On stochastic differential inclusions of Leontieff type

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Content

Introduction

The notion of mean derivatives was introduced by E. Nelson in the 60s of the twentieth century for the needs of the so-called stochastic mechanics created by him.
The new method for studying dynamically distorted signals in electronic devices was developed based on algebraic differential equations called Leontieftype equations
In this work we study a Leontief-type inclusion with symmetric first-order mean derivative

Some facts from the theory of matrices

Definition

The expression $\lambda A + B$, where λ - a real or complex parameter is called the matrix pencil. The polynomial $\det(\lambda A + B)$ is called the characteristic polynomial of the pencil. The pencil is called regular, if its characteristic polynomial is not identically equal to zero.

Some facts from the theory of matrices

Definition

If the matrix pencil satisfies equation

$$rank(A) = deg(det(\lambda A + B)), \tag{1}$$

then we say that pencil satisfies the rank-degree condition.

Some facts from the theory of matrices

If the matrix pencil $\lambda A + B$ is regular, there exist two non-degenerate linear operators P (acts from the left side) and Q (acts from the right side), that reduce the matrices A and B to the canonical quasi-diagonal form. Provided that the pencil satisfies the rank-degree condition, in the canonical quasi-diagonal form the matrices PAQ and PBQ look like this

$$PAQ = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}; \;\; PBQ = \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix}, \tag{2}$$

where J is a certain non-degenerate matrix.

Preliminaries on the mean derivatives

Definition

(i) Forward mean derivative $D\xi(t)$ of $\xi(t)$ at time $t \in [0, T)$ - is an L_1-random variable of the form

$$D\xi(t) = \lim_{\Delta t \to +0} E_t^{\xi}\left(\frac{\xi(t+\Delta t) - \xi(t)}{\Delta t}\right) \tag{3}$$

(ii) Backward mean derivative $D_*\xi(t)$ of $\xi(t)$ at time $t \in (0, T]$ - is an L_1-random variable

$$D_*\xi(t) = \lim_{\Delta t \to +0} E_t^{\xi}\left(\frac{\xi(t) - \xi(t-\Delta t)}{\Delta t}\right) \tag{4}$$

Preliminaries on the mean derivatives

Definition

(iii) For an L^1-stochastic process $\xi(t)$, $t \in [0, T]$, its quadratic mean derivative $D_2\xi(t)$, is defined by the formula

$$D_2\xi(t) = \lim_{\triangle t \to +0} E_t^{\xi}\left(\frac{(\xi(t+\triangle t) - \xi(t))(\xi(t+\triangle t) - \xi(t))^*}{\triangle t}\right), \tag{5}$$

where $(\xi(t+\triangle t) - \xi(t))$ - is a column vector and $(\xi(t+\triangle t) - \xi(t))^*$ - is its conjugate, i.e., the row vector.

Preliminaries on the mean derivatives

$$\begin{cases} D_S\xi(t) = a(t,\xi(t)), \\ D_2\xi(t) = \alpha(t,\xi(t)), \end{cases} \tag{6}$$

First-order stochastic differential equation with symmetric mean derivative.

$$\begin{cases} D_S\xi(t) \in \mathbf{a}(t,\xi(t)), \\ D_2\xi(t) \in \boldsymbol{\alpha}(t,\xi(t)), \end{cases} \tag{7}$$

First-order stochastic differential inclusion with symmetric mean derivative.

Preliminaries on the mean derivatives

Definition

A differential inclusion (7) is said to have a solution with initial condition $\xi_0 \in \mathbb{R}^n$ if there exists a probability space and a random process $\xi(t)$ defined on it and taking values in $\mathbb{R}^n$ such that $\xi(0) = \xi_0$ and $\xi(t)$ satisfies the inclusion (7) a.s.

Definition

A perfect solution of an inclusion is a stochastic process with continuous sample trajectories such that it is a solution in the sense of the definition above, and the measure corresponding to it on the space of continuous curves is a weak limit of the measures generated by solutions of the sequence of diffusion-type Ito equations with continuous coefficients.

The main result

$$\begin{cases} D_S\left(A\xi(t) + \int_0^t w(s)ds\right) \in B\xi(t) + F(t) + w(t), \\ D_2\xi(t) = \Lambda(t). \end{cases} \tag{8}$$

Here $\Lambda = Q \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} Q^*$

1. Matrices A and B describe the device
2. deterministic multivalued function $F(t)$ - is the ingoing signal
3. the stochastc process $\xi(t)$ - is the outgoing signal.
4. wiener process w - ingoing "noise"

The main result

Definition

A multivalued map $F : X \to Y$ is called upper semicontinuous at a point $x \in X$ if for any open set $V \in Y$ such that $F(x) \in V$, there exists a neighborhood $U(x)$ of a point x such that $F(U(x)) \in V$.

Definition

For a given $\varepsilon > 0$, a continuous one-to-one mapping $f_e : X \to Y$ is called the ε-approximation of the multivalued mapping $F : X \to Y$ if the graph of the mapping f as a set in $X \times Y$ lies in an ε-neighborhood of the graph of F.

The main result

Theorem

Let A, B be arbitrary matrices, A be singular, B be non-singular. Assume that they form a pencil satisfying the rank-degree condition and let $F : [0, T] \to \mathbb{R}^n$ be an upper semicontinuous multivalued mapping with closed convex images, then any sequence of the ε-approximations $\varepsilon > 0$ of the multivalued map $F(t)$ generates a perfect solution of the stochastic differential inclusion (8) with any initial condition $\xi(0) = \xi_0$.

The main result

Lets consider some cost criterion:

$$G(\xi(\cdot)) = \int\limits_0^T g(t, \xi(t))dt \qquad (9)$$

Theorem

Among the perfect solutions of inclusion (8), there is a solution ξ on which the value of G is minimal.

The idea of proof

We introduce $\eta(t) = Q^{-1}\xi(t)$, then the system (8) takes the form

$$\begin{cases} D_S\left(AQ\eta(t) + \int_0^t w(s)ds\right) \in BQ\eta(t) + F(t) + w(t), \\ D_2\eta(t) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{10}$$

We multiply both parts of the first line of inclusion by P on the left:

$$\begin{cases} PAQD_S\eta(t) + Pw(t) \in PBQ\eta(t) + PF(t) + Pw(t), \\ D_2\eta(t) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \tag{11}$$

The idea of proof

Now from (2) it follows that (11) can be written as

$$\begin{cases} \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix} D_S\eta(t) \in \begin{pmatrix} J & 0 \\ 0 & I_{n-d} \end{pmatrix} \eta(t) + PF(t), \\ D_2\eta(t) = \begin{pmatrix} I_d & 0 \\ 0 & 0 \end{pmatrix}. \end{cases} \quad (12)$$

Denote the multivalued mapping $PF(t)$ by $\tilde{F}$ and note that (12) can be split into two independent systems

$$\begin{cases} D_S\eta^{(1)}(t) \in J\eta^{(1)}(t) + \tilde{F}^{(1)}(t) \\ D_2\eta^{(1)}(t) = I_d \end{cases} \quad (13)$$

in $\mathbb{R}^d$ and

$$\begin{cases} \eta^{(2)}(t) \in -\tilde{F}^{(2)}(t) \\ D_2\eta^{(2)}(t) = 0 \end{cases} \quad (14)$$

in $\mathbb{R}^{n-d}$.

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