Groups $(2, 3, 7; n)$, their quotients and related number-theoretic questions

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The groups $(k, l, m; n)$

Definition

$$(k, l, m; n) = \langle x, y \mid x^k = y^l = (xy)^m = [x, y]^n = 1 \rangle$$

The symbol $(k, l, m; n)$ appeared first in the paper

H. S. M. Coxeter. The abstract groups $G^{m,n,p}$. *Trans. Amer. Math. Soc.* 45 (1939), 73–150.

We deal mostly with the case $(2, 3, 7; n)$.

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A $(2,3,7)$-generated group is a non-trivial quotient of the so-called triangle group
$T(2,3,7) = \langle x, y \mid x^2 = y^3 = (xy)^7 = 1 \rangle$.

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A finite (2,3,7)-generated group is called a Hurwitz group.

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The group $(2, 3, 7; 11)$

D. F. Holt, W. Plesken, B. Souvignier. Constructing a representation of the group $(2, 3, 7; 11)$. *J. Symbolic Comput.*, 24 (1997), no. 3–4, 489–492.

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- An element $(2, 3, 7; 11)$ of infinite order is found.

M. C. Tamburini, M. Vsemirnov. Irreducible (2,3,7)-subgroups of $\mathrm{PGL}_n(F)$, $n \leq 7$, II. *J. Algebra* 321 (2009), no. 8, 2119–2138.

Pairs of matrices $x, y \in \mathrm{SL}_7(F)$ satisfying

- $x^2 = y^3 = (xy)^7 = 1$,
- $\langle x, y \rangle$ is absolutely irreducible

are classified up to conjugation.

7-dimensional representations of $(2, 3, 7; n)$

Theorem (V.)

There exist 7-dimensional complex representations of the groups $(2, 3, 7; n)$ for $n = 4, 6, 7, \ldots$. Moreover, the images of $(2, 3, 7; n)$ are infinite for $n \geq 10$.

For $n = 4, 6, 7, 8$, we have well-known representations of $\mathrm{PSL}_2(7)$, $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(13)$, $\mathrm{PSL}_2(7).2^6$.
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Open Question

Are these representations of $(2,3,7;n)$ faithful for $n \geq 10$.

Alternating groups as quotients of $(2,3,7;n)$

In general, the group $(2,3,7;n)$ can be "large".

G. Higman obtained that all but finitely many alternating groups are quotients of $(2,3,7;n)$ for $n = 60060 = 2^2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$.

Later G. Higman proved the same for $n = 1980 = 2^2 \cdot 3^2 \cdot 5 \cdot 11$.

Finally, M. Conder showed that all but finitely many alternating groups are quotients of $(2,3,7;n)$ for $n = 84$.

M.D.E. Conder, A question by Graham Higman concerning quotients of the $(2,3,7)$ triangle group. *J. Algebra* 141 (1991), 275–286.

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The answer is known except one case.

Open Question

Is the group $(2, 3, 13; 4)$ finite or infinite?

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The following quotients are known:

- $\mathrm{PSL}_3(3)$,
- $\mathrm{PSL}_3(3).2^{12}$,
- $\mathrm{PSL}_2(25)$,
- $\mathrm{PSL}_2(25) \times \mathrm{PSL}_3(3)$,
- $\mathrm{PSL}_2(25) \times (\mathrm{PSL}_3(3).2^{12})$.

$\mathrm{PSL}_2(q)$ as quotients of $(2, 3, 7; n)$

D. F. Holt, W. Plesken. A cohomological criterion for a finitely presented group to be infinite. *J. Lond. Math. Soc. (2)*, 45, no.3 (1992), 469–480.

D. Holt and W. Plesken raised the following problem.

Show that for any sufficiently large n there is a prime power q, such that $\mathrm{PSL}_2(q)$ is generated by x and y satisfying $x^2 = y^3 = (xy)^7 = 1$ and the order of $[x, y]$ *is exactly* n.

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Theorem (Macbeath, 1969)

The groups $\mathrm{PSL}_2(q)$ are Hurwitz precisely when

- $q = p$, p is prime, $p \equiv 0, \pm 1 \pmod 7$;
- $q = p^3$, p is prime, $p \equiv \pm 2, \pm 3 \pmod 7$;

Up to conjugation, there are three classes of Hurwitz generators for $q \neq 7$, just one class for $q = 7$.

Theorem (V., 2018)

For any $n \notin \{1, 2, 3, 5, 8, 12, 18, 28, 30\}$, there exists q, such that $\mathrm{PSL}_2(q)$ is generated by x and y satisfying $x^2 = y^3 = (xy)^7 = 1$ and the order of $[x, y]$ *is exactly* n.

M. Vsemirnov. On the primitive divisors of the recurrent sequence $u_{n+1} = (4\cos^2(2\pi/7) - 1)u_n - u_{n-1}$ with applications to group theory. *Science China Mathematics*, 61, no. 11 (2018), 2101–2110.

Number-theoretic interpretation

Let $\theta = 2\cos(2\pi/7)$.

Consider the generalized quaternion algebra $\left(\frac{-1,\theta^2-3}{\mathbb{Q}(\theta)}\right)$ and the order

$$\mathcal{H} = \left\{ x_0 + x_1 i + x_2 j + x_3 k \;\middle|\; \begin{array}{l} 2x_s \in \mathbb{Z}[\theta],\ x_0 - x_3 - x_2\theta \in \mathbb{Z}[\theta], \\ x_1 + x_2 - x_3\theta \in \mathbb{Z}[\theta] \end{array} \right\}.$$

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One can show that $T(2,3,7)$ is isomorphic to the projective image of $\mathcal{H}_1^*$, the group of quaternions of norm 1 in $\mathcal{H}$.

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Prime ideals $\mathfrak{p}$ in $\mathbb{Z}[\theta]$ have norm $N(\mathfrak{p}) = p$, if $p \equiv 0, \pm 1 \pmod 7$, $N(\mathfrak{p}) = p^3$, if $p \equiv \pm 2, \pm 3 \pmod 7$. Since any quaternion algebra over a finite field $\mathbb{F}_q$ is isomorphic to $M_2(\mathbb{F}_q)$, for an odd prime ideal $\mathfrak{p}$ we have a natural homomorphism

$$\mathcal{H}_1^* \to \mathrm{SL}_2(q), \qquad q = N(\mathfrak{p}).$$

Macbeath's theorem says that this map is onto.

Some unexpected applications to number theory

Macbeath's theorem can be restated in the following way: Given a prime ideal $\mathfrak{p}$ in $\mathbb{Z}[\theta]$, $\mathfrak{p} \neq (2)$ and the numbers a_0, a_1, a_2, $a_3 \in \mathbb{Z}[\theta]$ such that

$$a_0^2 + a_1^2 - (\theta^2 - 3)(a_2^2 + a_3^2) \equiv 1 \pmod{\mathfrak{p}},$$

one can find x_0, x_1, x_2, $x_3 \in \mathbb{Z}[\theta]$ such that

$$x_s \equiv a_s \pmod{\mathfrak{p}}, \quad s = 0, 1, 2, 3.$$

and

$$x_0^2 + x_1^2 - (\theta^2 - 3)(x_2^2 + x_3^2) = 1.$$

All known proofs are via group theory.

Open Question

Is there a number-theoretic proof of this fact?

Notation:

$\epsilon = \exp(2\pi i/7)$,

$\theta = \epsilon + \epsilon^{-1}$,

α_1, α_2 are the two roots of $\alpha^2 - (\theta^2 - 1)\alpha + 1 = 0$,

$$u_n = \frac{\alpha_1^n - \alpha_2^n}{\alpha_1 - \alpha_2}.$$

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$$u_{n+1} = (\theta^2 - 1)u_n - u_{n-1}, \quad u_2 = \theta^2 - 1, \quad u_1 = 1$$

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In particular, for all $n \geq 0$ we have $u_n \in \mathbb{Z}[\theta]$.

Definition

A prime $\pi \in \mathbb{Z}[\theta]$ is a primitive divisor of u_n if $\pi \mid u_n$ but $\pi \nmid u_m$ for all $0 < m < n$.

Theorem (V., 2018)

For any fixed n the following conditions are equivalent:

(i) there exist q and $x, y \in \mathrm{PSL}_2(q)$ such that $\langle x, y \rangle = \mathrm{PSL}_2(q)$, $x^2 = y^3 = (xy)^7 = 1$ and the order of $[x, y]$ is n;

(ii) u_n has a primitive prime divisor.

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If $n \notin \{1, 2, 3, 5, 8, 12, 18, 28, 30\}$, then u_n has a primitive prime divisor.

Further possible directions

Let R be a word in the alphabet X, Y, X^{-1}, Y^{-1}. Assume that R has infinite order in $T(2,3,7) = \langle X, Y | X^2 = Y^3 = (XY)^7 = 1 \rangle$ (that is R is not identity in $T(2,3,7)$ and R is not conjugate to X, Y, Y^{-1}, $(XY)^i$).

Open Question

- Prove that for all sufficiently large n, the group $\langle X, Y | X^2 = Y^3 = (XY)^7 = R^n = 1 \rangle$ projects onto $\mathrm{PSL}_2(q)$ for some q.
- Find a uniform tight lower bound for n (independent of R).

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The problem is related to existence of primitive prime divisors of *arbitrary* second-order recurrences defined over $\mathbb{Z}[\theta]$.

$(2,3,7;2p)$ Đ¸ $G_2(p)$.

Theorem (V., 2006)

For any $p \geq 5$, the group $G_2(p)$ is an epimorphic image of $(2,3,7;2p)$.

$$x = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}.$$

The above generators (modulo p) satisfy further relations.

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Example 1. Let $g = ((xy)^2xy^2)^2(xyxy^2)^2$. We have $g^{e(p)} = 1$, where

$$e(p) = \begin{cases} (p-1)/4 & \text{if } p \equiv 1, 9 \pmod{20}; \\ (p-1)/2 & \text{if } p \equiv 11, 19 \pmod{20}; \\ (p+1)/2 & \text{if } p \equiv 3, 7, 13, 17 \pmod{20}. \end{cases}$$

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Example 2. Let $g = xy^2(xy)^2(xy^2)^2xyxy^2(xy)^2$. We have $g^{e(p)} = 1$, where

$$e(p) = \begin{cases} p-1 & \text{if } p \equiv 1, 4, 16, 25, 31 \pmod{33}; \\ p+1 & \text{if } p \equiv 5, 14, 20, 23, 26 \pmod{33}; \\ 2(p+1) & \text{if } p \equiv 7, 10, 13, 19, 28 \pmod{33}; \\ 2(p-1) & \text{if } p \equiv 2, 8, 17, 29, 32 \pmod{33} \text{ and } p \neq 2. \end{cases}$$

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$$e(p) = \begin{cases} (p-1)/4 & \text{if } p \equiv 1, 9 \pmod{20}; \\ (p-1)/2 & \text{if } p \equiv 11, 19 \pmod{20}; \\ (p+1)/2 & \text{if } p \equiv 3, 7, 13, 17 \pmod{20}. \end{cases}$$

Example 2. Let $g = xy^2(xy)^2(xy^2)^2xyxy^2(xy)^2$. We have $g^{e(p)} = 1$, where

$$e(p) = \begin{cases} p-1 & \text{if } p \equiv 1, 4, 16, 25, 31 \pmod{33}; \\ p+1 & \text{if } p \equiv 5, 14, 20, 23, 26 \pmod{33}; \\ 2(p+1) & \text{if } p \equiv 7, 10, 13, 19, 28 \pmod{33}; \\ 2(p-1) & \text{if } p \equiv 2, 8, 17, 29, 32 \pmod{33} \text{ and } p \neq 2. \end{cases}$$

Open Question

Complete the set of relations $X^2 = Y^3 = (XY)^7 = [X, Y]^{2p} = 1$ to obtain a presentation of $G_2(p)$.