Quantile hedging of Asian call options in (B, S) market with transaction costs

Sergey Gondin
joint work with Evgeny Pchelintsev

Tomsk State University

June 3, 2025
Divnomorskoe

Problem Statement

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a stochastic basis, $(B_t)_{0 \le t \le 1}$ is a risk-free asset, $(S_t)_{0 \le t \le 1}$ is a risky asset:

$$\begin{cases} dS_t = mS_t dt + \sigma S_t dW_t, \ S_0 > 0, \\ B_t = 1. \end{cases} \tag{1}$$

Let $\varepsilon \in (0,1)$. We need to find the smallest value $\widetilde{V}_0$ such that there exists an admissible strategy $(\widetilde{V}_0, \gamma^\varepsilon)$ satisfying the inequality

$$\mathbb{P}\left[\widetilde{V}_0 + \int_0^1 \gamma_u^\varepsilon dS_u \ge H\right] \ge 1 - \varepsilon, \tag{2}$$

where

$$H = \left(\int_0^1 S_t dt - K\right)_+ .$$

Related Works

- Föllmer H. Quantile hedging / H. Föllmer, P. Leukert // Finance Stochast. – 1999. – Vol. 3. – P. 251–273.
- Pergamenshchikov S. Limit theorem for Leland's strategy // Ann. Appl. Probab. – 2003. – Vol. 13, is. 3. – P. 1099–1118.
- Мурзинцева А. А. Задача хеджирования азиатских опционов купли с транзакционными издержками / А. А. Мурзинцева, С. М. Пергаменщиков, Е. А. Пчелинцев // Теория вероятностей и ее применения. – 2023. – Т. 68, № 2. – С. 253–276.

When passing to the equivalent martingale measure $\mathbb{Q}$, defined by the equality

$$\left.\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{P}}\right|_t = \exp\left\{-\frac{m}{\sigma}W_t - \frac{1}{2}\left(\frac{m}{\sigma}\right)^2 t\right\}, \tag{3}$$

the SDE describing the dynamics of the price of a risky asset will take the following form:

$$dS_t = \sigma S_t dW_t^{\mathbb{Q}}. \tag{4}$$

For this equation we can also write the solution:

$$S_t = S_0 \exp\left\{\sigma W_t^{\mathbb{Q}} - \frac{\sigma^2}{2}t\right\}, \tag{5}$$

where the process $W_t^{\mathbb{Q}} = W_t + \frac{m}{\sigma}t$ is a Wiener process with respect to the measure $\mathbb{Q}$.

Quantile Hedging of Asian Option

Problem (2) can be reduced to the following:

$$\begin{cases} \mathbb{E}^{\mathbb{Q}}\left[H\mathbf{1}_A\right] \longrightarrow \min_{A\in\mathcal{F}}, \\ \mathbb{P}\left[A\right] \geq 1-\varepsilon. \end{cases} \tag{6}$$

We have the following forms of set $A \in \mathcal{F}$:

$$A = \{W_1^{\mathbb{Q}} < b\}, \text{ when } m \leq \sigma^2$$

and

$$A = \{W_1^{\mathbb{Q}} < b_1\} \cup \{W_1^{\mathbb{Q}} > b_2\}, \text{ when } m > \sigma^2.$$

The values b, b_1, b_2 are determined from the equation $\mathbb{P}[A] = 1-\varepsilon$.

Quantile Hedging of Asian Option

Apply the martingale representation theorem to $M_t = \mathbb{E}^{\mathbb{Q}}[H\mathbf{1}_A \mid \mathcal{F}_t]$. Then:

$$M_t = \mathbb{E}^{\mathbb{Q}}\left[H\mathbf{1}_A\right] + \int_0^t g_s dW_s^{\mathbb{Q}}, \tag{7}$$

where g_t – $\mathbb{F}$ – adopted process, which is defined as

$$g_t = \frac{d}{dt}\langle M, W^{\mathbb{Q}}\rangle_t. \tag{8}$$

Put

$$\gamma_t^{\varepsilon} = \frac{g_t}{\sigma S_t} \text{ and } \beta_t^{\varepsilon} = \mathbb{E}^{\mathbb{Q}}\left[H\mathbf{1}_A\right] + \sigma\int_0^1 \gamma_t^{\varepsilon} S_t dW_t^{\mathbb{Q}} - \gamma_t^{\varepsilon} S_t. \tag{9}$$

Thus, we obtain a quantile self-financing strategy $\pi^{\varepsilon} = (\beta_t^{\varepsilon}, \gamma_t^{\varepsilon})_{0\le t\le 1}$.

$$d\widetilde{V}_t = \sigma\gamma_t^{\varepsilon} S_t dW_t^{\mathbb{Q}} = \gamma_t^{\varepsilon} dS_t, \qquad \widetilde{V}_0 = \mathbb{E}^{\mathbb{Q}}\left[H\mathbf{1}_A\right]. \tag{10}$$

Quantile Hedging of Asian Option

In our case

$$M_t = \widetilde{G}(t, \xi_t, S_t), \qquad \text{where} \quad \widetilde{G}(t, x, y) = \mathbb{E}^{\mathbb{Q}}\left[(x + y\eta_v - K)_+ \mathbf{1}_A\right], \tag{11}$$

where $\xi_t = \int_0^t S_u du$, $\eta_v = \int_0^v \exp\{\sigma W_u^{\mathbb{Q}} - \frac{\sigma^2 u}{2}\} du$, $v = 1 - t$.
Then

$$\gamma_t^{\varepsilon} = \widetilde{G}'_y(t, \xi_t, S_t). \tag{12}$$

Proposition

The function $\widetilde{G}(t, x, y)$ is an unique classical solution of the PDE

$$\begin{cases} \widetilde{G}'_t(t, x, y) + y\widetilde{G}'_x(t, x, y) + \dfrac{\sigma^2 y^2}{2}\widetilde{G}''_{yy}(t, x, y) = 0, \\ \widetilde{G}(1, x, y) = (x - K)_+ \mathbb{Q}(A). \end{cases} \tag{13}$$

Quantile Price for Asian Option

Now, $\forall a \in \mathbb{R}$ we define

$$F(a) = \int_0^1 \exp\{\sigma \widetilde{B}_u + \sigma u a\} du, \tag{14}$$

where $\widetilde{B}_u = W_u^{\mathbb{Q}} - uW_1^{\mathbb{Q}} - \sigma u/2$.

Theorem 1

If $m \leq \sigma^2$ *then*

$$C_\varepsilon = \int_{K/S_0}^{\infty} (S_0 z - K)\rho(z) dz. \tag{15}$$

If $m > \sigma^2$ *then*

$$C_\varepsilon = \int_{K/S_0}^{\infty} (S_0 z - K)\psi(z) dz. \tag{16}$$

Quantile Price for Asian Option

Here

$$\rho(z) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\varphi(a(z))}{F_1(a(z))}\mathbf{1}_{\{z<F(b)\}}\right], \tag{17}$$

$$\psi(z) = \mathbb{E}^{\mathbb{Q}}\left[\frac{\varphi(a(z))}{F_1(a(z))}\mathbf{1}_{\{z<F(b_1)\}\cup\{z>F(b_2)\}}\right], \tag{18}$$

$\varphi(\cdot)$ is the density function of the normal distribution with parameters $(0, 1)$, $F_1(a)$ is the derivative of the function $F(a)$ with respect to parameter a.

Quantile Hedging of Asian Option with Transaction Costs

The value of the portfolio at time t with initial capital $\widetilde{V}_0$ is described as follows:

$$\widetilde{V}_t^n = \widetilde{V}_0 + \int_0^t \gamma_u^n \, \mathrm{d}S_u - \kappa_n J_n, \tag{19}$$

where κ_n is the proportional transaction coefficient, J_n is a total trading volume

$$J_n = \sum_{j=1}^{n} S_{t_j} |\gamma_{t_j}^n - \gamma_{t_{j-1}}^n|. \tag{20}$$

In this case, the volume of the risky asset in the strategy π^ε is defined as:

$$\gamma_t^n = \sum_{j=1}^{n} \mathbf{1}_{\{t_{j-1} \leqslant t < t_j\}} \widehat{G}_y'(t_{j-1}, \xi_{t_{j-1}}, S_{t_{j-1}}), \; t_j = \frac{j}{n}, \; j = 0, \ldots, n, \tag{21}$$

where $\widehat{G}(t, x, y)$ is the solution to problem (13), with $\widehat{\sigma}^2 = \sigma^2 + \sigma\sqrt{n}\kappa_n\sqrt{\frac{8}{\pi}}$ volatility parameter.

Theorem 2

Let the transaction coefficient κ_n *be such that*

$$\limsup_{n\to\infty} n^{7/18}\kappa_n = 0. \tag{22}$$

Then the portfolio value (19) *for strategy* (21) *converges in probability to the modified payoff function, i.e.*

$$\mathbb{P}\text{-}\lim_{n\to\infty} \widetilde{V}_1^n = H\mathbf{1}_A. \tag{23}$$

In the case with transaction costs, the price of an Asian option is determined as follows

$$\widehat{C}_\varepsilon = \begin{cases} \displaystyle\int_{K/S_0}^{\infty} (S_0 z - K)\widehat{\rho}(z)dz, \ \sigma^2 \geq m, \\ \displaystyle\int_{K/S_0}^{\infty} (S_0 z - K)\widehat{\psi}(z)dz, \ \sigma^2 < m. \end{cases} \tag{24}$$

Here $\widehat{\rho}(\cdot)$ and $\widehat{\psi}(\cdot)$ are densities with parameter $\widehat{\sigma}$, defined in (17) and (18).

Quantile Hedging of Asian Option with Transaction Costs

Theorem 3

If $\lim\limits_{n\to\infty} \sqrt{n}\kappa_n = \infty$, *then*

$$\lim_{n\to\infty} \widehat{C}_\varepsilon = S_0. \tag{25}$$

If $\lim\limits_{n\to\infty} \sqrt{n}\kappa_n = 0$, *then*

$$\lim_{n\to\infty} \widehat{C}_\varepsilon = C_\varepsilon. \tag{26}$$

Monte-Carlo Simulations

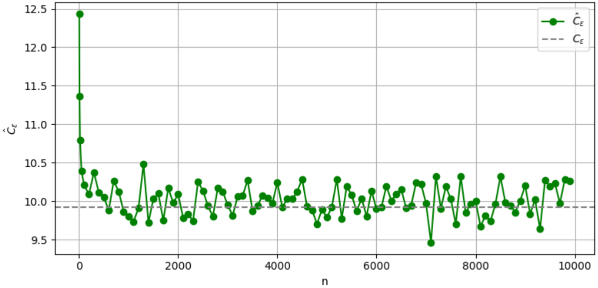
Table 1: Comparison of Asian option prices for $S_0 = 100$, $K = 100$,$\sigma = 0.5$, $m = 0.25$ ($m \leq \sigma^2$).

ε	0.1	0.05	0.025	0.01
C_*	11.49	11.49	11.49	11.49
C_ε	8.76	10.09	10.77	11.14

Table 2: Comparison of Asian option prices for $S_0 = 100$, $K = 100$, $\sigma = 0.5$, $m = 1$ ($m > \sigma^2$).

ε	0.1	0.05	0.025	0.01
C_*	11.5	11.5	11.5	11.5
C_ε	10.78	11.14	11.31	11.42

Monte-Carlo Simulations



Pic. 1: Convergence of $\hat{C}_\varepsilon$ for $\varepsilon = 0.05$, $m = 0.25$, $\sigma = 0.5$, and $\kappa_n = 1/n^{3/2}$.

Thanks for your attention!