International conference
“Number-theoretic aspects of linear algebraic groups and algebraic varieties: results and prospects”
dedicated to 85-th anniversary of academician V.P. Platonov
(June 16–19, 2025, Steklov Mathematical Institute, Moscow)

On the unboundedness of period lengths of functional continued fractions in a hyperelliptic field

Gleb Fedorov

Sirius University

fedorov.gv@talantiuspeh.ru

June, 17, 2025

Abstract

The report is devoted to joint results with V.P. Platonov concerning the problem of the unboundedness of period lengths of continued fractions of elements from a hyperelliptic field.

The famous Abel theorem establishes a criterion for the existence of elements in a hyperelliptic field that have a periodic expansion into a continued fraction. Subsequently, a significant number of studies were aimed at studying the problem of periodicity of functional continued fractions, including obtaining upper bounds on possible period lengths.

Until now, the problem of the finiteness of the set of possible period lengths of continued fractions for a given hyperelliptic field has remained open. In our report, we will present results that give a negative solution to this problem.

The report is based on joint articles with V.P. Platonov [1], [2].

---

[1] Platonov V.P., Fedorov G.V., Asymptotic behavior of period lengths of continued fractions in a hyperelliptic field // Izvestiya. Mathematics. 2025 (in press).

[2] Platonov V.P., Fedorov G.V., Continued fractions in hyperelliptic fields with an arbitrarily large period length // Doklady Mathematics. 2024. Vol. 109, no. 2. P. 147–151.

Field of formal power series

Let $\mathbb{K}$ be an arbitrary field, $\mathrm{Char}\,\mathbb{K} \neq 2$. Consider

$$\mathbb{K}((1/x)) = \left\{ \sum_{j=s}^{\infty} b_j \left(\frac{1}{x}\right)^j \mid b_j \in \mathbb{K},\ \ b_s \neq 0,\ \ s \in \mathbb{Z} \right\}.$$

For $\beta \in \mathbb{K}((1/x))$ we denote:

$$\beta = \sum_{j=s}^{\infty} b_j \left(\frac{1}{x}\right)^j, \quad [\beta] = \sum_{j=s}^{0} b_j \left(\frac{1}{x}\right)^j, \quad v_\infty(\beta) = s, \quad b_s \neq 0.$$

- $[\beta]$ — integer part of a power series.
- $v_\infty(\beta)$ — infinite valuation.

Construction of continued fraction

Let $\beta \in \mathbb{K}((1/x))$. Let's put

$$\beta_0 = \beta, \quad a_0 = [\beta_0], \quad M(a_0) = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \text{ – the simplest transformation.}$$

For each $j \in \mathbb{N}$ we set

$$\beta_j = M(a_{j-1})^{-1}[\beta_{j-1}] = \frac{1}{\beta_{j-1} - a_{j-1}}, \; a_j = [\beta_j], \; M(a_j) = \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix},$$

until $a_j \neq \beta_j$.

The result is an expansion of β into a functional continued fraction (finite or infinite), constructed in $\mathbb{K}((1/x))$:

$$\beta = [a_0; a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}}.$$

a_j — incomplete quotients, β_j — complete quotients.

Convergents

Let $\beta \in \mathbb{K}((1/x))$, $\beta = [a_0; a_1, \ldots]$. Let's denote

$$M_n = \prod_{i=0}^{n} M(a_j) = \prod_{i=0}^{n} \begin{pmatrix} a_j & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Then the following statements are true:

- $\beta = M_n[\beta_{n+1}] = \frac{p_n\beta_{n+1}+p_{n-1}}{q_n\beta_{n+1}+q_{n-1}}$;
- $\det M_n = (-1)^{n+1}$;
- $v_\infty^-(\beta - p_n/q_n) = -v_\infty(q_{n+1}) \to +\infty, \quad n \to \infty$;
- $\beta - p/q = O(x^{-2\deg q - 1}) \iff p/q$ – convergents for β;
- p_n/q_n – best approximation to β.

Field of functions

- Let $\mathcal{L} = \mathbb{K}(x)[y]/\left(y^2 - f(x)\right)$ be the function field of the curve $C : y^2 = f(x)$, where $f(x)$ is a square-free polynomial.
- If $\deg f$ is even and the leading coefficient of $f(x)$ is a full square in $\mathbb{K}$, then $\mathcal{L}$ can be embedded in $\mathbb{K}((1/x))$ in two ways.
- Depending on the embedding, the valuation v_∞ of the field $\mathbb{K}((1/x))$ is induced in two ways on the field $\mathcal{L}$: v_∞^-, v_∞^+, and

$$v_\infty^+(\beta) = v_\infty^-(\iota\beta),$$

  where $\beta \in \mathcal{L}$, $\iota : y \to -y$ is a hyperelliptic involution.
- Depending on the embedding, the elements of the field $\mathcal{L}$ can be expanded into two continued fractions in $\mathbb{K}((1/x))$, which correspond to the valuations v_∞^- and v_∞^+.

Abel's Theorem

Theorem (Abel, 1826; W. Schmidt, 2000)

The following conditions are equivalent:

- *the continued fraction of an element $\beta \in \mathcal{L}$ with discriminant $\mathbf{d}$ is quasi-periodic;*
- *there exists a non-trivial solution of the Pell-type functional equation*
$$\omega_1^2 - \omega_2^2 \mathbf{d} = \gamma,$$
with respect to the polynomials $\omega_1, \omega_2 \in \mathbb{K}[x]$, $\omega_2 \neq 0$, and $\gamma \in \mathbb{K}^*$.

Theorem

The following condition is equivalent to the previous ones:

- *there exists a non-trivial unit $\Omega_1 + \Omega_2\sqrt{f}$ of the ring $\mathcal{O}_\infty$ of integer elements of the hyperelliptic field $\mathcal{L}$ such that $\omega \mid \Omega_2$, where $\mathbf{d} = \omega^2 f$.*

Relation to the torsion problem in Jacobians

Theorem (Abel, 1826; W. Schmidt, 2000)

The following conditions are equivalent:

- *there exist elements in $\mathcal{L}$ with periodic continued fraction expansion;*
- *there exist elements in $\mathcal{L}$ with quasi-periodic continued fraction expansion;*
- *the divisor class $P_\infty^+ - P_\infty^-$ has finite order in the divisor class group $\Delta^\circ(\mathcal{L})$.*

If these conditions are satisfied, then the Jacobian J of $C:\ y^2 = f(x)$ has a torsion point.

Proposition

- *If the field $\mathcal{L}$ has quasi-periodic elements, then $\beta = \sqrt{f}$ is periodic.*
- *If $\mathbb{K} = \mathbb{F}_q$, then all nonrational elements of $\mathcal{L}$ are periodic.*

Lagrange's Theorem

Let $\mathbb{K} = \mathbb{F}_q$, $\beta \in \mathbb{K}((1/x))$.

Corollary

The following conditions are equivalent:

- *the continued fraction of β is periodic;*
- *the continued fraction of β is quasi-periodic;*
- *β is a quadratic irrationality.*

Analog of Lagrange's theorem

Theorem (Zannier, 2019)

The sequence of degrees of incomplete quotients functional continued fractions of non-rational elements of hyperelliptic fields is periodic.

$$\deg a_0, \deg a_1, \ldots, \deg a_m, \overline{\deg a_{m+1}, \ldots, \deg a_{m+N}}$$

The proof of this theorem does not allow us to draw conclusions about upper and lower bounds on the period length.

Main theorem

Question. Let us fix a hyperelliptic field $\mathcal{L}$, in which there are elements with periodic (quasi-periodic) expansion into a continued fraction. What can we say about the upper bound on the period length (or quasi-period length) of elements from $\mathcal{L}$?

Let $\mathbb{K}$ be a field of characteristic different from 2, and $\mathcal{L} = \mathbb{K}(x)(\sqrt{f})$ be a hyperelliptic field, whose ring of integers $\mathcal{O}_\infty$ has nontrivial units.

Theorem (Platonov, F., 2025)

For any arbitrarily large value of N there exists an element $\beta \in \mathcal{L}$ such that the period length of the continued fraction β is finite and greater than N.

Auxiliary results

For $A, B, C, D \in \mathbb{K}[x]$ and an element $\beta \in \mathbb{K}((1/x))$, we use the notation

$$\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)[\beta] = \frac{A\beta + B}{C\beta + D}.$$

The set

$$\Gamma = \left\{ \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \mid A, B, C, D \in \mathbb{K}[x],\ AD - BC \in \mathbb{K}^* \right\}$$

is a group acting on elements of $\mathbb{K}((1/x))$ by rule above. This group is called the group of unimodular transformations.

- A transformation $M \in \Gamma$ is said to be *trivial* if $M \in GL_2(\mathbb{K})$.
- The action Γ is transitive: $(M_1 \cdot M_2)[\beta] = M_1\,[M_2[\beta]]$.

Serret's Theorem

Definition

We say $\alpha \sim \beta$*, if there exists* $M \in \Gamma$*, such that*

$$\alpha = M[\beta] = \frac{A\beta + B}{C\beta + D}, \quad \textit{where} \;\; M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Theorem (V. Schmidt, 2000)

Let $\alpha, \beta \in \mathbb{K}((1/x)) \setminus \mathbb{K}(x)$*. The following conditions are equivalent:*

- $\alpha \sim \beta$*;*
- $\alpha_n = c\beta_m$ *for some* $c \in \mathbb{K}^*$*.*

Reduced element and reduced transformations

An element $\beta \in \mathcal{L}$ is called reduced, if $v_\infty^-(\beta) > 0$ and $v_\infty^+(\beta) < 0$.

- There exists a number $j_0 \geq 0$, such that for $j \geq j_0$ all complete quotients β_j are reduced.
- If the continued fraction of β is quasi-periodic, the quasi-period begins with the number $\tau = j_0$, that is, with the number of the first reduced complete quotient β_τ.

The transformation $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ is called reduced if $\deg A > \deg B$ and $\deg A > \deg C$.

- Since $\det M \in \mathbb{K}^*$ for a reduced M such that $D \neq 0$, we have $\deg B > \deg D$ and $\deg C > \deg D$.
- The simplest transformations $M(a_j)$ are reduced for $j \geq 1$.
- The product of reduced transformations is reduced.

Decomposition of a unimodular transformation

For $b, c \in \mathbb{K}^*$, we define

$$I_2(c, b) = \begin{pmatrix} c & 0 \\ 0 & b \end{pmatrix}.$$

Proposition

Let $M \in \Gamma$ be the reduced transformation. Then the following statements hold.
1. There is a unique representation of the form

$$M = M(a_0) \cdot M(a_1) \cdot \ldots \cdot M(a_n) \cdot I_2(c, b),$$

where $a_j \in \mathbb{K}[x] \setminus \mathbb{K}$, $0 \leq j \leq n$, $b, c \in \mathbb{K}^*$.
2. Let $\beta \in \mathcal{L}$ be a reduced element. Then

$$M[\beta] = [a_0; a_1, \ldots, a_n, c\beta/b],$$

where $a_j \in \mathbb{K}[x] \setminus \mathbb{K}$, $0 \leq j \leq n$, $b, c \in \mathbb{K}^*$ are the same as in point 1.

We define the degree of transformation $M = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$ as

$$\deg M = \max\{\deg A, \deg B, \deg C, \deg D\}.$$

For $n \geq 0$, we denote

$$M_n = \prod_{j=0}^{n} M(a_j),$$

then

$$\beta = M_n[\beta_{n+1}], \quad \det M_n = (-1)^{n+1},$$

and $M_n \in \Gamma$ — unimodular transformation.

We have

$$M_n = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix},$$

where $p_n, q_n \in \mathbb{K}[x]$ satisfy the recurrence relation $x_n = a_n x_{n-1} + x_{n-2}$ with initial conditions

$$M_{-1} = \begin{pmatrix} p_{-1} & p_{-2} \\ q_{-1} & q_{-2} \end{pmatrix} = I_2(1, 1).$$

Let $\beta \in \mathcal{L} \setminus \mathbb{K}[x]$ is a root of the equation

$$\Lambda_2 X^2 + 2\Lambda_1 X + \Lambda_0 = 0,$$

where $\Lambda_0, \Lambda_1, \Lambda_2 \in \mathbb{K}[x]$ are collectively coprime polynomials.

Denote $d = \Lambda_1^2 - \Lambda_0 \Lambda_2$ — the reduced discriminant of β. Since $\beta \in \mathcal{L}$, then $d = \omega^2 f$ for some $\omega \in \mathbb{K}[x]$.

As we mentioned earlier, for the quasi-periodicity of continued fraction of β, it is necessary and sufficient that there be a non-trivial unit $U = \Omega_1 + \Omega_2\sqrt{f} \in \mathcal{O}_\infty$ such that $\omega \mid \Omega_2$.

For $n \in \mathbb{N}_0$ we define the matrices $M_\Lambda^{(n)}$ as follows:

$$M_\Lambda^{(n)} = \begin{pmatrix} \Lambda_2^{(n)} & \Lambda_1^{(n)} \\ \Lambda_1^{(n)} & \Lambda_0^{(n)} \end{pmatrix} = M_{n-1}^T M_\Lambda M_{n-1}, \quad M_\Lambda^{(0)} = M_\Lambda = \begin{pmatrix} \Lambda_2 & \Lambda_1 \\ \Lambda_1 & \Lambda_0 \end{pmatrix}.$$

Proposition

1. *For each* $n \in \mathbb{N}_0$ *we have* $\det M_\Lambda^{(n)} = -d$.
2. *For* $n \in \mathbb{N}_0$, *the complete quotients* β_n *satisfy the equation*

$$\begin{pmatrix} X & 1 \end{pmatrix} M_\Lambda^{(n)} \begin{pmatrix} X \\ 1 \end{pmatrix} = 0.$$

3. *The complete quotients* β_n *of the continued fraction of* β *are of the form* $\beta_n = A_\Lambda^{(n)}[Y]$, *where*

$$A_\Lambda^{(n)} = \begin{pmatrix} (-1)^n & -\Lambda_1^{(n)} \\ 0 & \Lambda_2^{(n)} \end{pmatrix}, \quad Y \text{ satisfies the equation } Y^2 = d.$$

Proposition

There is a number $\tau \in \mathbb{N}_0$, $\tau \leq 1 + \max\big(\deg \Lambda_2 - (\deg d)/2, 0\big)/2$ *such that for* $j \geq \tau$ *the following relations hold*

$$\deg \Lambda_0^{(j)} < (\deg d)/2, \;\; \deg \Lambda_1^{(j)} = (\deg d)/2, \;\; \deg \Lambda_2^{(j)} < (\deg d)/2.$$

Proposition

Let for a quadratic irrationality β *having the discriminant* d, *the continued fraction constructed in the field* $\mathbb{K}((1/x))$, *have the form* $\beta = [a_0; a_1, \ldots]$. *Then for* $j \geq \tau$ *the following estimates hold:*

$$\deg a_j \leq (\deg d)/2 - \deg \Lambda_2^{(j)}.$$

Proposition

Let for some number $s \geq \tau$ *we have* $\deg a_s = (\deg d)/2$ *then the following statements hold.*

1. The complete quotient β_s *has the form* $\beta_s = b(T + Y)$, *where* $b \in \mathbb{K}^*$, $T = [Y]_\infty^-$, $Y^2 = d$.

2. If the continued fraction β *is quasi-periodic with quasi-period length* μ *and quasi-period constant* c, *then the continued fraction* β *can be written as*

$$\beta = [a_0; a_1, \ldots, a_{s-1}, \overline{a_s, \ldots, a_{s+\mu-1}}^{e}],$$

where $e = c$ *or* $e = c^{-1}$ *and* $\deg a_{s+j} < (\deg d)/2$ *for* $j = 1, \ldots, \mu - 1$.

Theorem

Let the continued fraction $\beta \in \mathcal{L}$ be quasi-periodic

$$\beta = [a_0, \ldots, a_{\tau-1}, \overline{a_\tau, \ldots, a_{\tau+\mu-1}}^{c}].$$

Let the polynomial Θ_1 have the minimum degree among all nontrivial solutions $(\Theta_1, \Theta_2, \gamma)$ of the equation $\Theta_1^2 - \Theta_2^2 d = \gamma$. Then

$$\sum_{j=\tau}^{\tau+\mu-1} \deg a_j = \deg \Theta_1.$$

Corollary

The inequalities on the length of the quasi-period μ are valid

$$\frac{2 \deg \Theta_1}{\deg d} \le \mu \le \deg \Theta_1.$$

Primitive roots of unity and cyclotomic polynomials

- Let $p_{\mathbb{K}} = \operatorname{Char} \mathbb{K} \neq 2$ — characteristic of the field $\mathbb{K}$.
- We will assume that in the case $p_{\mathbb{K}} = 0$ the condition of the form $p_{\mathbb{K}} \nmid n$ is satisfied for all $n \in \mathbb{N}$.
- Denote $\mathbb{K}^{(n)}$ by the cyclotomic extension of the field $\mathbb{K}$ by the splitting field $x^n - 1$ over $\mathbb{K}$.
- Denote $\mathcal{S}_n = \mathcal{S}_n(\mathbb{K}^{(n)})$ by the set of distinct roots of 1 of degree n over the field $\mathbb{K}^{(n)}$.
- If $p_{\mathbb{K}} \nmid n$, then $\mathcal{S}_n$ is a cyclic group of order n, and $|\mathcal{S}_n| = n$.
- If $p_{\mathbb{K}} \mid n$, then $\mathcal{S}_n = \mathcal{S}_m$, where $n = p_{\mathbb{K}}^j m$, $m, j \in \mathbb{N}$ and m is not divisible by $p_{\mathbb{K}}$.
- Further, we will consider only those $n \in \mathbb{N}$ such that $p_{\mathbb{K}} \nmid n$.

- A generator ξ of $\mathcal{S}_n$ is called a primitive root of 1 of degree n over the field $\mathbb{K}$.
- Denote $\mathrm{E}_n = \mathrm{E}_n(\mathbb{K}^{(n)}) = \{\xi^j \mid 1 \le j \le n,\ (j, n) = 1\}$ — the set of primitive roots of 1 of degree n over the field $\mathbb{K}^{(n)}$.
- $|\,\mathrm{E}_n\,| = \varphi(n)$, where $\varphi(n)$ is the Euler phi-function.
- We define the n-th cyclotomic polynomial over $\mathbb{K}$

$$\Phi_n(x) = \prod_{\xi \in \mathrm{E}_n} (x - \xi).$$

- $\Phi_n(x) \in \mathbb{K}^{(0)}[x]$, where $\mathbb{K}^{(0)}$ is a prime subfield of $\mathbb{K}$.
- $\deg \Phi_n(x) = \varphi(n)$.

Lemma

$$x^n - 1 = \prod_{d|n} \Phi_d(x), \quad x^n + 1 = \prod_{d|q} \Phi_{2^{t+1}d}(x), \quad \text{where } n = 2^t q.$$

- For $n > 2$, let $\mathsf{E}'_n \subset \mathsf{E}_n$ be some of the subsets of E_n such that exactly one of the primitive roots ξ, ξ^{-1} belongs to E'_n.
- For example, if $p_{\mathbb{K}} = 0$ then $\mathsf{E}'_n = \{\xi \in \mathsf{E}_n \mid 0 < \mathrm{Arg}(\xi) < \pi\}$.
- Let $\overline{\Phi}_n(x, z)$ be the homogenization of $\Phi_n(x)$.

Lemma

$$\overline{\Phi}_1(z+x, z-x) = 2x, \quad \overline{\Phi}_2(z+x, z-x) = 2 = \overline{\Phi}_2(z-x, z+x),$$

$$\overline{\Phi}_n(z-x, z+x) = \overline{\Phi}_n(z+x, z-x) =$$
$$= \prod_{\xi \in \mathsf{E}'_n} \left(2(x^2+z^2) + (\xi+\xi^{-1})(x^2-z^2)\right), \; n > 2.$$

Proof. Let's split the factors into pairs and expand the brackets

$$\big((z+x) - \xi(z-x)\big)\big((z+x) - \xi^{-1}(z-x)\big) =$$
$$= (z+x)^2 + (z-x)^2 - (z^2-x^2)(\xi+\xi^{-1}) =$$
$$= \big((z-x) - \xi(z+x)\big)\big((z-x) - \xi^{-1}(z+x)\big).$$

For $n \geq 2$ we define

$$R_n(x,z) = \begin{cases} \prod_{\xi \in \mathrm{E}'_n} \left(2(x+z) + (\xi + \xi^{-1})(x-z)\right), & \text{if } n > 2, \\ 2, & \text{if } n = 1 \text{ or } n = 2. \end{cases}$$

By the lemma above we have

$$\overline{\Phi}_n(z+x, z-x) = R_n(x^2, z^2) = \overline{\Phi}_n(z-x, z+x).$$

- $R_n(x,z) \in \mathbb{K}^{(0)}[x,z]$ is a homogeneous polynomial.
- The definition of the polynomial $R_n(x,z)$ does not depend on the choice of $\mathrm{E}'_n \subset \mathrm{E}_n$.

Let us define the polynomials $T_n(x), Q_n(x) \in \mathbb{K}^{(0)}[x]$ such that

$$T_n(u^2) = \frac{1}{2}\big((1+u)^n + (1-u)^n\big), \quad Q_n(u^2) = \frac{1}{2u}\big((1+u)^n - (1-u)^n\big).$$

The identity is true

$$(1+u)^n = T_n(u^2) + uQ_n(u^2).$$

Denote $T_n^{\mathrm{Cheb}}(x)$, $U_n^{\mathrm{Cheb}}(x)$ — Chebyshev polynomials of the first and second kind, respectively. Then the relations hold

$$T_n^{\mathrm{Cheb}}(x) = T_n\left(1 - \frac{1}{x^2}\right), \quad U_n^{\mathrm{Cheb}}(x) = Q_n\left(1 - \frac{1}{x^2}\right).$$

Denote $\overline{T}_n(x,z)$ and $\overline{Q}_n(x,z)$ — homogeneous polynomials, corresponding to the polynomials $T_n(x), Q_n(x)$.

Proposition

Let $n = 2^t q$, where q is odd. The identities are valid

$$\overline{T}_n(x,z) = \frac{1}{2}\prod_{d|q} R_{2^{t+1}d}(x,z), \quad \overline{Q}_n(x,z) = \frac{1}{2}\prod_{d|n} R_d(x,z).$$

Proof. Let us prove the first equality. By the lemma we have

$$T_n(u^2) = \frac{1}{2}(1+u)^n\left(1+\left(\frac{1-u}{1+u}\right)^n\right) =$$

$$= \frac{1}{2}(1+u)^n\prod_{d|q}\Phi_{2^{t+1}d}\left(\frac{1-u}{1+u}\right) = \frac{1}{2}\prod_{d|q}\overline{\Phi}_{2^{t+1}d}(1-u,1+u),$$

$$\overline{\Phi}_j(1-u,1+u) = \prod_{\xi\in\mathrm{E}_j^{-1}}\left(2(u^2+1)+(\xi+\xi^{-1})(u^2-1)\right) = R_j(u^2,1), \;\; j>2.$$

Lemma

Let $\theta_1(x), \theta_2(x) \in \mathbb{K}[x]$ be coprime polynomials. Put $P_n(x) = R_n(\theta_1(x), \theta_2(x))$. Then for any $j, n \in \mathbb{N}$ such that $j < n$, $p_{\mathbb{K}} \nmid j$, $p_{\mathbb{K}} \nmid n$, the polynomials $P_j(x), P_n(x)$ are coprime.

Proof. For $2 < j < n$ we have

$$P_n(x) = \prod_{\xi \in \mathrm{E}'_n} \Big(2\big(\theta_1(x) + \theta_2(x)\big) + (\xi + \xi^{-1})\big(\theta_1(x) - \theta_2(x)\big)\Big).$$

Suppose that $x_0 \in \overline{\mathbb{K}}$ is a common root of $P_j(x)$ and $P_n(x)$, where $\overline{\mathbb{K}}$ is the algebraic closure of the field $\mathbb{K}$.
Then there exist $\xi_j \in \mathrm{E}'_j$, $\xi_n \in \mathrm{E}'_n$ such that

$$2\big(\theta_1(x_0) + \theta_2(x_0)\big) + (\xi_j + \xi_j^{-1})\big(\theta_1(x_0) - \theta_2(x_0)\big) = 0,$$
$$2\big(\theta_1(x_0) + \theta_2(x_0)\big) + (\xi_n + \xi_n^{-1})\big(\theta_1(x_0) - \theta_2(x_0)\big) = 0.$$

Since $\theta_1(x), \theta_2(x)$ are coprime, then $\xi_j + \xi_j^{-1} = \xi_n + \xi_n^{-1}$.
This is impossible, since ξ_j, ξ_n are primitive roots of degree $j < n$.

Proof of main theorem

Proof of main theorem. Part 1.

Let $\Psi_1 + \Psi_2\sqrt{f}$ is fundamental unit and

$$\Omega_1^{(j)} + \Omega_2^{(j)}\sqrt{f} = (\Psi_1 + \Psi_2\sqrt{f})^j.$$

Let $u = \Psi_2\sqrt{f}/\Psi_1$, then

$$\Omega_1^{(j)}+\Omega_2^{(j)}\sqrt{f} = (\Psi_1+\Psi_2\sqrt{f})^j = \Psi_1^j(1+u)^j = \Psi_1^j\Big(T_j(u^2)+uQ_j(u^2)\Big),$$

where the polynomials $T_j(x), Q_j(x) \in \mathbb{K}^{(0)}[x]$ were defined earlier.

Our main goal is to construct a sequence $\beta^{(k)} \in \mathcal{L}$, $k = 1, 2, \ldots$, for which the lengths of the quasi-periods $\mu^{(k)} \to \infty$ as $k \to \infty$.

It suffices to construct a sequence of discriminants $d^{(k)} = (\omega^{(k)})^2 f$, $\omega^{(k)} \in \mathbb{K}[x]$, possessing the following properties:

1) $\forall k \in \mathbb{N}$ there exists a minimal number $n_k \in \mathbb{N}$: $\omega^{(k)} \mid \Omega_2^{(n_k)}$;
2) the sequence $\{n_k\}_{k\in\mathbb{N}}$ strictly increases;
3) $\deg \Omega_1^{(n_k)} / \deg \omega^{(k)} \to \infty$ as $k \to \infty$.

Proof of main theorem. Part 2.

Consider a sequence of odd numbers $\{n_k\}_{k\in\mathbb{N}}$, $p_{\mathbb{K}} \nmid n_k$, for which $\varphi(n_k)/n_k \to 0$ as $k \to \infty$.

For example, we can take $n_k = p_1 \cdot p_2 \cdot \ldots \cdot p_k$ — the product of consecutive odd primes $p_1 = 3$, $p_2 = 5$, $\ldots$, skipping $p_{\mathbb{K}} = \operatorname{Char}\mathbb{K}$ if the characteristic of the field $\mathbb{K}$ is finite.

Let $n = n_k$ and $\omega^{(k)}(x) = R_n\big(\Psi_1^2(x), f(x)\Psi_2^2(x)\big)$.
Since $\Psi_1^2(x) - f(x)\Psi_2^2(x) \in \mathbb{K}^*$, then $\Psi_1^2(x), f(x)\Psi_2^2(x)$ are coprime. We have

$\omega^{(k)}(x) \mid \overline{Q}_n\big(\Psi_1^2(x), f(x)\Psi_2^2(x)\big) = \Omega_2^{(n_k)}(x)$, and

$\omega^{(k)}(x) \nmid \overline{Q}_j\big(\Psi_1^2(x), f(x)\Psi_2^2(x)\big)$ for all $j < n$ such that $p_{\mathbb{K}} \nmid j$.

Let $m = \deg \Psi_1(x)$ — the degree of the fundamental unit $\Psi_1 + \Psi_2\sqrt{f}$. Since $\deg \Phi_n(x) = \varphi(n)$, then for $n > 2$ we have

$$\deg R_n(x,z) = \frac{1}{2}\deg R_n(x^2,z^2) = \frac{1}{2}\deg \overline{\Phi}_n(z+x, z-x) \leq \frac{\varphi(n)}{2}.$$

From here we get the estimation

$$\deg \omega^{(k)}(x) = \deg R_n\big(\Psi_1^2(x), f(x)\Psi_2^2(x)\big) \leq 2m \deg R_n(x,z) \leq m\varphi(n_k).$$

Let $\beta^{(k)} \in \mathcal{L}$ be a root of the polynomial

$$\Lambda_2^{(k)} Y^2 + 2\Lambda_1^{(k)} Y + \Lambda_0^{(k)} = 0, \quad (\Lambda_2^{(k)}, \Lambda_1^{(k)}, \Lambda_0^{(k)}) = 1,$$

with reduced discriminant

$$d^{(k)}(x) = (\Lambda_1^{(k)})^2 - \Lambda_0^{(k)}\Lambda_2^{(k)} = \big(\omega^{(k)}(x)\big)^2 f(x),$$
$$\deg d^{(k)}(x) = 2\deg \omega^{(k)}(x) + \deg f \leq 2m\varphi(n_k) + \deg f.$$

Proof of main theorem. Part 4.

Let
$$\Theta_1^{(n)}(x) = \Omega_1^{(n)}(x), \quad \Theta_2^{(n)}(x) = \Omega_2^{(n)}(x)/\omega^{(k)}(x).$$
By the corollary, the following estimates are valid for the length of the quasi-period $\mu^{(k)}$ of the continued fraction of the element $\beta^{(k)}$:

$$\frac{n_k}{\varphi(n_k) + \deg f/(2m)} = \frac{2nm}{2m\varphi(n_k) + \deg f} \leq$$
$$\leq \frac{2\deg \Theta_1^{(n)}}{\deg d^{(k)}} \leq \mu^{(k)} \leq \deg \Theta_1^{(n)} = n_k m.$$

The quantities m and $\deg f$ in the field $\mathcal{L}$ take fixed values. Since $\varphi(n_k)/n_k \to 0$ as $k \to \infty$, we have $\mu^{(k)} \to \infty$ as $k \to \infty$.

Proof of main theorem. Part 5.

If we take

$$\Lambda_2^{(k)} = 1,\ \Lambda_1^{(k)} = 0,\ \Lambda_0^{(k)} = -d^{(k)},$$

then $\beta^{(k)} = \sqrt{d^{(k)}} = \omega^{(k)}(x)\sqrt{f(x)}$ and the element $\beta^{(k)}$ has a quasi-periodic expansion into a continued fraction.

The period length is equal to $\mu^{(k)}$ or equal to $2\mu^{(k)}$.

Thus, the same lower bound holds for the period length, therefore, the period length of the continued fraction of the element $\beta^{(k)} = \sqrt{d^{(k)}} \in \mathcal{L}$ tends to infinity as $k \to \infty$.

The main theorem is proven.

Corollary

Let $\mathcal{L}$ be a hyperelliptic field defined as in Theorem. Then, for any N, there is an element $\beta \in \mathcal{L}$ for which the sequence of degrees of partial quotients $\{\deg a_i\}$ has a period length greater than N.

Example

Let $\mathbb{K} = \mathbb{K}^{(\infty)}$, that is any cyclotomic extension $\mathbb{K}^{(n)} \subset \mathbb{K}$.
Let $\Psi_1^2(x) - f(x)\Psi_2^2(x) = \gamma \in \mathbb{K}^*$ and $\deg(\Psi_1(x)) = m$.
Put $\omega^{(1)}(x) = f(x)\Psi_2^2(x)$, $\omega^{(2)}(x) = \Psi_1^2(x)$, and for $k > 2$, $p_{\mathbb{K}} \nmid k$, define

$$\omega^{(k)}(x) = 2\big(\Psi_1^2(x) + f(x)\Psi_2^2(x)\big) + (\xi_k + \xi_k^{-1})\big(\Psi_1^2(x) - f(x)\Psi_2^2(x)\big),$$

where ξ_k — any primitive root of 1 of degree k. Then

$$\omega^{(k)}(x) = 2\Omega_1^{(2)} + \gamma(\xi_k + \xi_k^{-1}) \in \mathbb{K}[x], \quad \deg \omega^{(k)}(x) = 2m.$$

We have $\omega^{(k)}(x) \mid \overline{Q}_k\big(\Psi_1^2(x), f(x)\Psi_2^2(x)\big)$,
$\omega^{(k)}(x) \nmid \overline{Q}_j\big(\Psi_1^2(x), f(x)\Psi_2^2(x)\big)$ for all $j < k$ such that $p_{\mathbb{K}} \nmid j$.
Let $\beta^{(k)} = \sqrt{d^{(k)}}$, where $d^{(k)} = \big(\omega^{(k)}(x)\big)^2 f(x)$. Then

$$\frac{2km}{4m + \deg f} = \frac{2 \deg \Omega_1^{(k)}}{\deg d^{(k)}} \le \mu^{(k)} \le \deg \Omega_1^{(k)} = km.$$

So, the length of the quasi-period $\mu^{(k)}$ on average increases approximately linearly with respect to k as $k \to +\infty$.

Let $\mathbb{K} = \mathbb{Q}$, $f_a(x) = x^2 + a$, where $a \in \mathbb{Z}$ is a square-free integer different from 0. Consider the field $\mathcal{L}_a = \mathbb{Q}(x)(\sqrt{f_a})$.

Let us study experimentally the quantity

$$\rho_n = \sup_{k \leq n} \sup_{\omega | \Omega_2^{(k)}} \mu\left(\omega\sqrt{f_a}\right),$$

where $\mu\left(\beta\right)$ is the length of the quasi-period of the continued fraction of the element β.

Let M_k be the set of irreducible over $\mathbb{Q}$ factors of $\Omega_2^{(k)}(x)$.

We find $\mu_k = \max_{\omega \in \mathrm{M}_k} \mu\left(\omega\sqrt{f_a}\right)$ and $\rho_n(a) = \max_{k \leq n} \mu_k$ for $n = 1, 2, \ldots$.

The sequence $\{\rho_n(a)\}$ can be interpreted as *a sequence of local maxima of quasi-period lengths*.

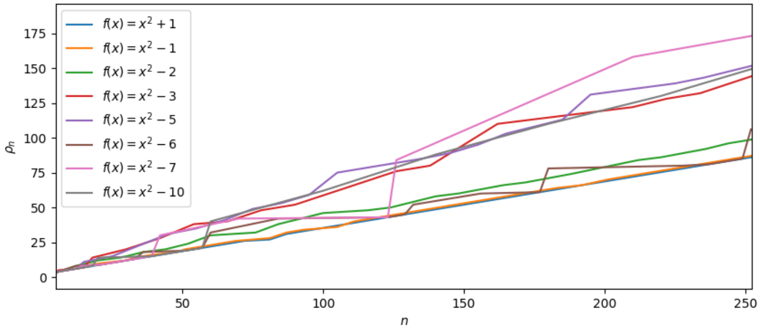

Figure: Comparison of sequences $\rho_n(a)$ of local maxima of quasi-period lengths.

The Figure shows a comparative analysis of the initial segments of the sequences $\{\rho_n(a)\}$ for $a \in \{-10, -7, -6, -5, -3, -2, -1, 1\}$.

Thanks for reading!

Gleb Fedorov
fedorov.gv@talantiuspeh.ru