Aggrandization of spaces: a general approach and applications

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We base our study recent work on the so-called local aggrandization of Lebesgue spaces and extend this approach to the case of arbitrary Banach spaces of functions on metric spaces. We show that grand spaces of holomorphic functions can be equivalently defined in terms of aggrandization associated only with the boundary.

[1]Alexey Karapetyants and Stefan Samko. Aggrandization of Spaces of Holomorphic Functions Reduces to Aggrandization on the Boundary. Mathematical Notes, 2024, Vol. 116, No. 6, pp. 1292–1305.

1. What is Grand Lebesgue space over finite measure domain Ω

Given an open set $\Omega \subset \mathbb{R}^n$, $|\Omega| = 1$, $1 < p < \infty$, and $\theta > 0$, the grand Lebesgue space $L^{p),\theta}(\Omega)$ consists of all measurable on Ω functions f such that

$$\|f\|_{p),\theta} = \sup_{0<\varepsilon<p-1} \left(\varepsilon^\theta \int_\Omega |f(x)|^{p-\varepsilon}\, dx \right)^{\frac{1}{p-\varepsilon}}. \qquad \mathbf{(1.1)}$$

The following equivalences for the norms in grand and small Lebesgue spaces are known:

$$\|f\|_{p),\theta} \simeq \sup_{0<t<1} \ln^{-\frac{\theta}{p}}\left(\frac{e}{t}\right)\left(\int_t^1 f_*(s)^p\, ds\right)^{\frac{1}{p}},$$

$$\|f\|_{(p,\theta} \simeq \int_0^1 \ln^{\frac{\theta}{p'}-1}\left(\frac{e}{t}\right)\left(\int_0^t f_*(s)^p\, ds\right)^{\frac{1}{p}} \frac{dt}{t}.$$

2. BASIC RELATIONS ($\theta = 1$)

For a finite measure domain Ω :

$$L^p \subset L^{p)} \subset L^{p-\varepsilon}$$

$$L^{p+\varepsilon} \subset L^{(p} \subset L^p$$

$$L^p \subset L^{p,\infty} \subset L^{p)}$$

$$L^p \subset L^p \log^{-1} L \subset L^{p)} \subset \cap_{\alpha>1} L^p \log^{-\alpha} L$$

3. Fundamental function

Lemma 3.1. *(S. Umarkhadzhiev). Let $1 < p < \infty$, $\theta > 0$, Ω be an open set in $\mathbb{R}^n$, $|\Omega| = 1$, and E be an open set in Ω with $|E| = \eta$. Then*

$$\|\chi_E\|_{p),\theta} = \begin{cases} e^{\frac{\theta}{p}W_-(-\frac{p\eta^{\frac{1}{\theta}}}{e})}, & 0 < \eta^{\frac{1}{\theta}} < \frac{1}{p-1}e^{-\frac{1}{p-1}}, \\ (p-1)^\theta \eta, & \frac{1}{p-1}e^{-\frac{1}{p-1}} \leqslant \eta^{\frac{1}{\theta}} < 1, \end{cases} \tag{3.2}$$

where W_- is the branch of Lambert function W, which yields

$$\|\chi_E\|_{p),\theta} \simeq \eta^{\frac{1}{p}} \ln^{-\frac{\theta}{p}} \frac{e}{\eta}, \tag{3.3}$$

and

$$\|\chi_E\|_{(p,\theta} \simeq \eta^{\frac{1}{p}} \ln^{\frac{\theta}{p'}} \frac{e}{\eta}. \tag{3.4}$$

4. Whether there is a difference between a space of holomorphic functions and the corresponding grand space

A natural question arises: whether there is a difference ? The answer is known for $\mathcal{A}^p(\mathbb{D})$ and $\mathcal{A}^{p),\theta}(\mathbb{D})$. More precisely,

$$\frac{1}{(1-z)^{\lambda}} \in \mathcal{A}^p(\mathbb{D}) \Leftrightarrow \lambda < \frac{2}{p},$$

see Duren and Schuster book (2004), while it is easily checked that

$$\frac{1}{(1-z)^{\frac{2}{p}}} \in \mathcal{A}^{p),\theta}(\mathbb{D}), \quad \theta = 1.$$

Moreover, we proved that

$$g_\theta(z) = \frac{1}{(1-z)^{\frac{2}{p}}} \ln^{\frac{\theta-1}{p}} \frac{e}{1-z} \in \mathcal{A}^{p),\theta}(\mathbb{D})$$

under the proper choice of the branch of logarithmic function.

5. Grand Lebesgue spaces in real analysis

Grand Lebesgue spaces in real analysis were introduced and studied in connection with the applications to PDE's.

(1) T.Iwaniec, C.Sbordone. On the integrability of the Jacobian under minimal hypotheses. Arch. Rational Mech. Anal. (1992) 119: 129.

(2) L.Greco, T.Iwaniec, C.Sbordone. Inverting the p-harmonic operator, Manuscr. Math., 92:2 (1997), 249-258.

(3) G. Di Fratta and A. Fiorenza. A direct approach to the duality of grand and small Lebesgue spaces, Nonlinear Anal., 70 (7) (2009), 2582–2592

(4) A. Fiorenza and G. E. Karadzhov. Grand and small Lebesgue spaces and their analogs, Z. Anal. Anwendungen, 23 (4) (2004), 657-681

(5) Etc...

6. Grand Lebesgue spaces in complex analysis

In the case of holomorphic functions grand spaces over the unit disc first appeared in [1] below (see also [2]).

(1) Karapetyants, A.N., Samko, S.G. On Grand and Small Bergman Spaces. Math Notes 104, 431–436 (2018).

(2) A. Karapetyants. On mixed norm holomorphic grand and small spaces. Complex Variables and Elliptic Equations. Volume 67, 2022 - Issue 3: Special Issue in honor of Prof. V. S. Rabinovich . Guest Editors: Alexey N. Karapetyants, Vladislav V. Kravchenko, R. Michael Porter and Sergii M. Torba.

(3) Karapetyants, A., Samko, S. Aggrandization of Spaces of Holomorphic Functions Reduces to Aggrandization on the Boundary. Math Notes 116, 1292–1305 (2024).

7. Ideas of aggrandization

(1) S. G. Samko and S. M. Umarkhadzhiev. On Iwaniec-Sbordone spaces on sets which may have infinite measure. Azerbaijan Journal of Mathematics (2011).

(2) ETC...

(3) Rafeiro, H., Samko, S. and Umarkhadzhiev, S. Local grand Lebesgue spaces on quasi-metric measure spaces and some applications. Positivity 26, 53 (2022).

(4) H. Rafeiro and S. Samko. Local grand variable exponent Lebesgue spaces. Z. Anal. Anwend. 42 (2023), no. 1/2, pp. 1–15

(5) S. G. Samko and S. M. Umarkhadzhiev. Local Grand Lebesgue spaces. Vladikavkaz Mathematical Journal 2021, Volume 23, Issue 4, P. 96–108.

8. What we prove: an example

Using the general method of local aggrandization, in particular for Bergman spaces (of holomorphic functions) we prove the following norm equivalence

$$\sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \left(\int_{\mathbb{D}} |f(z)|^{p-\varepsilon} \mathrm{dA}(z) \right)^{\frac{1}{p-\varepsilon}}$$
$$\approx \sup_{0<\varepsilon<l} \varepsilon^{\theta} \left(\int_{\mathbb{D}} |f(z)|^{p} (1-|z|)^{\lambda\varepsilon} \mathrm{dA}(z) \right)^{\frac{1}{p}}$$

for all $\lambda > 0$ and $l > 0$. In fact, we prove more general statement with $(1-|z|)^{\lambda\varepsilon}$ replaced by $a(1-|z|)^{\varepsilon}$, under some assumptions on the function a, called *aggrandizer*.

Moreover, we generalize this to the case of weighted Bergman and Bergman - Morrey spaces.

9. Preliminaries

For a function a positive on $\mathbb{R}_+$, its Matuszewska – Orlicz indices $m(a)$ and $M(a)$ are defined by

$$m(a) = \sup_{0<t<1} \frac{\ln\left(\overline{\lim}_{h\to 0}\frac{a(ht)}{a(h)}\right)}{\ln t},$$

and

$$M(a) = \sup_{t>1} \frac{\ln\left(\overline{\lim}_{h\to 0}\frac{a(ht)}{a(h)}\right)}{\ln t}.$$

Note that for $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}_+$:

$$m(t^\alpha) = \alpha, \;\; m(t^\alpha a(t)) = \alpha + m(a), \;\; m(a(t)^\beta) = \beta m(a),$$

$$m\left(\frac{1}{a}\right) = -M(a).$$

Matuszewska,W., Orlicz,W.: On some classes of functions with regard to their orders of growth. Studia Math. 26, 11–24 (1965). https://doi.org/10.4064/sm-26-1-11-24

10. Preliminaries: continuation

A positive function a on $(0, \mathcal{D})$ is called quasi-monotone if there exist $\alpha, \beta \in \mathbb{R}$ such that $a(t)t^{-\alpha}$ is almost increasing and $a(t)t^{-\beta}$ is almost decreasing. A quasi-monotone function has finite indices and

$$m(a) \;=\; \sup\{\alpha \,|\, a(t)t^{-\alpha} \;\text{ is almost increasing}\},$$

and

$$M(a) \;=\; \inf\{\beta \,|\, a(t)t^{-\beta} \;\text{ is almost decreasing}\}.$$

Everywhere in the sequel, when considering indices of a function, we suppose that it is quasi-monotone near the origin.

For a function $\varphi : [0, \infty) \to [0, \infty]$ we say that it satisfies the doubling condition if there exists $C_{(2)} > 0$ such that $\varphi(2t) \leqslant C_{(2)}\varphi(t)$, $t > 0$. A function φ satisfies the reverse doubling condition if there exists $c_{(2)} > 0$ such that $\varphi(t) \leqslant c_{(2)}\varphi(2t)$, $t > 0$.

11. Local aggrandization of function spaces over metric spaces: the "weighted" space

Let (Λ, d) be a metric space and

$$\mathcal{D} = \operatorname{diam} \Lambda, \quad 0 < \mathcal{D} \leqslant \infty.$$

Given a closed non empty subset $F \subset \Lambda$, denote

$$\delta_F(x) = \inf_{y \in F} d(x, y), \quad x \in \Lambda.$$

Let $X = X(\Lambda)$ be an arbitrary normed space of functions $f : \Lambda \to \mathbb{C}$ and $\| \cdot \|_X$ be its norm.

With the space X we associate the normed space $X_w = X(\Lambda, w)$ depending on a functional parameter $w : \Lambda \to \mathbb{R}_+$, and assume that

$$X_w \Big|_{w=1} = X.$$

By $\| \cdot \|_{X,w}$ we denote the norm in X_w. We assume that w is a non-negative function on Λ.

12. Local aggrandization of function spaces over metric spaces: Definition

Assumption 1: We assume that the space X_w possesses the following property (lattice property with respect to weights): for any two weights u, v such that $u \leqslant v$, the following inequality holds

$$\|f\|_{X,u} \leqslant \|f\|_{X,v}. \tag{12.5}$$

Definition 12.2. *Let $F \subset \Lambda$ be a closed non-empty set. For a positive almost increasing function $a \in L^\infty(0, \mathcal{D})$, $a(0+) = 0$, we define the local grand space*

$$X^g_{F,a,\theta} = X^g_{F,a,\theta}(\Lambda)$$

related to the space X, by the norm

$$\|f\|_{X^g_{F,a,\theta}} = \sup_{0<\varepsilon<l} \left(\varepsilon^\theta \|f\|_{X,(a\circ\delta_F)^\varepsilon}\right), \quad l > 0. \tag{12.6}$$

The function a used in definition of $X^g_{F,a,\theta}$ will be referred to as aggrandizer. The embedding

$$X \subset X^g_{F,a,\theta} \tag{12.7}$$

holds because $a \in L^\infty(0, \mathcal{D})$ by definition.

13. Basic properties of aggrandization

Lemma 13.3. *The space $X^g_{F,a,\theta}$ does not depend on the choice of $l > 0$, up to equivalence of norms:*

$$\|f\|_{X^g_{F,a,\theta}}\Big|_{l=l_1} \leqslant \|f\|_{X^g_{F,a,\theta}}\Big|_{l=l_2} \leqslant C\|f\|_{X^g_{F,a,\theta}}\Big|_{l=l_1}, \qquad l_1 < l_2.$$

In view of Lemma 13.3 we do not introduce the parameter l into the notation of the space $X^g_{F,a,\theta}$.

Lemma 13.4. *Let $s \in (0, \mathcal{D})$ be fixed. The norm (12.6) is equivalent to*

$$\sup_{0<\varepsilon<l} \left(\varepsilon^\theta \|\chi_{\{\delta_F(\cdot)<s\}} f\|_{X,(a\circ\delta_F)^\varepsilon}\right) + \|\chi_{\{\delta_F(\cdot)\geqslant s\}} f\|_X. \tag{13.8}$$

14. Basic properties of aggrandization

Theorem 14.5. *The following statements are valid.*

(1) *If there exists a number* $\alpha > 0$ *such that* $a(t) \leqslant Cb(t)^{\alpha}, \ \ t \in (0, \mathcal{D})$, *then*

$$X^g_{F,b,\theta} \hookrightarrow X^g_{F,a,\theta}.$$

(2) *If the function* a *is almost increasing near the origin, and* F_1 *and* F_2 *are closed non-empty sets such that* $F_1 \subseteq F_2 \subset \Lambda$, *then*

$$X^g_{F_1,a,\theta} \hookrightarrow X^g_{F_2,a,\theta}.$$

15. Basic properties of aggrandization

Theorem 15.6. *Let* $\nu > 0$. *Then*

$$X^g_{F,a,\theta} = X^g_{F,a^\nu,\theta}$$

up to equivalence of norms.

Theorem 15.7. *Let* a *and* b *be almost increasing functions on* $(0, \mathcal{D})$. *If* $m(a) > 0$ *and* $m(b) > 0$, *then*

$$X^g_{F,a,\theta} = X^g_{F,b,\theta}$$

up to equivalence of norms.

Proof.

$$c(\eta)t^{M(a)+\eta} \leqslant a(t) \leqslant C(\eta)t^{m(a)-\eta}, \quad t \in (0, \delta), \text{ where } \delta \in (0, \mathcal{D}).$$

$\square$

16. The embedding $X \subset X^g_{F,a,\theta}$ is strict, in general

We will need additional assumptions about the space X. Assume that characteristic functions of balls in Λ belong to X and

$$\|\chi_{B(x_0,r)}\|_X \leqslant \|\chi_{B(x_0,\rho)}\|_X, \quad r < \rho. \tag{16.9}$$

Assume that in the case of constant weights $w = C$ there exists a nondecreasing function $\varphi_X : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f\|_{X,C} \leqslant \varphi_X(C)\|f\|_X. \tag{16.10}$$

Note that

$$\varphi_X(C) = C^{\frac{1}{p}}$$

for Lebesgue and Morrey spaces if weight is interpreted as measure (we will give definitions of Lebesgue and Morrey spaces below).

17. THE EMBEDDING $X \subset X^g_{F,a,\theta}$ IS STRICT, IN GENERAL

It sufficies to consider $F = \{x_0\}$. Let us denote

$$\nu_X(\rho) = \|\chi_{B(x_0,\rho)}\|_X. \tag{17.11}$$

The function ν_X is known as fundamental function of X.

Theorem 17.8. *Assume that (16.9) holds. Let $x_0 \in \Lambda$. Assume that the norm of X is absolute continuous. The function*

$$f_0(x) = \nu_X\left(d(x_0, x)\right)^{-1}, \quad x \in \Lambda \tag{17.12}$$

does not belong to X.

18. Conditions on the space X, under which f_0 belongs to the corresponding grand space $X^g_{F,a,\theta}$.

Theorem 18.9. *Let $m(a) > 0$. Assume that $x_0 \in \Lambda$, $F = \{x_0\}$, and the properties (16.10) and (16.9) hold. Then the function f_0, defined in (17.12), belongs to $X^g_{F,a,\theta}$ provided the following condition is satisfied:*

$$\sup_{\varepsilon \in (0,r)} \left\{ \varepsilon^\theta \int_0^r \frac{\nu_X(t)}{\nu_X(\frac{t}{2})} \varphi_X(t^\varepsilon) \frac{dt}{t} \right\} < \infty. \tag{18.13}$$

Corollary 18.10. *Assume that $x_0 \in \Lambda$, $F = \{x_0\}$ and the properties (16.10) and (16.9) hold. Assume also that*

$$\int_0^\delta \nu_X(s) \frac{ds}{s} < \infty$$

for some $\delta > 0$, and the function ν_X possesses the doubling property in a right-sided neighbourhood of the origin. Then the function f_0 defined in (17.12) belongs to $X^g_{F,a,\theta}$ provided $\theta \geqslant 1$.

19. Preliminaries on spaces

Let $\Lambda = \mathbb{D}$ be the unit disc in $\mathbb{C}$. We identify $\mathbb{R}^2 \equiv \mathbb{C}$ so that $(x, y) = z$. Let $\mathrm{dA}(z) = \frac{1}{\pi} dx dy$. The norm in $L^p(\mathbb{D})$ is given by

$$\|f\|_{L^p(\mathbb{D})} = \left(\int_{\mathbb{D}} |f(z)|^p \mathrm{dA}(z) \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty. \tag{19.14}$$

Let $\rho = \rho(t)$. In the sequel we always assume that for arbitrary $\delta \in (0, 1)$

$$0 < \inf_{t \in (\delta, 1)} \rho(t) \leq \sup_{t \in (\delta, 1)} \rho(t) < \infty.$$

The weighted Lebesgue space $L^p(\mathbb{D}, \rho)$ is defined by the norm

$$\|f\|_{L^p(\mathbb{D}, \rho)} = \left(\int_{\mathbb{D}} |f(z)|^p \, \rho(1 - |z|) \, \mathrm{dA}(z) \right)^{\frac{1}{p}}, \quad 1 \leqslant p < \infty.$$

Let $\mathcal{A}^p(\mathbb{D})$ and $\mathcal{A}^p(\mathbb{D}, \rho)$ stand for the subspaces of $L^p(\mathbb{D})$ and $L^p(\mathbb{D}, \rho)$, respectively, which consist of functions holomorphic in $\mathbb{D}$.

20. PRELIMINARIES ON SPACES

For $1 < p < \infty$ and $\theta > 0$, the grand Lebesgue space $L^{p),\theta}(\mathbb{D})$ consists of all functions f measurable on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta}(\mathbb{D})} := \sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \|f\|_{L^{p-\varepsilon}(\mathbb{D})} < \infty, \tag{20.15}$$

and the grand weighted Lebesgue space $L^{p),\theta}(\mathbb{D}, \rho)$ consists of all functions f measurable on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta}(\mathbb{D},\rho)} := \sup_{0<\varepsilon<p-1} \varepsilon^{\theta} \|f\|_{L^{p-\varepsilon}(\mathbb{D},\rho)} < \infty. \tag{20.16}$$

The corresponding subspaces of holomorphic functions will be denoted by $\mathcal{A}^{p),\theta}(\mathbb{D})$ and $\mathcal{A}^{p),\theta}(\mathbb{D}, \rho)$, respectively.

21. THE GRAND MORREY SPACE

Let $D(z,r)$ denote the Euclidean disc in $\mathbb{C}$ with center z and radius r. The function φ is positive and bounded on $(0,1)$.

The Morrey space $L^{p,\varphi}(\mathbb{D})$ is defined as the set of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^{p,\varphi}(\mathbb{D})} := \sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}}\|f\|_{L^p(D(z,r)\cap\mathbb{D})} < \infty.$$

The grand Morrey space $L^{p),\theta,\varphi}(\mathbb{D})$ is defined as the set of all measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^{p),\theta,\varphi}(\mathbb{D})} := \sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}}\|f\|_{L^{p),\theta}(D(z,r)\cap\mathbb{D})} < \infty.$$

22. AGGRANDIZATION: THE CASE OF THE SPACE $L^p(\mathbb{D}, \rho)$

We take

$$F = \mathbb{T}, \quad \delta_{\mathbb{T}}(z) = 1 - |z|.$$

Recall that the aggrandizer $a(t) > 0$ for $t \in (0,1)$ and $a(0) = 0$. We find it convenient to use the notation $L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D}, \rho)$ for the aggrandized space for $L^p(\mathbb{D}, \rho)$. Same for $\mathcal{A}^{p)}_{\mathbb{T},a\theta}(\mathbb{D}, \rho)$. Hence,

$$\begin{aligned} \|f\|_{L^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)} &= \sup_{0<\varepsilon<l} \left(\varepsilon^{\theta} \|f\|_{L^p(\mathbb{D},\rho),(a\circ\delta_{\mathbb{T}})^{\varepsilon}}\right) \\ &= \sup_{0<\varepsilon<l} \left(\varepsilon^{\theta} \int_{\mathbb{D}} |f(z)|^p (a\circ\delta_{\mathbb{T}})(z)^{\varepsilon} \rho(1-|w|) \mathrm{dA}(z)\right)^{\frac{1}{p}}, \end{aligned}$$

where $l > 0$. When dealing with the space $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D}, \rho)$ we will assume that $l < p - 1$ (we can do this in view of Lemma 13.3).

Theorem 23.11. *The following statements hold true.*

(1) *Let there exist* $\delta > 0$ *such that*

$$\int_0^1 \frac{\rho(t)}{a(t)^\delta} dt < \infty, \tag{23.17}$$

Then

$$\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D}, \rho) \hookrightarrow \mathcal{A}^{p),\theta}(\mathbb{D}, \rho). \tag{23.18}$$

24. AGGRANDIZATION: THE CASE OF THE SPACE $L^p(\mathbb{D}, \rho)$

(2) *Let any of the two following conditions be satisfied:*

(a) *the function $\rho = \rho(t)$ is almost decreasing and satisfying the reverse doubling condition and there exist $\eta > 0$, $c_a > 0$ and $\varepsilon_0 \in (0, p-1)$ such that*

$$a(t) \leqslant c_a \left(t^{\frac{2}{p}} \rho(t)^{\frac{1}{p-\varepsilon_0}}\right)^{\eta}, \qquad t \in (0,1); \tag{24.19}$$

(b) *the function $\rho = \rho(t)$ is almost increasing and $M(\rho) < p-1$, and there exist $\eta > 0$ and $c_a > 0$, such that*

$$a(t) \leqslant c_a \left(t^{\frac{2}{p}} \rho(t)^{\frac{1}{p}}\right)^{\eta}, \qquad t \in (0,1). \tag{24.20}$$

Then

$$\mathcal{A}^{p),\theta}(\mathbb{D}, \rho) \hookrightarrow \mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D}, \rho). \tag{24.21}$$

25. AGGRANDIZATION: THE CASE OF THE SPACE $L^p(\mathbb{D},\rho)$

The case of a power weight is of particular interest.

Theorem 25.12. *Let* $\rho(t) = t^\gamma$, $-1 < \gamma < p-1$, *and there exists* $\delta > 0$ *such that*

$$\int_0^1 \frac{t^\gamma dt}{a(t)^\delta} < \infty,$$

and there exist $C > 0$ *and* $\eta > 0$ *such that*

$$a(t) \leqslant Ct^\eta.$$

Then the space $\mathcal{A}^{p),\theta}(\mathbb{D},\rho)$ *coincides with the space* $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D},\rho)$, *up to equivalence of norms.*

We single out particular corollary for the unweighed case: $\rho = 1$. We use symbols $\mathcal{A}^{p),\theta}(\mathbb{D})$ and $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D})$ for the unweighed spaces.

Theorem 25.13. *The space* $\mathcal{A}^{p),\theta}(\mathbb{D})$ *coincides with the space* $\mathcal{A}^{p)}_{\mathbb{T},a,\theta}(\mathbb{D})$ *with any aggrandizer* a *satisfying* $m(a) > 0$, *up to equivalence of norms.*

26. Aggrandization: The case of Morrey space $L^{p,\varphi}(\mathbb{D})$.

Recall that

$$F = \mathbb{T}, \quad \delta_{\mathbb{T}}(z) = 1 - |z|.$$

We find it convenient to use the notation $L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$ for the aggrandized space for $L^{p,\varphi}(\mathbb{D})$. Same for $\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$. Hence:

$$\|f\|_{L^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})} = \sup_{0<\varepsilon<l} \left(\varepsilon^{\theta} \|f\|_{L^{p,\varphi}(\mathbb{D}),(a\circ\delta_{\mathbb{T}})^{\varepsilon}}\right)$$
$$\equiv \sup_{0<\varepsilon<l} \varepsilon^{\theta} \sup_{z\in\mathbb{D},r\in(0,1)} \frac{\varphi(r)}{r^{\frac{2}{p}}} \left(\int_{D(z,r)\cap\mathbb{D}} |f(w)|^p (a\circ\delta_{\mathbb{T}})(w)^{\varepsilon} \mathrm{dA}(w)\right)^{\frac{1}{p}},$$

where $l > 0$. Here we again assume that $l < p-1$.

27. AGGRANDIZATION: THE CASE OF MORREY SPACE $L^{p,\varphi}(\mathbb{D})$.

Theorem 27.14. *The following statements hold true.*

(1) *Let there exist* $\delta > 0$ *such that*

$$\int_0^1 \frac{dt}{a(t)^\delta} < \infty.$$

Then

$$\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D}) \hookrightarrow \mathcal{A}^{p),\theta,\varphi}(\mathbb{D}).$$

(2) *Let there exist* $\beta > 0$ *and* $c_{a,\varphi} > 0$ *such that*

$$a(t)^\beta \leqslant c_{a,\varphi}\varphi(t).$$

Then

$$\mathcal{A}^{p),\theta,\varphi}(\mathbb{D}) \hookrightarrow \mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D}).$$

Corollary 27.15. *Let the both assumptions in* (1) *and* (2) *of Theorem 27.14 are satisfied. Then the space* $\mathcal{A}^{p),\theta,\varphi}(\mathbb{D})$ *coincides with the space* $\mathcal{A}^{p),\varphi}_{\mathbb{T},a,\theta}(\mathbb{D})$, *up to equivalence of norms.*

28. Growth of holomorphic functions

The classical $\mathcal{A}^p(\mathbb{D})$ result reads as

$$|f(z)| \leqslant \frac{\|f\|_{L^p(\mathbb{D})}}{(1-|z|)^{\frac{2}{p}}}, \quad f \in \mathcal{A}^p(\mathbb{D}), \quad z \in \mathbb{D}, \ \ p > 0. \tag{28.22}$$

Theorem 28.16. *The following statements hold.*

(1) *Let* $f \in \mathcal{A}^p(\mathbb{D}, \rho)$, $1 < p < \infty$. *Let* $\rho = \rho(t)$ *be either almost decreasing and satisfying the reverse doubling condition, or almost increasing and* $M(\rho) < p-1$. *Then*

$$|f(z)| \leqslant \frac{\|f\|_{L^p(\mathbb{D},\rho)}}{(1-|z|)^{\frac{2}{p}}\rho(1-|z|)^{\frac{1}{p}}}, \quad z \in \mathbb{D}. \tag{28.23}$$

(2) *Let* $f \in \mathcal{A}^{p,\varphi}(\mathbb{D})$. *Then*

$$|f(z)| \leqslant \frac{\|f\|_{L^{p,\varphi}(\mathbb{D})}}{\varphi(1-|z|)}, z \in \mathbb{D}.$$

THANK YOU!