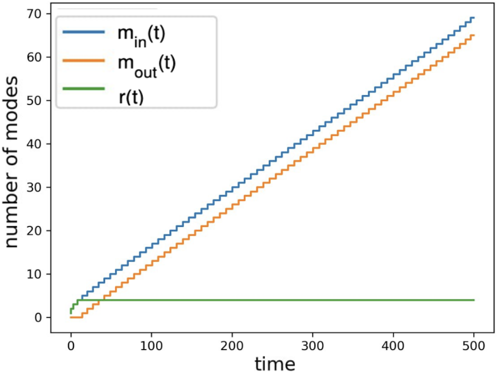
Emerging integrals of motion: $\hat{\kappa}_1^{out\dagger}\hat{\kappa}_1^{out}, \ldots, \hat{\kappa}_{m_{out}}^{out\;\dagger}\hat{\kappa}_{m_{out}}^{out}$

Relevant modes

Number of modes: $\underbrace{r(t_k^{out})}_{\text{relevant modes}} = \underbrace{m_{in}(t_k^{out})}_{\text{coupled modes}} - \underbrace{(k-1)}_{\text{decoupled modes}}$

Full system evolution over $[t_k^*, t_k^{out}]$ with: $i\partial_t|\Psi(t)\rangle = \hat{H}_{eff}(t)|\Psi(t)\rangle$

$$\hat{H}_{eff}(t) = \hat{H}_S(t) + \sum_{k=1}^{r(t)} \left[\hat{J}_y \chi_k(t)\hat{\kappa}_l^{rel\dagger} + \hat{J}_y \chi_k^*(t)\hat{\kappa}_l^{rel}\right] - \sum_{kl=1}^{r(t)} D_{kl}(t)\hat{\kappa}_l^{rel\dagger}\hat{\kappa}_k^{rel}$$



Decoherent histories, effective modeling dynamics

Before t_k^{out} the mode κ_k^{out} was coupled to the OQS:

Schmidt decomposition of state system + environment:

$$|\Psi(t_k^{out})\rangle = \sum_q C_q^{(k)} \times \underbrace{|\Psi_{q,A}(t_k^{out})\rangle_{rel}}_{\text{system + relevant modes}} \otimes \underbrace{|\Psi_{q,B}(t_k^{out})\rangle_{\kappa_k^{out}}}_{\text{mode } \kappa_k^{out}}$$

Von Neumann measurement (quantum jump):

$$|\Psi(t_k^{out})\rangle \to |\Psi_{q,A}(t_k^{out})\rangle_{rel} \text{ with probability } |C_q^{(k)}|^2$$

How many modes was decoupled — so many quantum jumps was occured:

$$|\Psi(t_1^{out})\rangle \to |\Psi_{(q_1),A}(t_1^{out})\rangle_{rel}$$

$$|\Psi_{(q_1),A}(t_2^{out})\rangle_{rel} \to |\Psi_{(q_1 q_2),A}(t_2^{out})\rangle_{rel}$$

$$|\Psi_{(q_1 q_2),A}(t_k^{out})\rangle_{rel} \to |\Psi_{(q_1 q_2 q_3),A}(t_k^{out})\rangle_{rel} \text{ etc}$$

Decoherent histories, effective modeling dynamics

$$|\Psi(t_1^{out})\rangle \to |\Psi_{(q_1),A}(t_1^{out})\rangle_{rel}$$

$$|\Psi_{(q_1),A}(t_2^{out})\rangle_{rel} \to |\Psi_{(q_1 q_2),A}(t_2^{out})\rangle_{rel}$$

$$|\Psi_{(q_1 q_2),A}(t_k^{out})\rangle_{rel} \to |\Psi_{(q_1 q_2 q_3),A}(t_k^{out})\rangle_{rel} \text{ etc}$$

Decoherent history — history of choices: $h = (q_1, q_2, \ldots, q_k)$

with probability $P(q_1, q_2, \ldots, q_k) = \prod_{k:\, t_k^{out} \le t} |C_{q_k}(k|q_1, \ldots, q_{k-1})|^2$

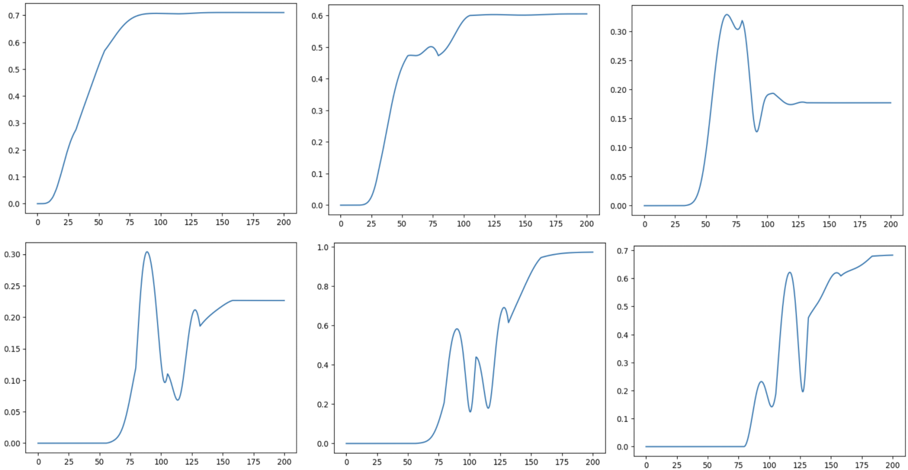
The average over all decoherent histories h up to time t reproduces the reduced density matrix:

$$\rho_{rel}(t) = \langle \left[|\Psi^h_{coll}(t)\rangle_{rel}\ {}_{rel}\langle\Psi^h_{coll}(t)| \right] \rangle_{\text{all histories}}$$

Efficient anzat

The emerging invariant structure of entanglement:

$$|\Psi(t)\rangle = \sum_{\mathbf{q_1},.,q_k} c_{q_{\mathbf{1}}}(1)..c_{q_k}(k|q_1,.,q_{k-1})\ |\Psi_{rel}^{(q_{\mathbf{1}}..q_k)}(t)\rangle\ \ |\kappa_1^{out}(t_1^{out})\rangle..|\kappa_{m_{out}(t)}^{out}(t_k^{out})\rangle$$



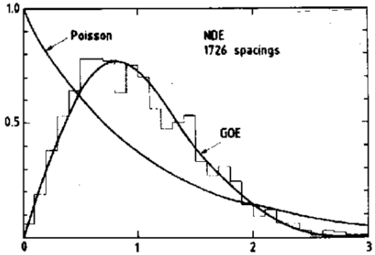
Energy level statistics:

Chaotic quantum system — **Wigner-Dyson** statistics:

$$P_{chaos}(\omega) = \frac{\pi}{2}\omega \exp(-\frac{\pi}{4}\omega^2)$$

Integrable quantum system — **Poisson** statistics:

$$P_{int}(\omega) = e^{-\omega}, \quad \text{there} \quad \omega = (E_{n+1} - E_n)/\langle\omega\rangle$$



Quantum chaos

Out-of-time ordered correlators (OTOC):

$$C_{ij}(t) = \langle [\hat{q}_i(t), \hat{p}_j(0)]^2 \rangle \approx \hbar^2 \{q_i(t), p_j(0)\}^2 =$$

$$= \hbar^2 \left| \frac{\partial q_i(t)}{\partial q_j(0)} \right|^2 \approx \hbar^2 \frac{\parallel \delta z(t) \parallel^2}{\parallel \delta z(0) \parallel^2} \approx \hbar^2 e^{2\Lambda t}$$

Loschmidt echo:

$$\mathcal{M}_L = |\langle \Psi_0 | e^{\frac{i}{\hbar}\hat{H}t} e^{-\frac{i}{\hbar}\hat{H}'t} | \Psi_0 \rangle|^2$$

The crucial role of the environment

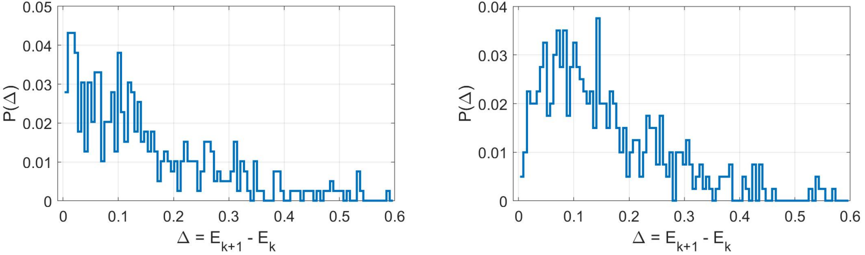
1) T. A. Brun, "An example of the decoherence approach to quantum dissipative chaos", Phys.Lett.A **206** 167 (1995) and related works

2) M. Berry, "Chaos and the semiclassical limit of quantum mechanics (is the moon there when somebody looks?)", Quantum Mechanics: scientific perspectives on Divine Action eds: R. J. Russell, P. Clayton, K. Wegter-McNelly and J. Polkinghorne (Vatican Observatory CTNS publications 2001)

Quantum kicked top

$$\hat{H}_S(t) = \frac{p}{\tau}\hat{J}_y + \frac{K}{2j}\left(\hat{J}_z - \beta\right)^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau) \qquad (\hbar = 1)$$

$$\vec{J} = (J_x, J_y, J_z),\ [J_i, J_j] = i\epsilon_{ijk}J_k$$



Crossover between Poisson statistics ($K = 2$) and Wigner-Dyson statistics ($K = 3$)

$\beta = 0.1,\ \tau = 1,\ p = 1.7$

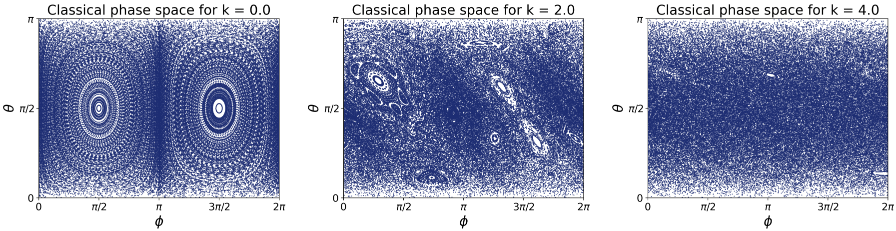
Phase space

The evolution for one period describe by Floquet operator:

$$\hat{\mathcal{F}} = e^{-i\frac{K}{2j}\left(\hat{J}_z-\beta\right)^2} e^{-i\alpha\hat{J}_y}$$

The stroboscopic evolution for angular momentum:

$$\hat{J}_{i,n} = \hat{\mathcal{F}}^{\dagger n}\hat{J}_{i,0}\hat{\mathcal{F}}^{n}$$



Entropy production as marker of quantum chaos

$$\hat{H}_S(t) = \frac{p}{\tau}\hat{J}_y + \frac{K}{2j}\left(\hat{J}_z - \beta\right)^2 \sum_{n=-\infty}^{\infty} \delta(t - n\tau)$$

$$\hat{H}(t) = \hat{H}_S(t) + \hat{J}_y\, h\, \left(\hat{a}_0{}^\dagger + \hat{a}_0\right) + \sum_{n=0}^{\infty}\left(\epsilon_n \hat{a}_n^\dagger \hat{a}_n + h_n \hat{a}_{n+1}^\dagger \hat{a}_n + h_n \hat{a}_n^\dagger \hat{a}_{n+1}\right)$$

Ergodicity for trajectories: $\langle \Delta S \rangle \approx \overline{\Delta S}$

$\langle ... \rangle$ is averaging over all trajectories at t_k (for one quantum jump)
$\overline{...}$ is averaging within one sufficiently long trajectory over all quantum jumps (all choices)

$$\langle S(t_k) \rangle = \frac{1}{N} \sum_{i=1}^{N} S_i(t_k)\,, \ \ \overline{S}_i = \frac{1}{T} \sum_{k=1}^{T} S_i(t_k)$$

When n quantum jumps (choices $q_1, ..., q_n$) have already happened and the moment of the next jump has come t_{n+1}:

$$|\Psi(t_{n+1})\rangle = \sum_i c_i |\Psi_{rel}(t_{n+1}, q_1, ..., q_n)\rangle \otimes |\kappa_n^{out}(t_{n+1})\rangle$$

The entropy for one jump increases: $\Delta S = -\sum_i |c_i|^2 \ln(|c_i|^2)$

The average entropy production for one trajectory per quantum jump :

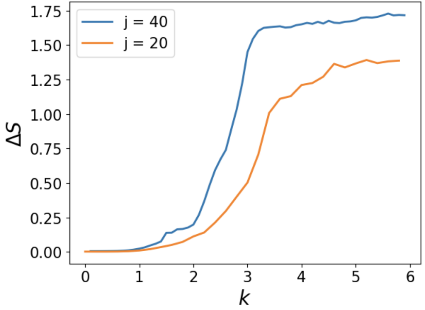
$$\langle \Delta S \rangle = \frac{1}{N} \sum_N \Delta S$$

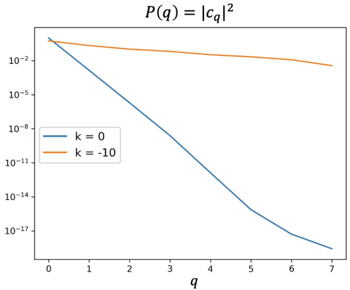
Entropy production as marker of quantum chaos

$$\hat{H}(t) = \hat{H}_S(t) + \hat{J}_y\, h\, \left(\hat{a}_0{}^{\dagger} + \hat{a}_0\right) + \sum_{n=0}^{\infty} \left(\epsilon_n \hat{a}_n^{\dagger}\hat{a}_n + h_n \hat{a}_{n+1}^{\dagger}\hat{a}_n + h_n \hat{a}_n^{\dagger}\hat{a}_{n+1}\right)$$

Non-Markovian environment:

$$\epsilon_n = 1\,,\;\; h_n = 0.2\,,\;\; h = 0.05$$





Ancilla-based approaches, simplest decoherent histories

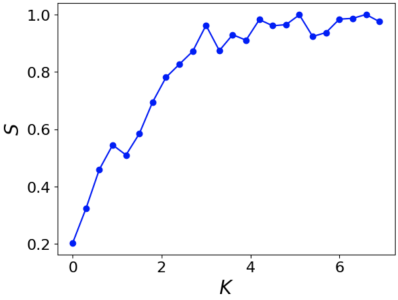


Figure: Entropy production in ancilla models vs. kicking strength

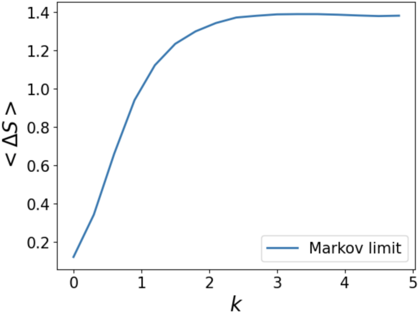


Figure: Entropy production in Markov limit $\epsilon_n = 4, h_n = 2, h = 0.5$

Global entanglement

Quantum kicked top as N qubit system:

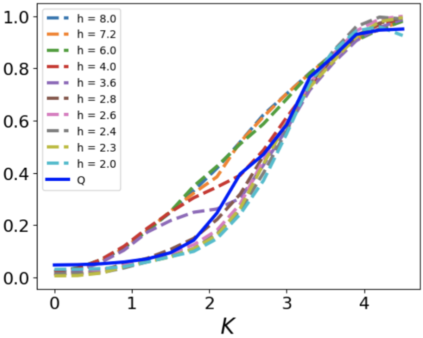
$$J_\alpha = \frac{1}{2}\sum_{i=1}^{N=2j} \sigma_{i,\alpha}$$

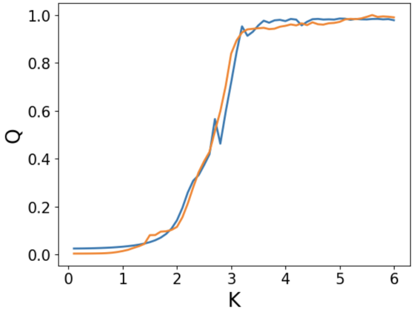
The Meyer-Wallach measure of global entanglement:

$$Q = 2\left(1 - \frac{1}{n}\sum_{k=0}^{n-1} Tr(\rho_k^2)\right) = 1 - \frac{4}{(2j+1)^2}\left(\langle J_x\rangle^2 + \langle J_y\rangle^2 + \langle J_z\rangle^2\right)$$

Comprasion, two regimes

The left figure for j = 40, the right for j = 20





Here h is the spectral bandwidth

Conclusions

- An efficient ansatz for full many-body quantum system
- A new method for diagnosing quantum chaos
- The normalized entropy production in the non-Markovian regime agrees with the Meyer–Wallach measure

Thank you for your attention:)

Supplement: Algoritm of finding forward light cone

$$|\phi(t)\rangle = \hat{a}_0^\dagger(t)\,|vac\rangle \;= \sum_{k=0} \phi_k\,|k\rangle$$

$$|\kappa_j\rangle = \hat{\kappa}_j^\dagger\,|vac\rangle \;= \sum_{k=0}^{\infty} U_{jk}\,\hat{a}_k^\dagger|vac\rangle = \sum_{k=0}^{\infty} U_{jk}\;|k\rangle$$

$$\int_0^t |C_{\kappa_j}(\tau)|^2 d\tau - a_{cut} > 0 \quad \rightarrow \quad g_+(\kappa_j, t) = \langle\kappa_j|\rho_+(t)|\kappa_j\rangle - a_{cut}$$

$$\int_0^t |C_{\kappa_j}(\tau)|^2 d\tau = \int_0^t \langle 0|[\hat{a}_0(\tau), \hat{\kappa}_j^\dagger][\hat{a}_0(\tau), \hat{\kappa}_j^\dagger]^\dagger|0\rangle d\tau =$$

$$= \langle\kappa_j|\int_0^t d\tau|\phi(\tau)\rangle\langle\phi(\tau)|\kappa_j\rangle = \langle\kappa_j|\rho_+(t)|\kappa_j\rangle$$

Analogous:

$$\int_t^T |C_{\kappa_j^{in}}(\tau)|^2 d\tau - a_{cut} < 0 \quad \rightarrow \quad g_-(\kappa_j^{in}, t) = \langle\kappa_j^{in}|\underbrace{\rho_-(t)}_{\int_t^T d\tau|\phi(\tau)\rangle\langle\phi(\tau)|}|\kappa_j^{in}\rangle - a_{cut}$$

Algorithm for finding the forward light cone

When solving a many-body problem on the interval $[0, T]$, it is initially necessary to evaluate:
$g_+(\chi, T) > 0$
density matrix $\rho_+(T)$ is calculated, and the most significant mode is found:

$$\rho_+(T)|\phi_l\rangle = \pi_l|\phi_l\rangle$$

Only the statistically significant modes are retained: $\pi_l/\pi_1 > r_{cut}$: $\underbrace{\phi_1, ..., \phi_{m_{in}(T)}}_{m_{in}(T)\text{modes}}$

To stretch the light cone and find one with minimal spread — one can apply a unitary transformation such that the least significant mode couples the latest.

To do this, the eigenvalues of $\rho_+(T)$ can be computed backward in time

At time T — all modes are coupled, there are $m_{in}(T)$ of them.

At each moment τ the following is calculated: $\rho_+(\tau)|\tilde{\phi}_k(\tau)\rangle = \pi_k(\tau)|\tilde{\phi}_k(\tau)\rangle$

When the condition for being inside the forward light cone is violated: $\pi_k(t_k^{in})/\pi_1(t_k^{in}) < r_{cut}$, the mode decouples and is added to the stream of incoming modes: $\kappa_{m(T)}^{in} = \tilde{\phi}_{m(T)}(t_{m(T)}^{in})$.

This happens at each time point t_k^{in}.

By recursively repeating these steps, all coupled modes κ_k^{in} and their coupling times t_k^{in} are determined:

$$\begin{pmatrix} \kappa_1^{in} \\ \vdots \\ \kappa_{m_{in}(T)}^{in} \end{pmatrix} = W \begin{pmatrix} \phi_1^{in} \\ \vdots \\ \phi_{m_{in}(T)}^{in} \end{pmatrix} = \begin{pmatrix} \chi_{1,0}^{in} & \chi_{1,1}^{in} & \cdots \\ \chi_{2,0}^{in} & \chi_{2,1}^{in} & \cdots \\ \cdots & \cdots & \ddots \end{pmatrix} \begin{pmatrix} \phi_1^{in} \\ \vdots \\ \phi_{m_{in}(T)}^{in} \end{pmatrix}$$

Algorithm for finding the backward light cone

Initially, $\rho_-(t)$ is a matrix of size $m_{in}(t) \times m_{in}(t)$, projected onto the subspace of incoming modes κ_1^{in}, ..., $\kappa_{m_{in}(t)}^{in}$ that have coupled by time t.

At each time moment from $t = 0$ to $t = T$ with step dt, its eigenvalues can be calculated: $\rho_-(t)|\tilde{\phi}_k^-(t)\rangle = \pi_k^-(t)|\tilde{\phi}_k^-(t)\rangle$

When $\pi_{m_{in}(t)}/\pi_1 < r_{cut}$ — the mode corresponding to $\pi_{m_{in}(t)}$ irreversibly decouples from the OQS.

As soon as a mode decouples, the dimension of the matrix ρ_- decreases by 1 at each decoupling time t_k^{out}. It is projected onto the subspace of $m_{in}(t_k^{out}) - m_{out}(t_k^{out})$ modes, i.e., the relevant subspace.

The irreversibly outgoing modes are obtained recursively from the stream of incoming modes via unitary transformations:

$$\begin{pmatrix} \hat{\kappa}_1^{out+} \\ \hat{\kappa}_1^{rel+} \\ \vdots \\ \hat{\kappa}_{m_{in}(t_1^{out})-1}^{rel+} \end{pmatrix} = U_1 \begin{pmatrix} \hat{\kappa}_1^{in+} \\ \vdots \\ \hat{\kappa}_{m_{in}(t_1^{out})}^{in+} \end{pmatrix}, \ldots, \begin{pmatrix} \hat{\kappa}_k^{out+} \\ \hat{\kappa}_1^{rel+} \\ \vdots \\ \hat{\kappa}_{r(t_k^{out})-1}^{rel+} \end{pmatrix} = U_k \begin{pmatrix} \hat{\kappa}_1^{rel+} \\ \vdots \\ \hat{\kappa}_{r(t_k^{out})}^{rel+} \end{pmatrix}$$