Coordinate Algebras of Algebraic Groups: Generators and Relations

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Number-theoretic aspects of linear algebraic groups and algebraic varieties: results and prospects

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Below all theorems with unspecified authorship are taken from the papers

V. L. Popov, *On the equations defining affine algebraic groups*, Pacific J. Math.

V. L. Popov, *Group varieties and group structures*, Izv. Math.

V. L. Popov, *Closed orbits of Borel subgroups*, Math. USSR-Sb.

Notation and convention

- k an **algebraically closed field**
- Below all algebraic varieties are **taken over** k and all algebraic groups are **taken in the sense of Chevalley–Borel–Humphreys–Springer**, i.e., are **reduced (smooth)**
- G a **connected algebraic group**.

Main question

By definition, an algebraic group is endowed with both the structures of a **group** and an **algebraic variety**. Since these structures agree, there must be a **dependence between them**.

Main question

Whence the following general

Main Question

To what extent do these structures determine each other?

Explicitly or implicitly, it has long been considered in the literature, in particular, in the classical papers of A. Weil, C. Chevalley, A. Borel, A. Grothendieck, M. Rosenlicht, M. Lazard, . . .

One half of the Main Question is

Question G⇒A

To what extent does **structure of algebraic variety** determines **group structure** ?

The following classical theorems provide striking **examples**, in which, in Shafarevich's words,

algebra is determined by **geometry**.

Question G⇒A

Theorem (C. Chevalley, A. Weil)

If the underlying variety of G is **projective** (*i.e., G is an abelian variety*), *then G is* **commutative**.

Question G⇒A

Theorem (A. Weil, 1948)

Abelian varieties *are isomorphic* $\iff$ *their* **underlying varieties** *are isomorphic.*

Theorem (C. Chevalley, 1950, M. Lazard, 1955)

G *is* **unipotent** $\iff$ *its* **underlying variety** *is isomorphic to* $\mathbb{A}^{\dim G}$.

And the following more recent generalizations of these theorems:

Theorem (G. A. Dill, 2021)

In the case of $\underline{\mathrm{char}(k) = 0}$, **connected commutative algebraic groups** *are isomorphic* $\iff$ *their* **underlying varieties** *are isomorphic.*

Theorem

G is **solvable and affine** $\iff$ *its* **underlying variety** *is isomorphic to $\mathbb{A}_*^p \times \mathbb{A}^q$ for some p, q, where $\mathbb{A}_*$ is the punctured affine line*

Geometry does not completely determine algebra:

- The case of **connected unipotent** algebraic groups:

For $n \geqslant 7$, the structures of (automatically unipotent) algebraic groups on $\mathbb{A}^n$ depend on parameters (moduli).

- The case of **connected reductive** algebraic groups G:

Notation:

- C^0 **the identity component of the center** of G,
- $S = [G, G]$ the **derived group** of G.

In general, G and $C^0 \times S$ are **not isomorphic** algebraic groups (example: $G = \mathrm{GL}_n$). However,

Theorem

The **underlying varieties** *of G and $C^0 \times S$ are* **isomorphic**.

Question G⇒A

Theorem

(a) *There are only* **finitely many** *(up to isomorphism) algebraic groups R whose* **underlying variety** *is isomorphic to that of G.*

(b) *Every such R is automatically* **reductive***.*

Remark

The number of R's in (a) **may exceed** 1 even if G is **semisimple**.

Example

Let $d \geqslant 2$ be an integer and $\varepsilon \in k$ a primitive d-th root of 1. Let

$$G_1 := \mathrm{PSL}_d \times \mathrm{SL}_d,$$
$$G_2 := (\mathrm{SL}_d \times \mathrm{SL}_d)/C$$

where C is the cyclic group generated by $\big(\mathrm{diag}(\varepsilon, \ldots, \varepsilon), \mathrm{diag}(\varepsilon, \ldots, \varepsilon)\big)$. One can show that G_1 and G_2 are **nonisomorphic** algebraic groups whose underlying varieties are **isomorphic**. For $d = 2$ this yields two groups

$$\mathrm{SO}_4 \quad \text{and} \quad \mathrm{PSL}_2 \times \mathrm{SL}_2.$$

The main purpose of this talk is to discuss **the other half of the Main Question**:

Question A⇒G

To what extent does

group structure determine **structure of algebraic variety**?

Our starting point for describing structure of algebraic variety is

Theorem (W. Chow, 1957)

The underlying variety of G is **quasi projective**, *i.e.*, **admits an embedding in a projective space**.

In view of this, below Question A⇒G is understood as

Main Problem

Are there **distinguished embeddings** *of the underlying variety of G in projective spaces and the* **equations of their images**, *which are* **canonically determined by the group structure** *of G?*

Main problem: abelian varieties

For abelian varieties, this Main problem is

the existence problem for canonically defined bases in linear systems and the problem of presentating homogeneous coordinate rings of ample invertible sheafs by generators and relations.

These problems were explored and solved by D. Mumford and G. R. Kempf (1966, 1989).

Main problem: affine algebraic groups

Our aim in this talk is to consider the Main Problem **for affine algebraic groups**. In this case it is naturally reformulated as

Main Problem for affine algebraic groups (geometric form)

Is there a **closed embedding** *of G into an* **affine** *space, which is* **canonically determined by the group structure***?*

The **algebraic counterpart of this geometric formulation** is

Main Problem for affine algebraic groups (algebraic form)

Is there a **presentation of the coordinate algebra** $k[G]$ **of** G **by generators and relations**, *which is* **canonically determined by the group structure***?*

Main problem: notation

Reduction of the Main problem for affine algebraic groups

Notation

- $\mathrm{Rad}_u(G)$ the **unipotent radical** of G,
- $R := G/\mathrm{Rad}_u(G)$ (it is a **connected reductive** group),
- C^0 the identity component of the center of R,
- $S = [R, R]$ the derived group of R

First step:

Although the algebraic groups G and $R \times \mathrm{Rad}_u(G)$ in general are **not isomorphic** (if $\mathrm{char}\, k > 0$), **their underlying varieties are always isomoprhic**. Since $\mathrm{Rad}_u(G)$ is **connected unipotent**, its underlying variety is an **affine space**. Hence presentation of G is obtained from that of R just by **adding several new variables**.

- Conclusion:

In the Main Problem for affine algebraic groups one may replace G by R, i.e., assume that G is reductive.

Second step:

Although the algebraic groups R and $C^0 \times S$ in general are **not isomorphic**, **their underlying varieties are always isomoprhic**. Since C^0 is a **torus**, its underlying variety is a **product of** d **copies of punctured affine line** for some d. Hence presentation of R is obtained from that of S just by **adding new variables** $\boldsymbol{x_1, y_1, \ldots, x_d, y_d}$ **and relations** $\boldsymbol{x_1 y_1 - 1, \ldots, x_d y_d - 1}$.

- Conclusion:

In the Main Problem for affine algebraic groups one may replace R by S, i.e., assume that G is semisimple.

Guiding example: $G = \mathrm{SL}_2$

We start describing our answer to the Main problem for affine algebraic groups with a **simple model example**, in which, however, all components of the general answer are visible

$\boxed{G = \mathrm{SL}_2}$

$k[x_1, x_2, x_3, x_4]$ the polynomial k-algebra in four variables x_i.
The usual presentation of $k[\mathrm{SL}_2]$ is given by the epimorphism

$$\mu\colon k[x_1, x_2, x_3, x_4] \to k[\mathrm{SL}_2], \quad \mu(x_i)\left(\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\right) = a_i,$$

We have $\ker \mu = (x_1x_4 - x_2x_3 - 1)$.

After rewriting, this presentation can be interpreted
in terms of the group structure of G. Namely, since

$$k[x_1, x_2, x_3, x_4] = k[x_1, x_3] \otimes_k k[x_2, x_4],$$

μ yields the k-algebra isomorphism

Guiding example: $G = \mathrm{SL}_2$

$$k[G] \cong \big(k[x_1, x_3] \otimes_k k[x_2, x_4]\big)/(x_1x_4 - x_2x_3 - 1).$$

Let U^+ and U^- be the following **opposite maximal unipotent subgroups** of G:

$$U^+ = \boxed{\begin{matrix} 1 & * \\ 0 & 1 \end{matrix}}, \quad U^- = \boxed{\begin{matrix} 1 & 0 \\ * & 1 \end{matrix}}.$$

and let

$$S^\pm = \{f \in k[G] \mid f(gu) \text{ for all } g \in G,\ u \in U^\pm\}.$$

Guiding example: $G = \mathrm{SL}_2$

Then

$$\mu\colon k[x_1, x_3] \xrightarrow{\cong} S^+,$$
$$\mu\colon k[x_2, x_4] \xrightarrow{\cong} S^-.$$

Therefore

$$k[G] \cong (\mathcal{S}^+ \otimes_k \mathcal{S}^-)/(f - 1),$$

where

$$f = \mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3).$$

Guiding example: $G = \mathrm{SL}_2$

This f **is defined in terms of the group structure** as follows:

- $\mathcal{S}^+$ and $\mathcal{S}^-$ are G-**stable regarding action by left translations**,
- f is the **unique element of $(\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$ determined by the conditions**
  - $f(e, e) = 1$,
  - $k[f] = (\mathcal{S}^+ \otimes_k \mathcal{S}^-)^G$.

General case: notation

Now we turn from this example to **considering the general case**.

Notation

- G **connected reductive** algebraic group,
- $k[G]$ **coordinate algebra of** G endowed with the G-**module structure** determined by **left translations**,
- B^+ and B^- a pair of **opposite Borel subgroups** of G,
- $U^\pm$ **unipotent radical** (=derived subgroup) of $B^\pm$

Consider in $k[G]$ the G-**stable subalgebras**

$$\mathcal{S}^{+} := \{f \in k[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U^{+}\},$$
$$\mathcal{S}^{-} := \{f \in k[G] \mid f(gu) = f(g) \text{ for all } g \in G, u \in U^{-}\},$$

and the natural **multiplication homomorphism** of k-algebras

$$\mu\colon \mathcal{S}^{+} \otimes_k \mathcal{S}^{-} \to k[G], \quad f_1 \otimes f_2 \mapsto f_1 f_2.$$

In 1992, for $k = \mathbb{C}$, the following conjectures have been put forward:

Conjectures (D. E. Flath and J. Towber)

(S) *The homomorphism* μ *is* **surjective**.

(K) *The ideal* $\ker \mu$ *in* $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ *is* **generated by** $(\ker \mu)^G$.

If Conjectures (S) and (K) are true, then the problem under consideration is **reduced** to the following two:

Problem (G) Find **generators of the** $(\ker \mu)^G$ canonically **determined by the group structure of** G.

Problem (P) Find **presentations of** $\mathcal{S}^{\pm}$ **by generators and relations** canonically **determined by the group structure of** G.

Below

"canonically determined by the group structure of G**"**
is abbreviated to **"canonically defined"**.

In 1992, D. E. Flath and J. Towber

- **proved Conjectures (S) and (K)**,
- **solved Problems (G), (P)**

for

$$\boxed{k = \mathbb{C}} \quad \text{and} \quad \boxed{G = \mathrm{SL}_n, \mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n}$$

by means of direct lengthy computations of some Laplace decompositions, minors, and algebraic identities between them.

We proved Conjectures (S) and (K) in full generality:

Theorem

Conjectures (S) and (K) **hold in full generality**, *i.e., for* **any field** $\boldsymbol{k}$ *and* **any connected reductive algebraic group** $\boldsymbol{G}$.

We also solved Problems (G) and (P).

Problem (G)

All our canonically defined generators of the k-algebra $(\ker \mu)^G$ are analogous to the element

$$\mu(x_1) \otimes \mu(x_4) - \mu(x_2) \otimes \mu(x_3) - 1$$

for $G = \mathrm{SL}_2$. Therefore **we call them**

$$\boxed{\mathrm{SL}_2\text{-type relations}}$$

of the sought-for canonical presentation of $k[G]$. Each of them is **inhomogeneous of degree** 2.

If G is semisimple, **there is a natural bijection between the set of these relations and**

the Hilbert basis $\mathscr{H}$ of the monoid of dominant weights of G

(i.e., the minimal system of generators of this monoid).

Precise description of the SL_2-type relations is given below.

Problem (P)

Special case:

- G is **semisimple**,
- monoid of dominant weights is **freely generared**, i.e.,

$$\text{cardinality of } \mathscr{H} = \operatorname{rank} G \qquad (*)$$

(for instance, **$(*)$ holds if G is simply connected**).

In this case, Problem (P) was solved by B. Kostant **assuming $\mathrm{char}\, k = 0$**.

Our solution holds **for any $\mathrm{char}\, k$**.

All our relations are **homogeneous of degree** 2. We call them

Plücker-type relations

of the sought-for canonical presentation of $k[G]$ because $\mathcal{S}^{\pm}$ for $G = \mathrm{SL}_n$ is the **coordinate algebra of the affine cone over the flag variety**, and for $\mathrm{char}\, k = 0$, the set of all these relations is the k-linear span of the classical Plücker-type relations determining this cone (they are obtained by Hodge).

The complete set of these relations is a canonically defined **union of finite-dimensional vector spaces indexed by the elements of** $\mathscr{H} \times \mathscr{H}$. Different spaces have zero intersection.

Precise description of the Plücker-type relations is given below.

General case

For arbitrary connected semisimple G, let $\tau\colon \widetilde{G} \to G$ be the universal covering. Then

- $\widetilde{G}$ is **simply connected semisimple**,
- $G = \widetilde{G}/\ker\tau$,
- $\ker\tau$ is a **finite central subgroup**.

Then the canonical presentation of $\mathcal{S}^{\pm}$ for $\widetilde{G}$ is given by the above, so Problem (P) for G is reduced to finding a presentation for the invariant algebra of finite abelian group $(\mathcal{S}^{\pm})^{\ker\tau}$.

We now turn to describing precise formulations of the solution to Problems (G) and (P).

Notation and convention

- $T := B^+ \cap B^-$ the maximal torus of G.
- $\mathrm{X}(T) := \mathrm{Hom}_{\mathrm{alg}}(T, \mathbb{G}_m)$ (additively written)
- t^λ the value of $\lambda \in \mathrm{X}(T)$ at $t \in T$.
- $\mathrm{X}(T)_+ \subset \mathrm{X}(T)$ the monoid of dominant weights of T determined by B^+.
- w_0 the longest element of the Weyl group of T.
- $\dot{w}_0$ a representative of w_0 in the normalizer of T.
- $\lambda^* := -w_0(\lambda) \in \mathrm{X}(T)_+$ for every $\lambda \in \mathrm{X}(T)_+$.

For every $\lambda \in \mathrm{X}(T)$, consider the following **weight spaces** of the T-actions on $\mathcal{S}^{\pm}$ by **right translations**:

$$\mathcal{S}^+(\lambda) := \{f \in \mathcal{S}^+ \mid f(gt) = t^{\lambda} f(g) \text{ for all } g \in G, t \in T\},$$
$$\mathcal{S}^-(\lambda) := \{f \in \mathcal{S}^- \mid f(gt) = t^{w_0(\lambda)} f(g) \text{ for all } g \in G, t \in T\}.$$

Properties:

- $\mathcal{S}^{\pm}(\lambda)$ is a **finite-dimensional G-submodule** of the G-module $\mathcal{S}^{\pm}$.
- $\mathcal{S}^{-}(\lambda)$ is the **right translation** of $\mathcal{S}^{+}(\lambda)$ by $\dot{w}_0$ (hence $\mathcal{S}^{+}(\lambda)$ and $\mathcal{S}^{-}(\lambda)$ are **isomorphic** G-modules).
- $\mathcal{S}^{\pm}(\lambda) \neq 0 \iff \lambda \in \mathrm{X}(T)_{+}$.
- $\mathrm{soc}_G \mathcal{S}^{\pm}(\lambda)$ is a **simple G-module with the highest weight $\boldsymbol{\lambda}^*$**.
- If $\mathrm{char}\, k = 0$, then $\mathcal{S}^{\pm}(\lambda) = \mathrm{soc}_G \mathcal{S}^{\pm}(\lambda)$. If $\mathrm{char}\, k > 0$, then, in general, this equality does **not** hold.

Properties:

- The submodules $\mathcal{S}^{\pm}(\lambda)$ form an $\mathrm{X}(T)$-**grading of** $\mathcal{S}^{\pm}$, i.e.,

$$\mathcal{S}^{\pm} = \bigoplus_{\lambda \in \mathrm{X}(T)_+} \mathcal{S}^{\pm}(\lambda), \quad \mathcal{S}^{\pm}(\lambda)\mathcal{S}^{+}(\mu) \subseteq \mathcal{S}^{\pm}(\lambda+\mu).$$

- $\mathcal{S}^-$ is the **right translation** of $\mathcal{S}^+$ by $\dot{w}_0$ (hence $\mathcal{S}^+$ and $\mathcal{S}^-$ are G-**isomorphic** k-**algebras**).

Theorem (S. Ramanan and A. Ramanathan, 1985)

For every $\lambda, \gamma \in \mathrm{X}(T)_+$,

the linear span of $\mathcal{S}^{\pm}(\lambda)\mathcal{S}^{\pm}(\gamma)$ over k is equal to $\mathcal{S}^{\pm}(\lambda + \gamma)$.

Comment

The difficulty lies in the case of $\mathrm{char}\, k > 0$, because for $\mathrm{char}\, k = 0$, this statement follows immediately from the **simplicity** of every G-module $\mathcal{S}^{\pm}(\lambda)$.

Since $\mathrm{X}(T)_+$ is a **finitely generated** monoid, the existence of the $\mathrm{X}(T)_+$-grading on $\mathcal{S}^{\pm}$ combined with the above theorem yield

Corollary

$\mathcal{S}^{\pm}$ *is a* **finitely generated *k*-algebra**.

This corollary yields the following **geometric reformulation of Conjecture (S).**

Conjecture (S): geometric reformulation

We consider the **action of G on itself by left translations**.

By the above Corollary, there are

- an **irreducible affine algebraic variety X endowed with an action of G**,
- a **G-equivariant dominant morphism**

$$\alpha\colon G \to X$$

such that

$$\alpha^*\colon k[X] \to \mathcal{S}^+ \text{ is a } G\text{-isomorphism of algebras.}$$

Conjecture (S): geometric reformulation

Consider in X two points

$$\boxed{x := \alpha(e) \quad \text{and} \quad y := \dot{w}_0 \cdot x}$$

and the G-equivariant morphism

$$\boxed{\iota\colon G \to X \times X, \quad g \mapsto g \cdot (x, y).}$$

Conjecture (S): geometric reformulation

Geometric reformulation of Conjecture (S):

Conjecture (S) is **equivalent** *to the property:*

morphism ι is a closed embedding.

Comments

- U^+ and U^- are the G-stabilizers of x and y. From this and $U^+ \cap U^- = \{e\}$ **injectivity of** ι is deduced.

- From $U^+ \cap U^- = \{e\}$ is deduced that $\ker(d\iota)_e = \{0\}$ which implies **separability of** ι.

- **Closedness of the image of** $\boldsymbol{\iota}$ $(= G$-orbit of $(x, y))$ follows from the **stronger property**

  $$\boxed{\boldsymbol{B^+\text{-orbit of } (x, y) \text{ is closed,}}}$$

  which is deduced from the following **general criterion** that generalizes Rosenlicht's classical theorem on closedness of orbits of unipotent group:

Connected solvable affine algebraic groups: criterion of orbit closedness

Theorem

Let S be a **connected solvable affine** *algebraic group acting on an* **affine** *algebraic variety Z. For a point $z \in Z$, consider*

- *the* **orbit morphism** $\tau : S \to Z$, $s \mapsto s \cdot z$;
- *the semigroup F of* **characters of S contained in** $\tau^*(k[Z])$.

Then the following are equivalent:

- *the orbit $S \cdot z$ is* **closed** *in Z;*
- *F is a* **group***;*
- *the* **convex cone generated by F in the $\mathbb{Q}$-vector space** $\mathrm{Hom}_{\mathrm{alg}}(S, \mathbb{G}_m) \otimes_{\mathbb{Z}} \mathbb{Q}$ **is the $\mathbb{Q}$-linear span of** F.

Geometric reformulation of Conjecture (K):

Conjecture (K) is **equivalent** *to the property:*

for the comorphism $\iota^*\colon k[X \times X] \to k[G]$,

the ideal $\ker \iota^*$ is generated by its G-invariant elements.

We show that in fact the following **more general statement** holds:

Theorem

Let Z be an **affine algebraic variety** *endowed with an action of a* **reductive** *algebraic group H. Let $z \in Z$ be a point such that the orbit morphism*

$$\varphi\colon H \to Z,\quad h \mapsto h \cdot z$$

is a **closed embedding***. Then for the comorphism*

$$\varphi^*\colon k[Z] \to k[H],$$

the ideal $\ker \varphi^*$ in $k[Z]$ is generated by its H-invariant elements.

The following special elements $s_\lambda \in k[G \times G]$ are used in our description of the canonically defined generators of the k-algebra $(\ker \mu)^G$:

Theorem

For every $\lambda \in \mathrm{X}(T)_+$, there exists a **unique** *element*

$$\boxed{\begin{gathered} s_\lambda \in (\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*))^G \subseteq k[G \times G] \\ \textit{such that } s_\lambda(e, e) = 1. \end{gathered}}$$

Solving Problem (G): elements s_λ

These s_λ's can be described explicitly:

Theorem

- *For every $\lambda \in \mathrm{X}(T)_+$, there is a unique up to k-proportionality nondegenerate G-invariant bilinear pairing*

$$\mathcal{S}^+(\lambda) \times \mathcal{S}^-(\lambda^*) \to k.$$

- *If $\{f_1, \ldots, f_d\}$ and $\{h_1, \ldots, h_d\}$ are the bases of $\mathcal{S}^+(\lambda)$ and $\mathcal{S}^-(\lambda^*)$ dual with respect to such a pairing, then $\varepsilon := \sum_{i=1}^d f_i(e)h_i(e) \neq 0$ and*

$$\boxed{s_\lambda = \varepsilon^{-1}\Big(\sum_{i=1}^d f_i \otimes h_i\Big).}$$

Solving Problem (G): elements s_λ

For $\operatorname{char} k = 0$, there is a characterization of s_λ as **eigenvectors of some linear operators**. Namely, every $\mathcal{S}^{\pm}(\lambda)$ is a $\mathscr{U}$-module, where $\mathscr{U}$ is the universal enveloping algebra of $\operatorname{Lie} G$. Let $\{x_1, \ldots, x_n\}$ and $\{x_1^*, \ldots, x_n^*\}$ be the bases of $\operatorname{Lie} G$ dual with respect to the Killing form Φ. Consider on $\mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$ the linear operator

$$\Delta := \sum_{i=1}^{n} (x_i \otimes x_i^* + x_i^* \otimes x_i)$$

and dentify $\operatorname{Lie} T$ with its dual by means of Φ.

Theorem

The following properties of $t \in \mathcal{S}^+(\lambda) \otimes_k \mathcal{S}^-(\lambda^*)$ are equivalent:

- $t = s_\lambda$;
- $t(e, e) = 1$ *and* $\Delta(t) = -\big(\Phi(\lambda + \sigma, \lambda) + \Phi(\lambda^* + \sigma, \lambda^*)\big)t$ *where σ is the sum of all positive roots.*

The following theorem **solves Problem (G)**:

Theorem (Canonical generators of the algebra $(\ker \mu)^G$ and the ideal $\ker \mu$)

Let $\lambda_1, \ldots, \lambda_m$ be a system of generators of the monoid $\mathrm{X}(T)_+$. Then

$$\boxed{s_{\lambda_1} - 1, \ldots, s_{\lambda_m} - 1}$$

is a **system of generators of the ideal $\ker \mu$ in $\mathcal{S}^+ \otimes_k \mathcal{S}^-$ and of the k-algebra $(\ker \mu)^G$**.

Remark

If G is **semisimple**, there is the **smallest** system of generators of $\mathrm{X}(T)_+$, namely, its **Hilbert basis**

$$\boxed{\mathscr{H} := \mathrm{X}(T)_+ \setminus 2\mathrm{X}(T)_+.}$$

This yields a **canonical** (uniquelly defined) system of generators of $\ker\mu$ and $(\ker\mu)^G$.

Below G is **semisimple**

Notation

- $\operatorname{Sym} \mathcal{S}^{\pm}(\lambda_i)$ the symmetric algebra of $\mathcal{S}^{\pm}(\lambda_i)$,
- $\operatorname{Sym}^m \mathcal{S}^{\pm}(\lambda_i)$ the mth symmetric power of $\mathcal{S}^{\pm}(\lambda_i)$,
- $\lambda_1, \ldots, \lambda_d$ all elements of the Hilbert basis $\mathscr{H}$ of $\mathrm{X}(T)_+$

Solving Problem (P): presentation of $\mathcal{S}^{\pm}$

The naturally $\mathbb{N}^d$-graded free commutative k-algebra

$$\mathcal{F}^{\pm} := \operatorname{Sym}\mathcal{S}^{\pm}(\lambda_1) \otimes_k \cdots \otimes_k \operatorname{Sym}\mathcal{S}^{\pm}(\lambda_d)$$

may be viewed as the **algebra of regular (polynomial) functions** $k[L^{\pm}]$ **on the vector space**

$$L^{\pm} := \mathcal{S}^{\pm}(\lambda_1)^* \oplus \cdots \oplus \mathcal{S}^{\pm}(\lambda_d)^*.$$

Let e_i be the ith unit vector of $\mathbb{N}^d$ and let $\mathcal{F}^{\pm}_{p,q}$ be the **homogeneous component of $\mathcal{F}^{\pm}$ of degree** $e_p + e_q$. Then

$$\mathcal{F}^{\pm}_{p,q} = \begin{cases} \mathcal{S}^{\pm}(\lambda_p) \otimes_k \mathcal{S}^{\pm}(\lambda_q) & \text{if } p \neq q, \\ \operatorname{Sym}^2 \mathcal{S}^{\pm}(\lambda_p) & \text{if } p = q. \end{cases}$$

Replacing tensor multiplication by usual multiplication of functions yields the natural **G-equivariant homomorphism of k-algebras**

$$\boxed{\phi^{\pm}\colon \mathcal{F}^{\pm} \to \mathcal{S}^{\pm}}.$$

Its restriction to $\mathcal{F}^{\pm}_{p,q}$ is a **homomorphism of G-modules**

$$\boxed{\psi^{\pm}_{p,q}\colon \mathcal{F}^{\pm}_{p,q} \to \mathcal{S}^{\pm}(\lambda_p + \lambda_q).}$$

By the above theorem of S. Ramanan, A. Ramanathan, $\phi^{\pm}$ and $\psi^{\pm}_{p,q}$ are **surjective**.

Since $\mathcal{F}^{\pm}$ is a **polynomial algebra** and $\phi^{\pm}$ is **surjective**,

> **finding a presentation of $\mathcal{S}^{\pm}$ by generators and relations means describing the ideal $\ker \phi^{\pm}$.**

Theorem (G. R. Kempf and A. Ramanathan, 1987)

Let G be a **connected semisimple** *group such that the Hilbert basis $\mathscr{H}$* **freely generates** *the monoid $\mathrm{X}(T)_+$ (i.e., $d = \dim T$). Then*

- *The ideal $\ker \phi^{\pm}$ of the $\mathbb{N}^d$-graded k-algebra $\mathcal{F}^{\pm}$ is* **homogeneous**.
- *The ideal $\ker \phi^{\pm}$ is* **generated by the union of all its homogeneous components of the total degree** 2.
- *These homogeneous components are* **precisely the subspaces**

$$\boxed{\ker \psi^{\pm}_{p,q}, \quad 1 \leqslant p \leqslant q \leqslant d.}$$

Comment 1

This canonical presentation of $\mathcal{S}^{\pm}$ is **redundant**: in the specified set of its relations **every space ker $\psi^{\pm}_{p,q}$ may be replaced by its basis**. A **canonical finding of such a basis** is given within the framework of **standard monomial theory** (the Littelmann path model).

Comment 2

If $\operatorname{char} k = 0$, then

- $\mathcal{S}^{\pm}(\lambda_p + \lambda_q)$ is the **Cartan component** of the G-module $\mathcal{F}^{\pm}_{p,q}$,
- $\psi^{\pm}_{p,q}$ is the **projection to the Cartan component parallel to the unique G-stable direct complement of this component**.
- $\ker \psi^{\pm}_{p,q}$ admits the following description. Let $\{x_1, \ldots, x_n\}$ and $\{x_1^*, \ldots, x_n^*\}$ be the dual bases of $\operatorname{Lie} G$ with respect to the Killing form Φ. Then $\ker \psi^{+}_{p,q}$ is the **image of the linear operator**

$$\Big(\sum_{s=1}^{n}(x_s \otimes x_s^* + x_s^* \otimes x_s)\Big) - 2\Phi(\lambda_p^*, \lambda_q^*)\mathrm{id}$$

on the vector space $\mathcal{F}^{\pm}_{p,q}$.

Comment 3

The classification of groups for which the monoid of dominant weights is freely generated by its Hilbert basis is known:

Theorem (R. Steinberg, 1975)

The following properties are **equivalent**:

- *The Hilbert basis* $\mathscr{H}$ **freely generates** $\mathrm{X}(T)_+$.
- $G = G_1 \times \cdots \times G_s$ *where every* G_i *is either a* **simply connected simple** *algebraic group or* **isomorphic to** SO_{n_i} **for an odd** n_i.

Corollary

If G *is* **simply connected**, *then* $\mathscr{H}$ **freely generates** $\mathrm{X}(T)_+$.

Example

To illustrate the above, now we explicitly describe the canonical presentation of $k[G]$ by generators and relations for

$$\boxed{G = \mathrm{SL}_n,\ n \geqslant 2, \text{ and } \operatorname{char} k = 0}$$

Example: $G = \mathrm{SL}_n$, $n \geqslant 2$, and $\mathrm{char}\, k = 0$

Take

- T the maximal torus of diagonal matrices in G;
- B^+ (resp., B^-) the Borel subgroup of lower (resp., upper) triangular matrices in G

Then

$$\mathscr{H} = \{\varpi_1, \ldots, \varpi_{n-1}\}, \quad \text{where}$$
$$\varpi_d \colon T \to k, \quad \mathrm{diag}(a_1, \ldots, a_n) \mapsto a_{n-d+1} \cdots a_n.$$

Example: $G = \mathrm{SL}_n$, $n \geqslant 2$, and $\operatorname{char} k = 0$

Every pair of integers $i_1, i_2 \in [1, n]$ determines the element $x_{i_1,i_2} \in k[G]$ given by

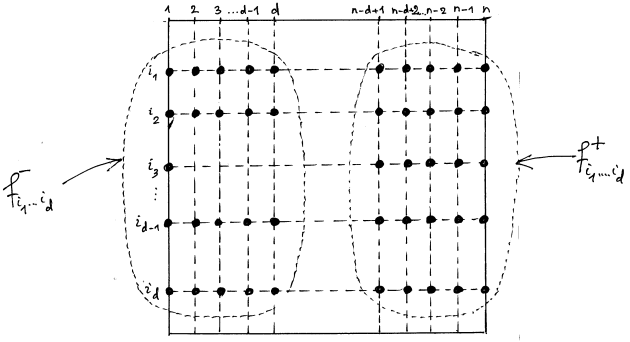
$$x_{i_1,i_2}\colon G \to k, \quad \begin{pmatrix} a_{1,1} & \ldots & a_{1,n} \\ \ldots & \ldots & \ldots \\ a_{n,1} & \ldots & a_{n,n} \end{pmatrix} \mapsto a_{i_1,i_2}.$$

The k-algebra $k[G]$ is generated by n^2 elements x_{i_1,i_2} .

Example: $G = \mathrm{SL}_n$, $n \geqslant 2$, and $\mathrm{char}\, k = 0$

Every integer $d \in [1, n-1]$ and sequence of integers $i_1 < \ldots < i_d$ from $[1, n]$ determine the following elements of $k[G]$:

$$f^{-}_{i_1,\ldots,i_d} := \det \begin{pmatrix} x_{i_1,1} & \ldots & x_{i_1,d} \\ \ldots & \ldots & \ldots \\ x_{i_d,1} & \ldots & x_{i_d,d} \end{pmatrix}, \quad f^{+}_{i_1,\ldots,i_d} := \det \begin{pmatrix} x_{i_1,n-d+1} & \ldots & x_{i_1,n} \\ \ldots & \ldots & \ldots \\ x_{i_d,n-d+1} & \ldots & x_{i_d,n} \end{pmatrix}.$$



Example: $G = \mathrm{SL}_n$, $n \geqslant 2$, and $\operatorname{char} k = 0$

For every fixed d,

$$\boxed{\text{all } f^{+}_{i_1,\ldots,i_d} \text{ form a basis of } \mathcal{S}^{+}(\varpi_d),}$$

$$\boxed{\text{all } f^{-}_{i_1,\ldots,i_d} \text{ form a basis of } \mathcal{S}^{-}(\varpi_d).}$$

This yields:

Generators

Let d runs over all integers from $[1, n-1]$ and $i_1, \ldots, i_d$ over all increasing sequences of integers from $[1, n]$. Then

- *the set $\{f^{+}_{i_1,\ldots,i_d}\}$ generates $\mathcal{S}^{+}$,*
- *the set $\{f^{-}_{i_1,\ldots,i_d}\}$ generates $\mathcal{S}^{-}$,*
- *the set $\{f^{+}_{i_1,\ldots,i_d}, f^{-}_{i_1,\ldots,i_d}\}$ generates $k[G]$.*

Geometric meaning

$\mathcal{S}^{\pm}$ is the coordinate algebra of the affine cone over the variety of complete flags in k^n embedded in the projective space by means of the composition of the product of Plücker embeddings of the Grassmannians and the subsequent Segre embedding

Relations

Let $\mathcal{F}$ be the polynomial k-algebra in variables $x^{+}_{i_1,\ldots,i_d}$ and $x^{-}_{i_1,\ldots,i_d}$, where d runs over all integers from $[1, n-1]$ and $i_1, \ldots, i_d$ over all increasing sequences of integers from $[1, n]$. Then the above yields:

The homomorphism

$$\phi\colon \mathcal{F} \to k[G], \quad x^{+}_{i_1,\ldots,i_d} \mapsto f^{+}_{i_1,\ldots,i_d}, \;\; x^{-}_{i_1,\ldots,i_d} \mapsto f^{-}_{i_1,\ldots,i_d},$$

is **surjective***.*

Relations are the **generators of the ideal** $\ker \phi$.

Plücker-type relations

These are the basic relations involving **only** $x^+_{\dots}$ or **only** $x^-_{\dots}$.

Notation

Let $i_1, \dots, i_d$ be a sequence of integers from $[1, n]$ and let $j_1, \dots, j_d$ be the nondecreasing sequence obtained from it by a permutation σ. We put

$$x^{\pm}_{i_1,\dots,i_d} = \begin{cases} \mathrm{sgn}(\sigma)\, x^{\pm}_{j_1,\dots,j_d} & \text{if } i_p \neq i_q \text{ for all } p \neq q, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem (W. V. D. Hodge, 1942, 1943)

The Plücker-type relations are

$$\sum_{l=1}^{q+1} (-1)^l x^{\pm}_{i_1,\ldots,i_{p-1},j_l} x^{\pm}_{j_1,\ldots,\widehat{j_l},\ldots,j_{q+1}},$$

where p and q run over all integers from $[1, n-1]$, $p \leqslant q$, and $i_1, \ldots, i_{p-1}$ and $j_1, \ldots, j_{q+1}$ run over all increasing sequences of integers from $[1, n]$.

SL_2-type relations

These are the basic relations involving **both** $x^+_{\dots}$ and $x^-_{\dots}$

Notation

Let $\boldsymbol{i}$ be an increasing sequence $i_1, \dots, i_d$ of integers from $[1, n]$. We denote by $\boldsymbol{i}^*$ the unique increasing sequence $j_1, \dots, j_{n-d}$ of integers from $[1, n]$ whose intersection with $i_1, \dots, i_d$ is **empty**. Let $\mathrm{sgn}(\boldsymbol{i}, \boldsymbol{i}^*)$ be the sign of the permutation $(i_1, \dots, i_d, j_1, \dots, j_{n-d})$.

Theorem (D. E. Flath, J. Towber, 1992)

The basic SL_2*-type relations are*

$$s_{\varpi_1} - 1, \ldots, s_{\varpi_{n-1}} - 1,$$

where

$$s_{\varpi_d} = \sum_{\boldsymbol{i}} \operatorname{sgn}(\boldsymbol{i}, \boldsymbol{i}^*) x^-_{\boldsymbol{i}} x^+_{\boldsymbol{i}^*},$$

and $\boldsymbol{i}$ *in the sum runs over all increasing sequences* $i_1, \ldots, i_{n-d}$ *of integers from* $[1, n]$.