## Semi-global fields and Hasse principle

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#### Hasse-Minskowski's theorem

- Let k be a number field,
- Two (finite dimensional, regular) quadratic forms  $q_1$  and  $q_2$  are isometric if and only if there are isometric over each completion  $k_v$  of k.
- A quadratic form q over k is isotropic if and only if it is isometric over each  $k_v$ .
- ▶ The first statement is about  $O(q_1)$ -torsors when the second statement is about a twisted flag variety.

#### Hasse-Minskowski's theorem II

It rephrases as follows.

- The map  $H^1(k, \mathcal{O}(q_1)) \to \prod_{\nu} H^1(k_{\nu}, \mathcal{O}(q_1))$  has trivial kernel (and is injective).
- Let  $X = \{q = 0\}$  be the projective quadric associated to q. Then X satisfies the Hasse principle, that is,

$$X(k) \neq \emptyset \iff \prod_{v} X(k_v) \neq \emptyset.$$

- Note that X is a twisted flag variety under the special orthogonal group SO(q), i.e.  $X_{k_s} \cong SO(q)_{k_s}/P$  where P is a  $k_s$ -parabolic of the semisimple group SO(q).
- What happens for other groups?

## Hasse principle for semisimple simply connected groups

Let G be a semisimple simply connected k-group, e.g. a form of  $SL_n$ ,  $Spin_n$ , etc...

- ▶ (1) The map  $H^1(k,G) \to \prod_{\nu} H^1(k_{\nu},G)$  is injective.
- $\triangleright$  (2) Let X be a projective G-homogeneous variety. Then X satisfies the Hasse principle, that is,

$$X(k) \neq \emptyset \iff \prod_{v} X(k_v) \neq \emptyset.$$

- ▶ The proof of (1) is case by case, Kneser for the classical groups, Harder for the exceptional groups, Chernousov for the type  $E_8$ .
- ▶ The proof of (2) is uniform from (1) (Harder).
- What about other fields?

## Semi-global fields

First there are similar results over  $\mathbb{F}_q(t)$  (Harder). There are also results for  $\mathbb{C}(x,y)$ ,  $\mathbb{C}((x,y))$  and other fields. We are interested in semi-global fields.

- ► Let *T* be a complete DVR of fraction field *K* and of residue field *k*.
- Let X/K be a smooth projective geometrically connected curve, the function field F = K(X) is a semi-global field.
- ► There are different sets of valuations of *F* to consider : discrete valuations arising from closed points of *X*, discrete valuations, rank one valuations, arbitrary valuations.
- There is a nice analysis in a paper of Harbater, Hartmann, Karemaker and Pop.

#### Divisorial valuations

F = K(X) is a semi-global field.

- ▶ Given an a regular projective T-model  $\mathfrak{X}$  of X, each point  $x \in \mathfrak{X}^{(1)}$  defines a divisorial valuation  $v_x$  (discrete).
- ► The set of valuations we consider is then that of divisorial valuations for all such models. It is then quite large.
- Example  $F = \mathbb{Q}_p(t)$ ,  $X = \mathbb{P}^1_{\mathbb{Q}_p}$  and the integral model  $\mathbb{P}^1_{\mathbb{Z}_p}$ . We can take the point  $0 \in \mathbb{P}^1(K)$  leading then to the completion  $\mathbb{Q}_p((t))$  which is a two-local field.
- We can take the divisor  $\mathbb{P}^1_{\mathbb{F}_p}$  of  $\mathbb{P}^1_{\mathbb{Z}_p}$ . The associate valuation is the Gauss valuation on  $\mathbb{Q}_p(t)$ ; its completion has residue field  $\mathbb{F}_p(t)$ .

### The conjectures

We assume here that K is a non-archimedean local field. We state conjectures of Colliot-Thélène, Parimala and Suresh (CT-P-S, 2012). Once again F = K(X).

- **Conjecture 1**: For G/F semisimple simply connected, the map  $H^1(F,G) \to \prod_{v \in \Omega} H^1(F_v,G)$  is injective.
- **Conjecture 2**: Local-global principle for twisted flag varieties: let X/F be a projective G-homogeneous space, then the local-global principle holds for X, that is,

$$X(F) \neq \emptyset \iff \prod_{\nu} X(F_{\nu}) \neq \emptyset.$$

#### Known cases

- Conjecture 1 is known for classical groups (Parimala-Suresh), constant groups (CT-P-S).
- Conjecture 2 is known for (higher) Severi-Brauer varieties and projective quadrics (CT-P-S, Reedy-Suresh).
- ightharpoonup Corollary : u(F) = 8.
- This is the u-invariant of Kaplansky, it means that 8 is the supremum of dimensions of F-anisotropic quadratic forms.

#### *u*-invariant

- ► Corollary : u(F) = 8.
- This follows from the fact that  $u(F_v) = 8$  for each divisorial valuation of F.
- ► This result was known before: Parimala-Suresh and independently Leep (2008) and then the patching approach of Harbater-Krashen-Hartmann (H-H-K, 2009).
- The advantage of the above (and H-H-K) method is to work for K arbitrary, namely u(F) = 2 u(K).

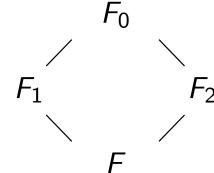
#### Main result

- ▶ Theorem (-, Parimala, 2023) For K arbitrary, the local-global principle holds for projective homogeneous spaces over F = K(X) with respect to divisorial valuations.
- ► Some small characteristics are excluded and the proof is uniform.
- ► On the first conjecture (*K* local field), there are several evidences but no general result yet.

# Why projective homogeneous varieties are simpler than torsors?

We will examine first the case of "patching fields".

► We are given a diagram of fields

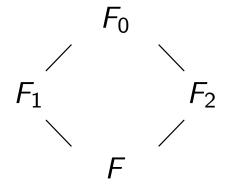


where  $F = F_1 \cap F_2$  and  $F_0$  is generated by  $F_1$  and  $F_2$ .

- We say this is a patching diagram if  $GL_n(F_0) = GL_n(F_1) GL_n(F_2)$  for each  $n \ge 1$ .
- ▶ Equivalently this is to say that the category Vect(F) is equivalent to the category  $Vect(F_1) \times_{Vect(F_0)} Vect(F_2)$  (H-H).
- It follows that the category of affine F-varieties (resp. algebraic groups, etc...) is equivalent to the fiber product of the categories of affine  $F_1$ -varieties and affine  $F_2$ -varieties over that of affine  $F_0$ -varieties.

# Patching

We are given a diagram of patching fields.



- ► The category Vect(F) is equivalent to the category  $Vect(F_1) \times_{Vect(F_0)} Vect(F_2)$ .
- It follows that the category of affine F-varieties (resp. algebraic groups, etc...) is equivalent to the fiber product of the categories of affine  $F_1$ -varieties and affine  $F_2$ -varieties over that of affine  $F_0$ -varieties.
- ightharpoonup Let G be an affine F-group. We claim that there is a bijection

$$G(F_1)\backslash G(F_0)/G(F_2)\stackrel{\sim}{\longrightarrow} \ker\Bigl(H^1(F,G) o H^1(F_1,G) imes H^1(F_2,G)\Bigr).$$

# Patching II

Let G be an affine F-group.

► There is a bijection

$$G(F_1)\backslash G(F_0)/G(F_2)\stackrel{\sim}{\longrightarrow} \ker\Bigl(H^1(F,G) o H^1(F_1,G) imes H^1(F_2,G)\Bigr).$$

- ▶ Given  $g \in G(F_0)$  we can patch the trivial G-torsors  $E_1 = G_{F_1}$  and  $E_2 = G_{F_2}$  with g. This provides a G-torsor over F.
- If E is G—torsor over F equipped with trivializations  $\phi_1: G_{F_1} \xrightarrow{\sim} E_{F_1}$  and  $\phi_2: G_{F_2} \xrightarrow{\sim} E_{F_2}$ , then the endomorphism  $\phi_1^{-1} \circ \phi_2$  on  $G_{F_0}$  is the left translation by an element  $g \in G(F_0)$ .
- Summarizing the study of the triviality of the kernel rephrases as the study of the decomposition  $G(F_0) = G(F_1) G(F_2)$ .

# Patching III

Now we assume that G is reductive and we are given a projective G-variety Z.

- ▶ Assume that  $Z(F_1) \neq \emptyset$  and  $Z(F_2) \neq \emptyset$ .
- ▶ We pick  $z_1, z_2$  so that  $z_1 = g \cdot z_2$  for  $g \in G(F_0)$ .
- ▶ If  $G(F_0) = G(F_1) G(F_2)$  we write  $g = g_1 g_2$ , so that  $g_1^{-1} \cdot z_1 = g_2 \cdot z_2 \in Z(F_1) \cap Z(F_2) = Z(F)$ . Thus  $Z(F) \neq \emptyset$ .
- We consider now the  $F_2$ -parabolic subgroup  $P_2 = \operatorname{Stab}_{G_{F_2}}(z_2)$ . If  $G(F_0) = G(F_1) G(F_2) P_2(F_0)$ , I claim that it still works.
- ▶ Indeed write  $g = g_1 g_2 p_2$  so that  $g_1^{-1} . z_1 = g_2 . p_2 . z_2 = g_2 . z_2 \in Z(F)$ .

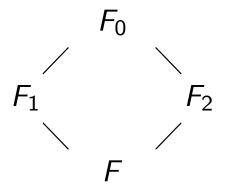
## Kneser-Tits subgroups

As before let  $P_2$  be a  $F_2$ -parabolic subgroup of G. We are interested in decomposition  $G(F_0) = G(F_1) G(F_2) P_2(F_0)$ .

- The essential case is when G is semisimple simply connected case, absolutely almost F-simple,  $F_1$ -isotropic and  $F_2$ -isotropic.
- ▶ We deal with the Kneser-Tits subgroup  $G(F_0)^+$  of  $G(F_0)$ . We know that  $G(F_0) = G(F_0)^+ P_2(F_0)$  so that is enough to show that  $G(F_0)^+ \subset G(F_1) G(F_2)$ .
- ► This is much easier since  $G(F_0)^+$  is well-understood in certain cases and also have nice generators.

#### Back to the H-H-K method

We deal with the special case of patching diagram



where  $F_0$  is complete for a discrete valuation containing K as valued subfield and sharing the same unformizing parameter  $t \in T$ . We denote by  $\widehat{R}_0$  the ring of integers of  $F_0$ .

- We assume furthermore that  $F_1$  is dense in  $F_0$  and that t-adically complete T-submodules  $V \subset F_1 \cap \widehat{R}_0$  and
- ▶  $W \subset F_2 \cap \widehat{R_0}$  satisfying the following conditions :

$$V + W = \widehat{R}_0; \tag{1}$$

$$V \cap t\widehat{R}_0 = tV$$
 and  $W \cap t\widehat{R}_0 = tW$ . (2)

- ▶ **Theorem** (-, Parimala)  $G(F_0)^+ = G(F_1)^+ G(F_2)^+$ .
- ▶ This was known when G is F-rational (H-H-K).

#### Back to the H-H-K method, II

- ▶ **Theorem** (-, Parimala)  $G(F_0)^+ = G(F_1)^+ G(F_2)^+$ .
- ightharpoonup The case G isotropic is simpler.
- ▶ **Corollary**. Let Z be a projective G-homogenous variety such that  $Z(F_1) \neq \emptyset$  and  $Z(F_2) \neq \emptyset$ . Then  $Z(F) \neq \emptyset$ .
- ▶ This is the first step for the proof of the main Theorem.
- It concerns the case F = K(x),  $X = \mathbb{P}^1_K$ ,  $\mathfrak{X} = \mathbb{P}^1_T$ . We will associate fields  $F_1$ ,  $F_2$  and  $F_0$  to  $P = 0_k$  and  $U = \mathbb{P}^1_k \setminus \{P\} = \operatorname{Spec}(k[x^{-1}])$ .

# Patching again

- ►  $F = K(x), X = \mathbb{P}^1_K$ ,  $\mathfrak{X} = \mathbb{P}^1_T$ . We will associate fields  $F_1$ ,  $F_2$  and  $F_0$  to  $P = 0_k$  and  $U = \mathbb{P}^1_k \setminus \{P\} = \operatorname{Spec}(k[x^{-1}])$ .
- ▶ For each  $Q \in \mathfrak{X}$ , we put  $R_Q = \mathcal{O}_{\mathfrak{X},Q}$  and denote by  $\widehat{R}_Q$  its t-adic completion. We put  $F_1 = \operatorname{Frac}(\widehat{R}_P)$ .
- We put  $R_U = \bigcap_{Q \in U} R_Q \subset F$ . We denote by  $\widehat{R}_U$  the t-adic completion of  $R_U$  and put  $F_2 = F_U = \operatorname{Frac}(\widehat{R}_U)$ .
- ▶ We consider the prime ideal  $t\widehat{R}_P \subset \widehat{R}_P$  and denote by  $\widehat{R}_0$  the t-adic completion of  $(\widehat{R}_P)_{t\widehat{R}_P}$  and by  $F_0$  its fraction field.
- ▶ The claim is that  $(F, F_1, F_2, F_0)$  is a patching diagram which satisfies the additional assumptions [H-H].

## Patching again

- ▶ The claim is that  $(F, F_1, F_2, F_0)$  is a patching diagram which satisfies the additional assumptions [H-H].
- ► In the equicharacteristic case, we have

$$F_1 = k((t,x)), \quad F_2 = k(x)((t)), \quad F_0 = k((x))((t)).$$

- We have then a local-global principle for G-homogeneous space X over F with respect to the overfields  $F_1$  and  $F_2$  which are much simpler.
- ▶ The field  $F_2$  is of the shape  $F_v$  for a divisorial valuation v. This is not the case of  $F_1$ .

## Patching again

► In the equicharacteristic case, we have

$$F_1 = k((t,x)), \quad F_2 = k(x)((t)), \quad F_0 = k((x))((t)).$$

- We have then a local-global principle for the patching data attached to (P, U).
- By Weil restriction trick, we have a similar result for any curve X and any regular model.
- ▶ In other words, the above case is the essential one.
- It remains to explain how we can connect the field  $F_1$  and with divisorial valuations of F.

#### Towards tame loop torsors

- For simplicity I will explain how it works only for the field k((t,x)) which is the field of fractions of the regular local ring R = k[[t,x]].
- We blow up  $B = \operatorname{Spec}(k[[t,x]])$  at its closed point and the exceptional divisor gives rise to a canonical valuation  $v: k((t,x))^{\times} \to \mathbb{Z}$ . We have v(t) = v(x) = 1 and the residue field is k(u) where u is the image of  $\frac{t}{x}$ .
- We have  $F = k((t))(x) \subset F_1 = k((t,x))$  and v induces a divisorial valuation on F. Since F is dense in  $F_1$ , we have  $F_v = F_{1,v}$ .
- We are given now a projective G-variety Z over F having rational points everywhere locally for completions with respect to divisorial valuations.
- ▶ We have  $Z(F_2) \neq \emptyset$  and  $Z(F_v) \neq \emptyset$ . From  $Z(F_{1,v}) \neq \emptyset$ , we would like to show that  $Z(F_1) \neq \emptyset$  in order to apply the former corollary.
- Here start new technicalities.

#### Tame loop torsors

- A first step is to reduce to the case when G arises from a reductive scheme  $\mathfrak{G}$  over  $k[[t,x]][t^{-1},x^{-1}]$ . This is not enough; we need it to arise from a so-called *tame loop object*.
- ▶ For example,  $\mathfrak{G} = \mathsf{SO}(q)$  is a tame loop reductive group scheme if and only if the  $k[[t,x]][t^{-1},x^{-1}]$ -quadratic form is diagonalizable.
- In the case of tame loop objects, there is a fixed point theorem. If  $\mathfrak{Z}$  is a scheme of parabolic subgroups for  $\mathfrak{G}$ ,  $\mathfrak{Z}(F_{1,v}) \neq \emptyset$  is equivalent to  $\mathfrak{Z}\left(k[[t,x^{-1}]][t^{-1},x]\right) \neq \emptyset$ . A fortiori this is equivalent with  $\mathfrak{Z}(F_1) \neq \emptyset$ .
- This loop theory is inspired of our work with Chernousov and Pianzola for (loop) torsors on Laurent polynomials. It works as well in the arithmetic setting, e.g.  $\mathbb{Z}_p[[t]]\left[\frac{1}{p},\frac{1}{t}\right]$ .

► Thanks for attention.

Спасибо!