Applications of finiteness conjecture for groups with good reduction:

To vladimir Petrovich Platonov

on the occasion of his 85th birthday

(joint work with A. Rapinchuk , I. Rapinchuk)

Vladimir Chernousov
University of Alberta

Steklov Institute, Moscow: June 16, 2025

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Mordell conjecture: *A smooth projective curve C over a number field K of genus* $g \geqslant 2$ *has only finitely many K-rational points.*

Shafarevich conjecture: *For any* $g \geqslant 2$ *there exists only finitely many isomorphism classes of smooth projective curves over a number field K of genus g having good reduction at all* $v \notin S$.

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Remark. Of course a split group G has a good reduction for every v. Here are more examples.

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Let us now state an analogue of Shafarevich's conjecture for reductive groups.

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Remark. Let $S \subset S' \subset V$ where S' is also finite. Let n_S (resp. $n_{S'}$) be the number of isomorphism classes of K-forms of G which have good reduction for all $v \notin S$ (resp. $\notin S'$).

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Thus, there are natural bijections between 3 sets:

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One then writes $A = J(D, \mu)$ and says that A arises from D and μ via the first Tits construction.

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The natural map

$$H^1(K, F_4) = \coprod_{\Omega} H^1(\mathcal{O}_v, \mathcal{G}_\Omega)_{\mathrm{an}} \to H^1(k_v, G_0)$$

is a bijection.

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In the second case we show that $[\xi]$ comes from a maximal anisotropic torus $S \subset G_0$ splitting over an unramified extension L/K of degree 3. Then assuming that G' doesn't admit good reduction we conclude with the use of the formula for g_3 that $g_3(G')$ is ramified which contradicts to the fact that by construction $g_3(G) = g_3(G')$ is unramified.