On the Cauchy problem for generalized stochastic and deterministic Korteweg-de Vries equations.

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ICSM-10
2025

Introduction

Generalized Korteweg-de Vries equation (GKdV)

$$u_t + (f(u))_x + u_{xxx} = 0, \quad u(x,0) = u_0(x), \; x \in R$$

Stochastic Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = f(x,t) + F(u)\frac{\partial^2 B}{\partial t \partial x}, \quad u(x,0) = u_0(x), x \in R$$

$$du + \left(\frac{\partial^3 u}{\partial x^3} + u\frac{\partial u}{\partial x}\right) dt = f(x,t)dt + F(u)dW, \; u(x,0) = u_0(x), x \in R$$

[1]. *A de Bouard, A Debussche* On the Stochastic Korteweg–de Vries Equation // Journal of Functional Analysis, Volume 154, Issue 1, 1 April 1998, Pages 215-251.
[2]. *Debussche A.* Numerical simulation of the stochastic Kordeweg-de Vries equation/ A. Debussche, J. Printems //Physica D 134. – 1999. – 200-226.

Goals and objectives of the project

The purpose of this research

is to construct analytical solutions of sGKdV equations using solutions of corresponding classical equations and using the technique of symmetric integrals and to study the effect of noise on various terms of equations [3][4]:

1. noise acts simultaneously on the nonlinear and dispersion terms of the equation;
2. noise only affects the dispersion term;
3. noise only affects the nonlinear term;
4. noise affects the right side of the equation.

[3] Nasyrov F., Suchkova D. *ON THE CAUCHY PROBLEM OF STOCHASTIC AND DETERMINISTIC EQUATIONS OF KORTEWEG DE VRIES TYPE* Journal of Math. S. 10.1007/s10958-025-07683-7. 2025.
[4] Suchkova D.A. *On Generalized Stochastic Korteweg–de Vries Equations with Time Noise.* Abstracts of papers presented at ICSM-9,

Case 1. Noise acts simultaneously on the nonlinear and dispersion terms of the equation

Stochastic GKdV with time noise in the nonlinear and the dispersion term

$$u_t + ((f(u))_x + u_{xxx})(1 + V'(t)) = 0, \quad u(x,0) = u_0(x), \ x \in R$$

$$d(u)_t + [(f(u))_x + u_{xxx}]dt + [(f(u))_x + u_{xxx}] * dV(t) = 0. \qquad (1.1)$$

1.

$$(u)_t dt = u_t dt + u_v * dV(t) \quad [5] \qquad (1.2)$$

$$[u_t + (f(u))_x + u_{xxx}]dt + [u_v + (f(u))_x + u_{xxx}] * dV(t) = 0. \qquad (1.3)$$

The solution of (1.3) is a function of three variables $u(x, t, V(t))$, $u(x, 0, V(0)) = u_0(x)$, where $u(x, t, v)$ is some sufficiently smooth function whose differential with a symmetric integral satisfies equation (1.3).

[5] *Nasyrov F.S.* Local times, symmetric integrals and stochastic analysis. Moscow: Fizmatlit, 2011

Case 1. Noise acts simultaneously on the nonlinear and dispersion terms of the equation

Using the lemma on the equality of two integrals, equation (1.3) is equivalent to two relations:

$$u_t(x,t,v)|_{v=V(t)} + (f(u(x,t,V(t)))_x + u_{xxx}(x,t,V(t)) = 0, \tag{1.4}$$

$$u_v(x,t,v)|_{v=V(t)} + (f(u(x,t,v))_x|_{v=V(t)} + u_{xxx}(x,t,v)|_{v=V(t)} = 0. \tag{1.5}$$

2. From relations (1.4) and (1.5), we obtain the equation:

$$u_t = u_v, \tag{1.6}$$

the general solution of which has the form of a traveling wave $u(x,t,v) = \varphi(x,t+v)$. To find the function $\varphi(.)$, we substitute it into equation (1.5). The resulting deterministic equation coincides with the classical GKdV equation. In what follows, we will consider only deterministic versions of the equations, setting $v = V(t)$ in them.

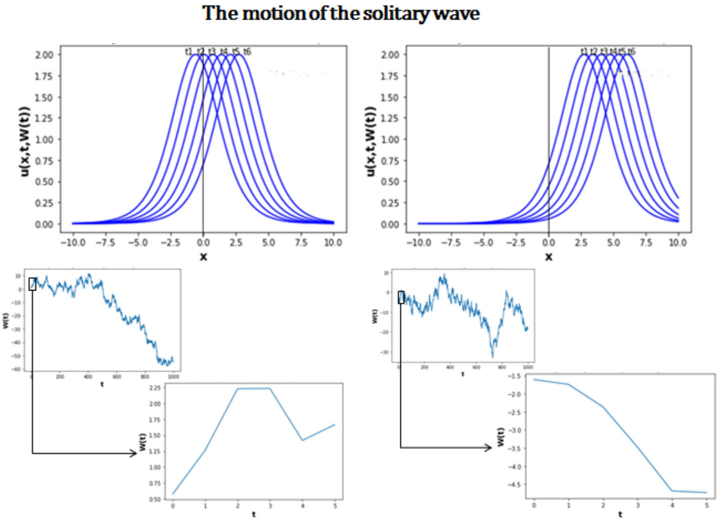
Case 1. Noise acts simultaneously on the nonlinear and dispersion terms of the equation

So we got the following result

Theorem 1.

Let $V(t)$, $V(0) = 0$ be a continuous function, and $\varphi(x,t)$ is a solution to the deterministic GKdV equation.
Then the function
$u(x, t, V(t)) = \varphi(x, t + V(t))$
is the solution stochastic GKdV equation with time noise in the nonlinear and the dispersion terms (1.1).

Case 1. Noise acts simultaneously on the nonlinear and dispersion terms of the equation



Case 2. Noise only affects the dispersion term

Stochastic GKdV with time noise in the dispersion term

$$u_t + (f(u))_x + V'(t)u_{xxx} = 0, \quad u(x,0) = u_0(x), \;\; x \in R$$

$$u(x,t) = u(x,t,V(t))$$

$$d(u)_t + (f(u))_x dt + u_{xxx} * dV(t) = 0, \quad u(x,0,V(0)) = u_0(x) \qquad (2.1)$$

1.

$$(u)_t dt = u_t dt + u_v * dV(t) \;\; [5] \qquad (2.2)$$

$$[u_t + (f(u))_x]dt + [u_v + u_{xxx}] * dV(t) = 0, \quad u(x,0,V(0)) = u_0(x) \quad (2.3)$$

The solution of (2.3) is a function of three variables $u(x,t,V(t))$, $u(x,0,V(0)) = u_0(x)$, where $u(x,t,v)$ is some sufficiently smooth function whose differential with a symmetric integral satisfies equation (2.3).

[5] *Nasyrov F.S.* Local times, symmetric integrals and stochastic analysis. Moscow: Fizmatlit, 2011

Case 2. Noise only affects the dispersion term

Using the lemma on the equality of two integrals, equation (2.3) is equivalent to two relations:

$$u_t(x,t,v)|_{v=V(t)} + (f(u(x,t,V(t)))_x = 0, \tag{2.4}$$

$$u_v(x,t,v)|_{v=V(t)} + u_{xxx}(x,t,V(t)) = 0. \tag{2.5}$$

In what follows, we will consider only deterministic versions of the equations, setting $v = V(t)$ in them.

$$u_t + (f(u))_x = 0, \tag{2.6}$$

$$u_v + u_{xxx} = 0, \tag{2.7}$$

2. Using the method of characteristics, we solve the Cauchy problem for the equation (2.6) at $v = 0$:

$$u(x,t,0) = u_0(x - f'(u(x,t,0))t). \tag{2.8}$$

Case 2. Noise only affects the dispersion term

3. $Ai(z) = \frac{1}{\pi}\int\limits_0^\infty \cos\left(\frac{y^3}{3} + yz\right) dy$ – the Airy function of the first kind [5].

The solution of the Cauchy problem for equation (2.7) is constructed using the Airy transform for the linearized KdV equation:

$$u(x,t,v) = (3v)^{-1/3}\int\limits_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3v)^{1/3}}\right) u(y,t,0)\, dy.$$

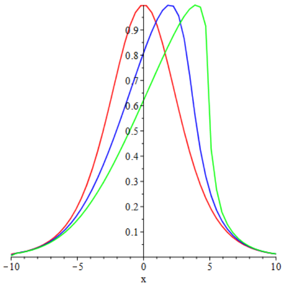
Theorem 2.

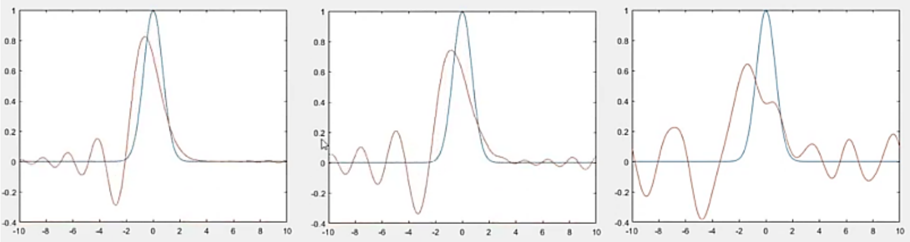
Let $V(t)$, $V(0) = 0$ be a continuous function. The Cauchy problem for (2.1) has a solution until the moment of destruction of the solution of the Cauchy problem for the inviscid Burgers equation:

$$u(x,t,V(t)) \\ = (3\ \mathrm{V(t)})^{-1/3}\int\limits_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3V(t))^{1/3}}\right) u_0(y - f'(u(y,t,0))t)\, dy.$$

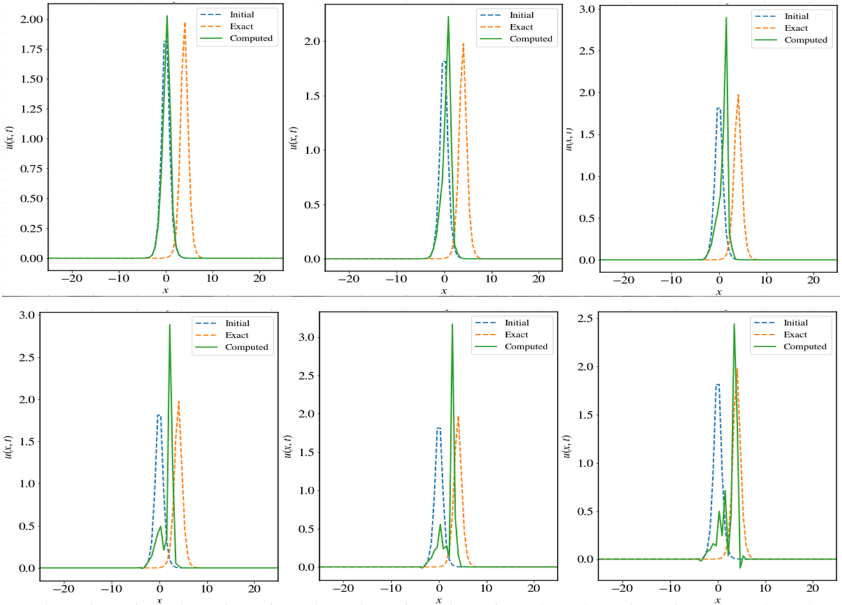
[5] *Olivier Vallee, Manuel Soares Airy functions and applications to physics.* — World Scientific Publishing Co. Pte.Ltd, 2004

Solutions of the inviscid Burgers and linearized Korteweg-de Vries equations





Case 2. Noise only affects the dispersion term



Case 3. Noise only affects the nonlinear term

Stochastic GKdV with time noise in the nonlinear term

$$u_t + V'(t)(f(u))_x + u_{xxx} = 0, \quad u(x,0) = u_0(x), \ x \in R$$

$$u(x,t) = u(x,t,V(t))$$

$$d(u)_t + (f(u))_x * dV(t) + u_{xxx}dt = 0, \quad u(x,0,V(0)) = u_0(x) \tag{3.1}$$

1.

$$(u)_t dt = u_t dt + u_v * dV(t) \ \ [5] \tag{3.2}$$

$$[u_t + u_{xxx}]dt + [u_v + (f(u))_x] * dV(t) = 0. \tag{3.3}$$

The solution of (3.3) is a function of three variables $u(x,t,V(t))$, where $u(x,t,v)$ is some sufficiently smooth function whose differential with a symmetric integral satisfies equation (3.3).

[5] *Nasyrov F.S.* Local times, symmetric integrals and stochastic analysis. Moscow: Fizmatlit, 2011

Case 3. Noise only affects the nonlinear term

Using the lemma on the equality of two integrals, equation (3.3) is equivalent to two relations:

$$u_t(x,t,v)|_{v=V(t)} + u_{xxx}(x,t,V(t)) = 0, \tag{3.4}$$

$$u_v(x,t,v)|_{v=V(t)} + (f(u))_x(x,t,V(t)) = 0. \tag{3.5}$$

In what follows, we will consider only deterministic versions of the equations, setting $v = V(t)$ in them.

$$u_t + u_{xxx} = 0, \tag{3.6}$$

$$u_v + (f(u))_x = 0, \tag{3.7}$$

2. Using the method of characteristics, we solve the Cauchy problem for the equation (3.7) at $t = 0$:

$$u(x,0,v) = u_0(x - f'(u(x,0,v))v). \tag{3.8}$$

Case 3. Noise only affects the nonlinear term

3. The solution of the Cauchy problem for equation (3.6) is constructed using the Airy transform for the linearized KdV equation:

$$u(x,t,v) = (3t)^{-1/3} \int_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3t)^{1/3}}\right) u(y,0,v)\,dy \qquad (3.9)$$

4. Let $V(t) \equiv t$ in problem (3.3), $\widetilde{u}(x,t) = u(x,0,t)$. Suppose that the function $\widetilde{u}(x,t)$ is a solution to the Cauchy problem

$$\widetilde{u}_t(x,t) + f'(\widetilde{u}(x,t))\widetilde{u}_x(x,t) = 0, \quad \widetilde{u}(x,0) = \varphi(x). \qquad (3.10)$$

Then the solution of the Cauchy problem for the deterministic GKdV equation has the form:

$$u(x,t) = (-3t)^{-\frac{1}{3}} \int_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3t)^{\frac{1}{3}}}\right) \widetilde{u}(y,t)\,dy. \qquad (3.11)$$

Case 3. Noise only affects the nonlinear term

Theorem 3.

Let $V(t)$, $V(0) = 0$ be a continuous function.

I. Then the Cauchy problem for the stochastic GKdV equation with time noise in the nonlinear term (3.1) has a solution until the moment of destruction of the solution of the Cauchy problem for the inviscid Burgers equation: $u(x, t, V(t))$

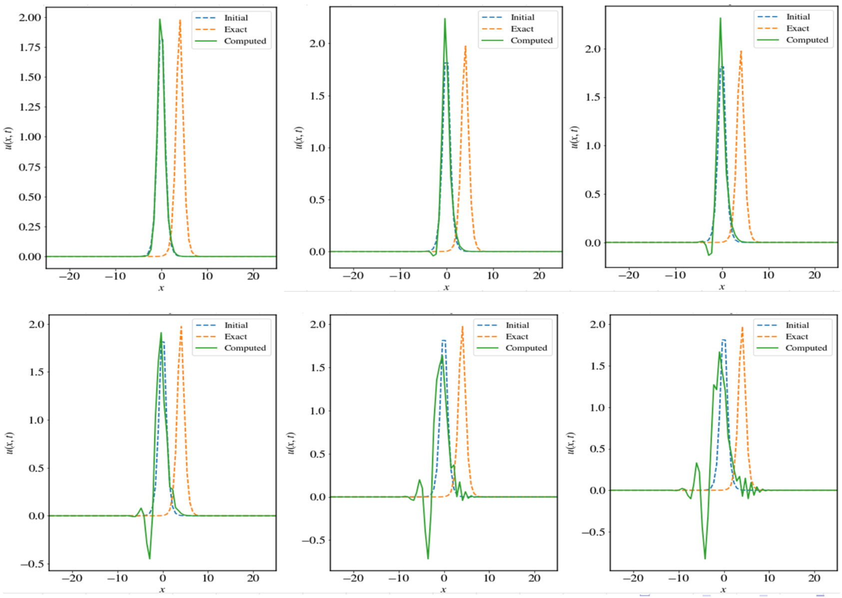
$$=(3\mathrm{t})^{-1/3} \int_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3t)^{1/3}}\right) u_0(y - f'(u(y, 0, V(t))V(t))\, dy.$$

II. Let $\widetilde{V}(t) \equiv t$ in Cauchy problem for the equation (3.1), then the function

$$u(x, t) = (-3t)^{-\frac{1}{3}} \int_{-\infty}^{+\infty} Ai\left(\frac{x-y}{(3t)^{\frac{1}{3}}}\right) \widetilde{u}(y, t)\, dy$$

defines the solution to the Cauchy problem for the deterministic GKdV equation.

Case 3. Noise only affects the nonlinear term



Case 4. Noise only affects the right side of the equation

Stochastic GKdV with time noise in the right side of the equation

$$u_t + (f(u))_x + u_{xxx} = r(x,t,V(t))V'(t), \quad u(x,0) = u_0(x), \ x \in R$$

$$u(x,t) = u(x,t,V(t))$$

$$d(u)_t + [(f(u))_x + u_{xxx}]dt - r(x,t,V(t,x)) * dV(t) = 0, \ \ u(x,0,V(0)) = u_0(x) \tag{4.1}$$

1.

$$(u)_t dt = u_t dt + u_v * dV(t) \tag{4.2}$$

$$(u_t + (f(u))_x + u_{xxx})dt + (u_v - r(x,t,V(t,x))) * dV(t) = 0. \tag{4.3}$$

The solution of (4.3) is a function of three variables $u(x,t,V(t))$, $u(x,0,V(0)) = u_0(x)$, where $u(x,t,v)$ is some sufficiently smooth function whose differential with a symmetric integral satisfies equation (4.3).

Case 4. Noise only affects the right side of the equation

$$u_t + (f(u))_x + u_{xxx} = 0, \tag{4.3}$$

$$u_v = r. \tag{4.4}$$

From equation (4.4) we obtain:

$$u(x,t,V(t)) = G(x,t,V(t)) + \widetilde{u}(x,t), \tag{4.5}$$

where $G(x,t,V(t)) = \int r(x,t,V(t))dv$, and $\widetilde{u}(x,t)$ – arbitrary function to be defined. Substituting expression (4.5) into (4.3), we obtain

$$\widetilde{u}_t + (f(\widetilde{u}))_x + \widetilde{u}_{xxx} = -[G_t + (f(G))_x + G_{xxx}], \tag{4.6}$$

here the right side of equation (4.6) is a known function.

Theorem 4.

Let $V(t)$, $V(0) = 0$ be a continuous function, $\widetilde{u}(x,t)$ be a solution to equation (4.6). Then formula (4.5) provides a solution to the stochastic GKdV with time noise in the right side of the equation.

- Debussche A., de Bouard A. *On the Stochastic Korteweg–de Vries Equation* // Journal of Funct. Anal. 1998. Vol. 154. Is 1. pp. 215-251.
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- Nasyrov F.S. *Local times, symmetric integrals and stochastic analysis* — Moscow: Fizmatlit. 2011. (in Russian)
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- Nasyrov F., Suchkova D. *ON THE CAUCHY PROBLEM OF STOCHASTIC AND DETERMINISTIC EQUATIONS OF KORTEWEG DE VRIES TYPE* Journal of Math. S. 10.1007/s10958-025-07683-7. 2025.
- Suchkova D.A. *On Generalized Stochastic Korteweg–de Vries Equations with Time Noise.* Abstracts of papers presented at ICSM-9, Theory Probab. and its appl., Vol. 69, No. 4, 2024.

Thank you for your attention!

Let $V(s)$, $s \in [0, +\infty)$, be an arbitrary continuous function, then *symmetric integral* is called

$$\int_0^t f(s, V(s)) * dV(s) =$$

$$\lim_{n\to\infty} \sum_k \frac{1}{\Delta t_k^{(n)}} \int_0^t f(s, V^{(n)}(s))\, ds \Delta V_k^{(n)}$$

$$= \lim_{n\to\infty} \int_0^t f(s, V^{(n)}(s))(V^{(n)})'(s)\, ds,$$

where $V^{(n)}(s)$ are broken lines constructed by the function $V(s)$ and partitions $\{t_k^{(n)}\}$ of the interval $[0, t]$ such that $\max_k (t_k^{(n)} - t_{k-1}^{(n)}) \to 0$ as $n \to \infty$.

We will say that for a pair of functions $(V(s), f(s,u))$ the condition (S) is true if the following assumptions are satisfied:
(a) $V(s)$, $s \in [0,t]$, is a continuous function,
(b) For a. u function $f(s,u)$, $s \in [0,t]$, is continuous on the right and has bounded variation,
(c) Total variation $|f|(t,u)$ with respect to variable s of function $f(s,u)$ on $[0,t]$ is locally summable with respect to variable u,
(d) For a.e. u $\int_0^t \mathbf{1}(s : V(s) = u)|f|(ds,u) = 0$, where $\mathbf{1}(A)$ is the indicator of set A, i.e. function equal to 1 on A and 0 outside A.
It is known that the condition (S) is a sufficient condition for the existence of symmetric integrals $\int_0^t f(s, V(s)) * dV(s)$.

Some properties of the symmetric integral.

1. Let condition (S) be true for $(V(s), f(s,u))$, then

$$\int_{t_0}^{t} f(s, V(s)) * dV(s) = \int_{V(t_0)}^{V(t)} f(t,u)\, du$$

$$- \int_R \int_{t_0}^{t} \kappa(u, V(t_0), V(s)) f(ds, u)\, du,$$

where $\kappa(u,a,b) = sgn(b-a)\mathbf{1}(a \wedge b < v < a \vee b)$. In particular, if $f(t,u)$ is absolutely continuous in t for a.e. u, then

$$\int_0^t f(s,V(s)) * dV(s) = \int_{V(0)}^{V(t)} f(t,u)\,du - \int_0^t \int_{V(0)}^{V(s)} f'_s(s,v)dvds.$$

2. Let the function $F(t,u)$ have continuous partial derivatives F'_t and F'_u, then there exists a symmetric integral $\int_0^t F'_u(s,V(s)) * dV(s)$ and the formula is valid

$$F(t,V(t)) - F(0,V(0))$$

$$= \int_0^t F'_s(s,V(s))\,ds + \int_0^t F'_u(s,V(s)) * dV(s).$$

Thus, for the symmetric integral there is a differential corresponding to the stochastic differential over the Wiener process with the Stratonovich integral.

3. **Lemma 1.** (on the equality of two integrals)

Let $x(s)$, $s \in [0, T]$, *be a continuous nowhere differentiable function, and let the continuous functions* $f(s, v)$ *and* $G(s, v)$, $(s, v) \in [0, T] \times R$, *satisfy the following assumptions:*

- *Function(* $G(s, x(s))$, $s \in [0, T]$, *be summable;*
- *Function* $f(s, v)$ *for each* s *is summable with respect to variable* $v \in R$ *and has continuous derivative* $f'_s(s, v)$ *satisfying condition* $\int_R \int_0^T |f'_s(s, v)| dsdv < \infty$.

Then the condition

$$\int_0^t f(s, x(s)) * dx(s) = \int_0^t G(s, x(s)) ds, \quad t \in [0, T],$$

is equivalent to the condition

$$f(s, x(s)) = G(s, x(s)) = 0, \quad s \in [0, T].$$