On the Maximum Principle with a Pathwise Cost Functional for One-dimensional Stochastic Differential Equations

Nasyrov F.S.

2025

Introduction 1. The deterministic case 1

The problem of the deterministic optimal control is to build the control $u(t)$ and trajectory $x(t)$ such that то minimize the cost functional

$$J(u(.)) = \int_0^T f_0(t, x(t), u(t))\, dt + h_0(x(0), x(T)),$$

here U is the class of admissible controls over conditions

$$\dot{x}(t) = f(t, x(t), u(t)),\ \ t \in [0, T],\ \ x(0) = x_0,$$

and control constrains $u(t) \in U$.
One of the methods for solving this problem is the Pontryagin maximum principle, in which this problem is reduced to solving a system of differential equations.

Introduction 2. The deterministic case 2

The function $H(x, u, t, \psi) = -f_0(t, x, u) + \psi f(t, x, u)$ is called *Hamilton-Pontryagin function*, and a function $u = u(t, x, \psi)$ is found from the condition $H(x, u(t, x, \psi), t, \psi) = \max_{u \in U} H(x, u, t, \psi)$, where $\psi(s)$ is *an adjoint variable*.

The boundary value problem of the maximum principle is an ODE system of equations

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t, x(t), \psi(t))), \\ \dot{\psi}(t) = -\psi(t) f'_x(t, x(t), u(t, x(t), \psi(t))) + (f_0)'_x(t, x(t), u(t, x(t), \psi(t))), \\ x(0) = x_0, \ \psi(T) = -(h_0)'_y(x(0), x(T)). \end{cases}$$

This system is called *the boundary value problem of the maximum principle* or *an (extended) Hamiltonian system.* Recall that deterministic optimal control $u(t)$ uses all function information $x(t)$ on the $[0, T]$. However, in many cases, information is available only up to the present time.

Main result 1

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ be a complete probabilistic space with filtration. Suppose that the filtration (F_t) is generated by a random process $V(s)$ with continuous realizations.
Consider a controlled SDE

$$x(t) = x_0 + \int_0^t b(s, x(s), u(s))\, ds + \int_0^t \sigma(s, x(s), u(s)) * dV(s),\ \ t \in [0, T], \tag{1}$$

with two cost functionals: *pathwise functionals*

$$J(u(.)) = \int_0^T f_0(t, x(t), u(t))\, dt + h_0(T, x(0), x(T)), \tag{2}$$

and *averaged functionals*

$$J_E(u(.)) = \mathrm{E}(J(u(.))). \tag{3}$$

The second integral on the right-hand side (1) is a symmetric integral over the process $V(s)$, the definition of which will be given later.

Main result 2

So we have two problems:

Problem 1. Minimize $J(u(.))$ over $\mathcal{U}$.
Problem 2. Minimize $J_E(u(.)) = E(J(u(.)))$ over $\mathcal{U}$.
Here and below we study optimization problems in the class of piecewise continuous non-anticipatory controls u satisfying condition $u(t) \in U$.
It is well known that in the case of the Wiener process $V(s) = W(s)$, this problem 2 can be solved using, for example, the maximum principle. As far as the author knows, the problem 1 in the first time setting has not been studied.
The main content of this research is an attempt to answer the following question:

Let $V(s)$ be an arbitrary process with continuous realizations. Is it possible to construct adapted controls for the path-wise optimization problem, which are preferable in comparison with those obtained in the stochastic maximum principle with the averaged functional?

Main result 3

Theorem. *Let the functions* $b(t,x,u)$, $\sigma(t,x,u)$, $f_0(t,x,u)$, $h_0(T,x(0),y)$ *be twice continuously differentiable with respect to all their variables, the function* $\sigma(t,x,u)\neq 0$ *for all* (t,x,u). *Let* $(x(s),\psi(s),C(s))$ *be the solution of the system*

$$\begin{cases} dx(s) = b(s,x(s),u(s))ds + \sigma(s,x(s),u(s)) * dV_(s), \\ d\psi(s)=[(f_0)'_x(s,x(s),u(s))-\psi(s)b'_x(s,x(s),u(s))]ds - C(s) * dV(s), \\ x(0) = x_0,\ \ \psi(T) = -h'_x(T,x(0),x(T)),\ \ s\in[0,T], \end{cases} \tag{1}$$

where is the function $u(s)=u^*(s,x,\psi)$ *is found from the maximum condition* $H^{(1)}(s,x,\psi,u^*(s,x,\psi))=\max_{u\in U} H^{(1)}(s,x,\psi,u)$, $H^{(1)}(s,x,\psi,u) = \psi b(s,x,u) - f_0(s,x,u)$. *Then system (1) is a boundary value problem of the maximum principle for the problem under study, i.e. the pair* $(x(s),\psi(s))$ *satisfies the necessary conditions of the maximum principle.*

Main result 4

This FBSD-system is investigated as follows. First, the second equation is reduced to solving a system of two first-order partial differential equations

$$\psi'_s(s,x) + \psi'_x(s,x)b(s,x(s),u^*(s,x,\psi(s,x)))$$

$$= (f_0)'_x(s,x,u^*(s,x,\psi(s,x))) - \psi(s,x)b'_x(s,x,u^*(s,x,\psi(s,x)));$$

$$\psi'_x(s,x)\sigma(s,x,u^*(s,x,\psi(s,x))) + C(s) = 0,$$

and then the first equation is reduced to solving a chain of two first-order ODEs.

This problem was previously solved (see [1], [2]) under the assumption that the coefficient $\sigma(s,x,u)$ does not depend on the control u.

REFERENCES

1. *Nasyrov F. S.* The pathwise determined maximum principle and symmetric integrals.// Journal of Mathematical Sciences. (doi: 10.1007/s10958-022-06144-9)
https://doi.org/10.1007/s10958-022-06144-9,
Springer Nature, 2023.

2. *Nasyrov F. S.* On the maximum principle for stochastic differential equations with a a path-wise cost functional // Journal of Mathematical Sciences. https://doi.org/10.1007/s10958-024-07038-8 (doi: 10.1007/s10958-024-07038-8)
Springer Nature, 2024.

3. *Nasyrov F. S.* Local times, symmetrical integrals and stochastic analyses. Moscow, Fizmatlit, 2011.

Thank you for your attention!

Symmetric integrals and SDE, Nasyrov F. [2005-2011]

Let us present some necessary information about symmetric integrals.
Let $V(s)$, $s \in [0, T]$, be an arbitrary continuous function, then the *symmetric integral* is defined as

$$\int_0^t f(s, V(s)) * dV(s) \overset{def}{=} \lim_{n\to\infty} \sum_k \frac{1}{\Delta t_k^{(n)}} \int_{[\Delta t_k^{(n)}]} f(s, V^{(n)}(s)) ds\, \Delta V_k^{(n)}$$

$$= \lim_{n\to\infty} \int_0^t f(s, V^{(n)}(s))(V^{(n)})'(s)\, ds,$$

where $V^{(n)}(s)$ is a broken line and $\max_k(t_k^{(n)} - t_{k-1}^{(n)}) \to 0$ as $n \to \infty$, where $V^{(n)}(s)$ is a broken line and $\max_k(t_k^{(n)} - t_{k-1}^{(n)}) \to 0$ as $n \to \infty$, and if the limit on the right side of the equality exists and does not depend on the choice of the sequence of partitions $T_n, n \in N$.

Some Properties of Symmetric Integrals

1. Suppose that a function $f(s,v)$ for a.e. v has a summable derivative $f'_v(s,v)$, then
$\int_{t_0}^{t} f(s,V(s)) * dV(s) = \int_{V(t_0)}^{V(t)} f(t,v)\,dv - \int_{t_0}^{t}\int_{V(t_0)}^{V(s)} f'_v(s,v)\,dv\,ds.$

2. Let the function $F(t,v)$ has continuous partial derivatives F'_t and F'_v, then
$F(t,V(t)) - F(0,V(0)) = \int_0^t F'_s(s,V(s))\,ds + \int_0^t F'_v(s,V(s)) * dV(s).$
Thus, for a symmetric integral, there is a differential corresponding to the stochastic differential with respect to the Stratonovich integral.

In particular, from the Ito formula with the Stratonovich integral applied to the random function
$\int_{W(t_0)}^{W(t)} f(t,v)\,dv = \int_{t_0}^{t} f(s,W(s)) * dW(s) + \int_{t_0}^{t}\int_{W(t_0)}^{W(s)} f'_v(s,v)\,dv\,ds$
we obtain formula for a symmetric integral.
Therefore, we will use the same notation for both types of integrals.

Symmetric integrals and SDE – 3

Consider a pathwise (deterministic) equation with a symmetric integral

$$\xi(t) - \xi(0) = \int_0^t \sigma(s, \xi(s)) * dV(s) + \int_0^t b(s, \xi(s))\, ds,\ \ \xi(0) = \xi_0. \quad (4)$$

Solution $\equiv$ any function $\xi(s) = \varphi(s, V(s)), s \in [0, T]$, $\xi(0) = \xi_0$, satisfying the following conditions:
1. $(V(s), \sigma(s, \varphi(s, v)))$ satisfy the condition (S) on the segment $[0, T]$;
2. $b(s, V(s), \xi(s))$ summable on the segment $[0, T]$;
3. Differential $d\xi(s)$ coincides with the right side of equation (4).
The function $\varphi(s, v)$ is called a *structure of the solution* of equation (4).
It is essential that equation (4) is a deterministic equation, therefore, in the case when $V(s)$ is a random process, this equation (4) is a pathwise equation, which means that *both the coefficients of the equation and the solution do not have to be predictable.*

Symmetric integrals and SDE

Theorem 1. *Let $\sigma(t,\varphi)$ and $b(t,\varphi)$ be continuous. Suppose that function $\varphi(t,v)$, $\varphi(0,V(0))=\xi(0)$, has continuous derivatives $\varphi'_t(t,v)$, $\varphi'_v(t,v)$ and satisfy the system:*

$$\varphi'_t(t,V(t)) = b(t,\varphi(t,V(t))), \quad t\in[0,T], \qquad (P1)$$

$$\varphi'_v(t,V(t)) = \sigma(t,\varphi(t,V(t))), \quad t\in[0,T]. \qquad (P2)$$

Then $\xi(t)=\varphi(t,V(t))$ is a solution of equation (4).

Theorem 1 allows us to find a solution to equation (4):
$\varphi'_v(t,v)=\sigma(t,\varphi(t,v)) \Rightarrow \varphi(t,v)\equiv\varphi(t,v+z(t))$
$\Rightarrow \varphi'_t(t,v)|_{v=z(t)+V(t)} + \sigma(t,\varphi(t,z(t)+V(t)))\,z'(t) =$
$b(t,\varphi(t,z(t)+V(t),u(t)), \quad \varphi(0,z(0)+V(0))=\xi(0).$

Theorem 2. *Let function $\sigma(t,\varphi)$, $b(t,\varphi)$ and their partial derivatives $\sigma'_t(t,\varphi)$, $\sigma'_\varphi(t,\varphi)$ and $b'_\varphi(t,\varphi)$ be continuous. Then, if a solution of the Cauchy problem (4) exists, then it is unique.*

Thank you for your attention!