Semi-global fields and Hasse principle

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June 19, 2025

Number-theoretic aspects of linear algebraic groups and algebraic varieties : results and prospects dedicated to 85-th anniversary of academician V.P. Platonov

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This is a report on joint work with R. Parimala (Emory). It is on arXiv :2406.17355 and 2301.07572.

Hasse-Minskowski's theorem

- Let k be a number field,
- Two (finite dimensional, regular) quadratic forms q_1 and q_2 are isometric if and only if there are isometric over each completion k_v of k.
- A quadratic form q over k is isotropic if and only if it is isometric over each k_v.
- The first statement is about $O(q_1)$-torsors when the second statement is about a twisted flag variety.

Hasse-Minskowski's theorem II

It rephrases as follows.

- The map $H^1(k, \mathrm{O}(q_1)) \to \prod_v H^1(k_v, \mathrm{O}(q_1))$ has trivial kernel (and is injective).
- Let $X = \{q = 0\}$ be the projective quadric associated to q. Then X satisfies the Hasse principle, that is,

$$X(k) \neq \emptyset \iff \prod_v X(k_v) \neq \emptyset.$$

- Note that X is a twisted flag variety under the special orthogonal group $\mathrm{SO}(q)$, i.e. $X_{k_s} \cong \mathrm{SO}(q)_{k_s}/P$ where P is a k_s-parabolic of the semisimple group $\mathrm{SO}(q)$.
- What happens for other groups ?

Hasse principle for semisimple simply connected groups

Let G be a semisimple simply connected k-group, e.g. a form of SL_n, Spin_n, etc...

- (1) The map $H^1(k,G) \to \prod_v H^1(k_v,G)$ is injective.
- (2) Let X be a projective G-homogeneous variety. Then X satisfies the Hasse principle, that is,

$$X(k) \neq \emptyset \iff \prod_v X(k_v) \neq \emptyset.$$

- The proof of (1) is case by case, Kneser for the classical groups, Harder for the exceptional groups, Chernousov for the type E_8.
- The proof of (2) is uniform from (1) (Harder).
- What about other fields ?

Semi-global fields

First there are similar results over $\mathbb{F}_q(t)$ (Harder). There are also results for $\mathbb{C}(x, y)$, $\mathbb{C}((x, y))$ and other fields. We are interested in semi-global fields.

- Let T be a complete DVR of fraction field K and of residue field k.
- Let X/K be a smooth projective geometrically connected curve, the function field $F = K(X)$ is a *semi-global field*.
- There are different sets of valuations of F to consider : discrete valuations arising from closed points of X, discrete valuations, rank one valuations, arbitrary valuations.
- There is a nice analysis in a paper of Harbater, Hartmann, Karemaker and Pop.

Divisorial valuations

$F = K(X)$ is a semi-global field.

- Given an a regular projective T-model $\mathfrak{X}$ of X, each point $x \in \mathfrak{X}^{(1)}$ defines a divisorial valuation v_x (discrete).
- The set of valuations we consider is then that of divisorial valuations for all such models. It is then quite large.
- Example $F = \mathbb{Q}_p(t)$, $X = \mathbb{P}^1_{\mathbb{Q}_p}$ and the integral model $\mathbb{P}^1_{\mathbb{Z}_p}$. We can take the point $0 \in \mathbb{P}^1(K)$ leading then to the completion $\mathbb{Q}_p((t))$ which is a two-local field.
- We can take the divisor $\mathbb{P}^1_{\mathbb{F}_p}$ of $\mathbb{P}^1_{\mathbb{Z}_p}$. The associate valuation is the Gauss valuation on $\mathbb{Q}_p(t)$; its completion has residue field $\mathbb{F}_p(t)$.

The conjectures

We assume here that K is a non-archimedean local field. We state conjectures of Colliot-Thélène, Parimala and Suresh (CT-P-S, 2012). Once again $F = K(X)$.

- **Conjecture 1** : For G/F semisimple simply connected, the map $H^1(F, G) \to \prod_{v \in \Omega} H^1(F_v, G)$ is injective.

- **Conjecture 2** : Local-global principle for twisted flag varieties : let X/F be a projective G-homogeneous space, then the local-global principle holds for X, that is,

$$X(F) \neq \emptyset \iff \prod_v X(F_v) \neq \emptyset.$$

Known cases

- Conjecture 1 is known for classical groups (Parimala-Suresh), constant groups (CT-P-S).
- Conjecture 2 is known for (higher) Severi-Brauer varieties and projective quadrics (CT-P-S, Reedy-Suresh).
- Corollary : $u(F) = 8$.
- This is the u-invariant of Kaplansky, it means that 8 is the supremum of dimensions of F-anisotropic quadratic forms.

u-invariant

- **Corollary :** $u(F) = 8$.
- This follows from the fact that $u(F_v) = 8$ for each divisorial valuation of F.
- This result was known before : Parimala-Suresh and independently Leep (2008) and then the patching approach of Harbater-Krashen-Hartmann (H-H-K, 2009).
- The advantage of the above (and H-H-K) method is to work for K arbitrary, namely $u(F) = 2\,u(K)$.

Main result

- **Theorem (-, Parimala, 2023)** For K arbitrary, the local-global principle holds for projective homogeneous spaces over $F = K(X)$ with respect to divisorial valuations.
- Some small characteristics are excluded and the proof is uniform.
- On the first conjecture (K local field), there are several evidences but no general result yet.

Why projective homogeneous varieties are simpler than torsors ?

We will examine first the case of “patching fields”.

- We are given a diagram of fields

$$
\begin{array}{ccccc}
 & & F_0 & & \\
 & \diagup & & \diagdown & \\
F_1 & & & & F_2 \\
 & \diagdown & & \diagup & \\
 & & F & &
\end{array}
$$

  where $F = F_1 \cap F_2$ and F_0 is generated by F_1 and F_2.
- We say this is a patching diagram if $\mathrm{GL}_n(F_0) = \mathrm{GL}_n(F_1)\,\mathrm{GL}_n(F_2)$ for each $n \geq 1$.
- Equivalently this is to say that the category $Vect(F)$ is equivalent to the category $Vect(F_1) \times_{Vect(F_0)} Vect(F_2)$ (H-H).
- It follows that the category of affine F-varieties (resp. algebraic groups, etc...) is equivalent to the fiber product of the categories of affine F_1-varieties and affine F_2-varieties over that of affine F_0-varieties.

Patching

We are given a diagram of patching fields.

$$\begin{array}{ccccc} & & F_0 & & \\ & \diagup & & \diagdown & \\ F_1 & & & & F_2 \\ & \diagdown & & \diagup & \\ & & F & & \end{array}$$

- The category $Vect(F)$ is equivalent to the category $Vect(F_1) \times_{Vect(F_0)} Vect(F_2)$.
- It follows that the category of affine F-varieties (resp. algebraic groups, etc...) is equivalent to the fiber product of the categories of affine F_1-varieties and affine F_2-varieties over that of affine F_0-varieties.
- Let G be an affine F-group. We claim that there is a bijection

$$G(F_1)\backslash G(F_0)/G(F_2) \xrightarrow{\sim} \ker\Big(H^1(F,G) \to H^1(F_1,G)\times H^1(F_2,G)\Big).$$

Patching II

Let G be an affine F-group.

- There is a bijection

$$G(F_1)\backslash G(F_0)/G(F_2) \xrightarrow{\sim} \ker\Big(H^1(F,G) \to H^1(F_1,G)\times H^1(F_2,G)\Big).$$

- Given $g \in G(F_0)$ we can patch the trivial G-torsors $E_1 = G_{F_1}$ and $E_2 = G_{F_2}$ with g. This provides a G–torsor over F.
- If E is G–torsor over F equipped with trivializations $\phi_1 : G_{F_1} \xrightarrow{\sim} E_{F_1}$ and $\phi_2 : G_{F_2} \xrightarrow{\sim} E_{F_2}$, then the endomorphism $\phi_1^{-1} \circ \phi_2$ on G_{F_0} is the left translation by an element $g \in G(F_0)$.
- Summarizing the study of the triviality of the kernel rephrases as the study of the decomposition $G(F_0) = G(F_1)\, G(F_2)$.

Patching III

Now we assume that G is reductive and we are given a projective G-variety Z.

- Assume that $Z(F_1) \neq \emptyset$ and $Z(F_2) \neq \emptyset$.
- We pick z_1, z_2 so that $z_1 = g \,.\, z_2$ for $g \in G(F_0)$.
- If $G(F_0) = G(F_1)\, G(F_2)$ we write $g = g_1\, g_2$, so that $g_1^{-1} \,.\, z_1 = g_2 \,.\, z_2 \in Z(F_1) \cap Z(F_2) = Z(F)$. Thus $Z(F) \neq \emptyset$.
- We consider now the F_2–parabolic subgroup $P_2 = \mathrm{Stab}_{G_{F_2}}(z_2)$. If $G(F_0) = G(F_1)\, G(F_2)\, P_2(F_0)$, I claim that it still works.
- Indeed write $g = g_1\, g_2\, p_2$ so that $g_1^{-1} \,.\, z_1 = g_2 \,.\, p_2 \,.\, z_2 = g_2 \,.\, z_2 \in Z(F)$.

Kneser-Tits subgroups

As before let P_2 be a F_2-parabolic subgroup of G. We are interested in decomposition $G(F_0) = G(F_1)\,G(F_2)\,P_2(F_0)$.

- The essential case is when G is semisimple simply connected case, absolutely almost F-simple, F_1-isotropic and F_2-isotropic.
- We deal with the Kneser-Tits subgroup $G(F_0)^+$ of $G(F_0)$. We know that $G(F_0) = G(F_0)^+\,P_2(F_0)$ so that is enough to show that $G(F_0)^+ \subset G(F_1)\,G(F_2)$.
- This is much easier since $G(F_0)^+$ is well-understood in certain cases and also have nice generators.

Back to the H-H-K method

We deal with the special case of patching diagram

$$\begin{array}{ccccc} & & F_0 & & \\ & / & & \backslash & \\ F_1 & & & & F_2 \\ & \backslash & & / & \\ & & F & & \end{array}$$

where F_0 is complete for a discrete valuation containing K as valued subfield and sharing the same unformizing parameter $t \in T$. We denote by $\widehat{R}_0$ the ring of integers of F_0.

- We assume furthermore that F_1 is dense in F_0 and that t-adically complete T-submodules $V \subset F_1 \cap \widehat{R}_0$ and
- $W \subset F_2 \cap \widehat{R}_0$ satisfying the following conditions :

$$V + W = \widehat{R}_0; \tag{1}$$

$$V \cap t\widehat{R}_0 = tV \quad \text{and} \quad W \cap t\widehat{R}_0 = tW. \tag{2}$$

- **Theorem** (-, Parimala) $G(F_0)^+ = G(F_1)^+ \, G(F_2)^+$.
- This was known when G is F–rational (H-H-K).

Back to the H-H-K method, II

- **Theorem** (-, Parimala) $G(F_0)^+ = G(F_1)^+ \, G(F_2)^+$.
- The case G isotropic is simpler.
- **Corollary**. Let Z be a projective G-homogenous variety such that $Z(F_1) \neq \emptyset$ and $Z(F_2) \neq \emptyset$. Then $Z(F) \neq \emptyset$.
- This is the first step for the proof of the main Theorem.
- It concerns the case $F = K(x)$, $X = \mathbb{P}^1_K$, $\mathfrak{X} = \mathbb{P}^1_T$. We will associate fields F_1, F_2 and F_0 to $P = 0_k$ and $U = \mathbb{P}^1_k \setminus \{P\} = \mathrm{Spec}(k[x^{-1}])$.

Patching again

- $F = K(x)$,$X = \mathbb{P}^1_K$, $\mathfrak{X} = \mathbb{P}^1_T$. We will associate fields F_1, F_2 and F_0 to $P = 0_k$ and $U = \mathbb{P}^1_k \setminus \{P\} = \mathrm{Spec}(k[x^{-1}])$.
- For each $Q \in \mathfrak{X}$, we put $R_Q = \mathcal{O}_{\mathfrak{X},Q}$ and denote by $\widehat{R}_Q$ its t-adic completion. We put $F_1 = \mathrm{Frac}(\widehat{R}_P)$.
- We put $R_U = \bigcap_{Q \in U} R_Q \subset F$. We denote by $\widehat{R}_U$ the t–adic completion of R_U and put $F_2 = F_U = \mathrm{Frac}(\widehat{R}_U)$.
- We consider the prime ideal $t\widehat{R}_P \subset \widehat{R}_P$ and denote by $\widehat{R}_0$ the t-adic completion of $(\widehat{R}_P)_{t\widehat{R}_P}$ and by F_0 its fraction field.
- The claim is that (F, F_1, F_2, F_0) is a patching diagram which satisfies the additional assumptions [H-H].

Patching again

- The claim is that (F, F_1, F_2, F_0) is a patching diagram which satisfies the additional assumptions [H-H].
- In the equicharacteristic case, we have

$$F_1 = k((t,x)), \quad F_2 = k(x)((t)), \quad F_0 = k((x))((t)).$$

- We have then a local-global principle for G-homogeneous space X over F with respect to the overfields F_1 and F_2 which are much simpler.
- The field F_2 is of the shape F_v for a divisorial valuation v. This is not the case of F_1.

Patching again

- In the equicharacteristic case, we have

$$F_1 = k((t, x)), \quad F_2 = k(x)((t)), \quad F_0 = k((x))((t)).$$

- We have then a local-global principle for the patching data attached to (P, U).
- By Weil restriction trick, we have a similar result for any curve X and any regular model.
- In other words, the above case is the essential one.
- It remains to explain how we can connect the field F_1 and with divisorial valuations of F.

Towards tame loop torsors

- For simplicity I will explain how it works only for the field $k((t,x))$ which is the field of fractions of the regular local ring $R = k[[t,x]]$.
- We blow up $B = \mathrm{Spec}(k[[t,x]])$ at its closed point and the exceptional divisor gives rise to a canonical valuation $v : k((t,x))^\times \to \mathbb{Z}$. We have $v(t) = v(x) = 1$ and the residue field is $k(u)$ where u is the image of $\frac{t}{x}$.
- We have $F = k((t))(x) \subset F_1 = k((t,x))$ and v induces a divisorial valuation on F. Since F is dense in F_1, we have $F_v = F_{1,v}$.
- We are given now a projective G–variety Z over F having rational points everywhere locally for completions with respect to divisorial valuations.
- We have $Z(F_2) \neq \emptyset$ and $Z(F_v) \neq \emptyset$. From $Z(F_{1,v}) \neq \emptyset$, we would like to show that $Z(F_1) \neq \emptyset$ in order to apply the former corollary.
- Here start new technicalities.

Tame loop torsors

- A first step is to reduce to the case when G arises from a reductive scheme $\mathfrak{G}$ over $k[[t,x]]\left[t^{-1},x^{-1}\right]$. This is not enough ; we need it to arise from a so-called *tame loop object*.
- For example, $\mathfrak{G} = \mathrm{SO}(q)$ is a tame loop reductive group scheme if and only if the $k[[t,x]]\left[t^{-1},x^{-1}\right]$-quadratic form is diagonalizable.
- In the case of tame loop objects, there is a fixed point theorem. If $\mathfrak{Z}$ is a scheme of parabolic subgroups for $\mathfrak{G}$, $\mathfrak{Z}(F_{1,v}) \neq \emptyset$ is equivalent to $\mathfrak{Z}\Big(k[[t,x^{-1}]]\left[t^{-1},x\right]\Big) \neq \emptyset$. A fortiori this is equivalent with $\mathfrak{Z}(F_1) \neq \emptyset$.
- This loop theory is inspired of our work with Chernousov and Pianzola for (loop) torsors on Laurent polynomials. It works as well in the arithmetic setting, e.g. $\mathbb{Z}_p[[t]]\left[\frac{1}{p},\frac{1}{t}\right]$.

- Thanks for attention.
- Спасибо !