Fractional telegraph equation
and
fractionally integrated telegraph processes

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Telegraph process

Let $\varepsilon = \varepsilon(t) \in \{0, 1\}$, $t \geq 0$, be a two-state càdlàg Markov process with the infinitesimal generator

$$\Lambda = \begin{pmatrix} -\lambda_0 & \lambda_0 \\ \lambda_1 & -\lambda_1 \end{pmatrix}, \qquad \lambda_0, \lambda_1 > 0.$$

Let $c_0, c_1 \in (-\infty, \infty)$, $c_0 > c_1$.

The two-state process $t \to c_{\varepsilon(t)}$, $t \geq 0$, is called a telegraph process.

Integrated telegraph process

The piecewise linear random process that arises after classical integration,

$$\mathscr{T}(t) = \int_0^t c_{\varepsilon(s)} \mathrm{d}s = \sum_{n=0}^{N(t)-1} c_{\varepsilon(\tau_n)} \Delta\tau_n + c_{\varepsilon(\tau_{N(t)})}(t - \tau_{N(t)}), \qquad \Delta\tau_n = \tau_{n+1} - \tau_n,$$

is called the *integrated telegraph process*. Here $\{\tau_n\}_{n\geq 1}$, $0 < \tau_1 < \tau_2 < \ldots$ are the switching times of $\varepsilon = \varepsilon(t)$, $\tau_0 = 0$, and $N = N(t)$ is the counting Poisson process.

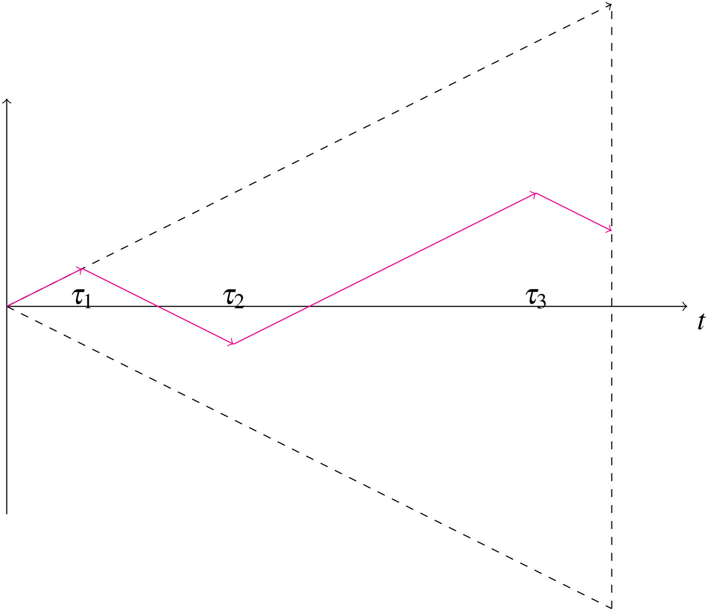

Figure: A sample path of the telegraph process $\mathscr{T}(t)$.

Process $\mathscr{T}(t)$ is not Markov, but the process $(\mathscr{T}(t), \varepsilon(t))$ is.

The transition matrix $\Pi(t) = (\pi_{ij}(t))_{i,j\in\{0,1\}}$, $\pi_{ij}(t) = \mathbb{P}_i\{\varepsilon(t) = j\}$, for the Markov process $\varepsilon = \varepsilon(t)$ has the form

$$\Pi(t) = \exp(t\Lambda) = \frac{1}{2\lambda}\begin{pmatrix}\lambda_1 + \lambda_0 e^{-2\lambda t} & \lambda_0 - \lambda_0 e^{-2\lambda t} \\ \lambda_1 - \lambda_1 e^{-2\lambda t} & \lambda_0 + \lambda_1 e^{-2\lambda t}\end{pmatrix}, \qquad 2\lambda := \lambda_0 + \lambda_1.$$

Here by $\mathbb{P}_i\{\cdot\}$, denotes the conditional probability at the initial state, $\mathbb{P}_i\{\cdot\} = \mathbb{P}\{\cdot \mid \varepsilon(0) = i\}$.

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Telegraph equation

In the symmetric case $c_0 = -c_1 = c, c > 0$, $\lambda_0 = \lambda_1 = \lambda$, the probability density function $p(t,x)$ of $\mathscr{T}(t)$ satisfies the telegraph (or damped wave) equation,

$$\frac{\partial^2 u}{\partial t^2}(t,x) + 2\lambda \frac{\partial u}{\partial t}(t,x) = c^2 \frac{\partial^2 u}{\partial x^2}(t,x), \qquad t > 0.$$

If $c, \lambda \to +\infty$ and $c^2/\lambda \to 1$ (Kac's scaling), then the telegraph equation formally becomes

$$\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(t,x), \qquad t > 0.$$

Moreover,

$$\mathscr{T} \xrightarrow{D} W.$$

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The fractional telegraph equation and related stochastic processes have become very popular for research since about the 1980s.

See, for example,

Wyss,W.: Fractional diffusion equation. J. Math. Phys. 27, 2782–2785 (1986)

Schneider, W.R., Wyss, W.: Fractional diffusion and wave equations. *J. Math. Phys.* 30, 134–144 (1989)

Fujita, Y.: Integrodifferential equation which interpolates the heat equation and the wave equation (I). *Osaka J. Math.* 27, 309–321 (1990)
Fujita, Y.: Integrodifferential equation which interpolates the heat equation and the wave equation (II). *Osaka J. Math.* 27, 797–804 (1990)

Anh,V.V., Leonenko, N.N.: Scaling laws for fractional diffusion-wave equation with singular data. *Statistics and Probability Letters* 48, 239–252 (2000)

Orsingher E., Beghin L. Time-fractional telegraph equations and telegraph processes with brownian time. *Probab. Theory Relat. Fields* 2004; 128, 141–160.
Orsingher E., Beghin L. Fractional diffusion equations and processes with randomly varying time. *Ann. Probab.* 2009; 37(1), 206–249.

Fractional telegraph equation

For example: the equation

$$\mathscr{D}_t^{2\alpha} u(t,x) + 2\lambda \cdot \mathscr{D}_t^{\alpha} u(t,x) = c^2 \frac{\partial^2 u}{\partial x^2}(t,x) \qquad \text{for } 0 < \alpha \leq 1,$$

can be solved by applying the Fourier transform.
In the particular case $\alpha = 1/2$, it is proved that the fundamental solution coincides with the distribution of the classically integrated telegraph process with Brownian time, i.e.

$$W(t) = \mathscr{T}(|B(t)|), \qquad t > 0,$$

E.Orsingher, L.Beghin (2004).
Here $|B(t)|$ is a reflecting Brownian motion independent of $\mathscr{T}$.

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Popular methodology for implementing fractionality

- Calculus:

classic equation $\Rightarrow$ fractional equation $\Rightarrow$ *analysis*

- Stochastics: Markov process $\Rightarrow$ Kolmogorov equation

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Fractional telegraph process

Instead of the classic integrated telegraph process $\mathscr{T}(t) = \int_0^t c_{\varepsilon(s)} \mathrm{d}s$, we study a fractionally integrated telegraph process.

Definition

The fractionally integrated telegraph process $\mathscr{F}^\alpha(t)$ is defined as

$$\mathscr{F}^\alpha(t) = I^\alpha[c_{\varepsilon(\cdot)}](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-r)^{\alpha-1} c_{\varepsilon(r)} \mathrm{d}r, \qquad 0 < t < \infty,$$

where $t \to c_{\varepsilon(t)}$ is a two-state telegraph process, $\alpha \in (0,1)$.

For any fixed $t > 0$, consider also the process

$$\mathscr{F}^\alpha(s,t) = \frac{1}{\Gamma(\alpha)} \int_0^s (t-r)^{\alpha-1} c_{\varepsilon(r)} \mathrm{d}r, \qquad 0 < s < t.$$

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Fractional telegraph process

The process $\mathscr{F}^{\alpha}(t)$ is a continuous piecewise-deterministic process of the form

$$\mathscr{F}^{\alpha}(t) = \begin{cases} \sum_{n=0}^{N(t)-1} \phi_n(\tau_{n+1};t) + \phi_{N(t)}(t;t), & \text{if } N(t) \geq 1 \ , \\ c_{\varepsilon(0)} \dfrac{t^{\alpha}}{\Gamma(1+\alpha)} = \phi_0(t;t), & \text{if } N(t) = 0, \end{cases}$$

where $\{\tau_n\}_{n\geq 1}$ are the switching times of $\varepsilon = \varepsilon(t)$, $\tau_0 = 0$, and the sequence of functions is defined by

$$\phi_n(s;t) = \frac{c_{\varepsilon(\tau_n)}}{\Gamma(\alpha)} \int_{\tau_n}^{s} (t-u)^{\alpha-1} \mathrm{d}u = c_{\varepsilon(\tau_n)} \cdot \frac{(t-\tau_n)^{\alpha} - (t-s)^{\alpha}}{\Gamma(1+\alpha)}, \ \tau_n < s \leq \tau_{n+1},$$

so that

$$\phi_n(\tau_{n+1};t) = c_{\varepsilon(\tau_n)} \cdot \frac{(t-\tau_n)^{\alpha} - (t-\tau_{n+1})^{\alpha}}{\Gamma(1+\alpha)}, \qquad n \leq N(t).$$

Fractional telegraph process

Equivalently,

$$\mathscr{F}^{\alpha}(t) = \frac{1}{\Gamma(1+\alpha)}\left[c_{\varepsilon(0)}t^{\alpha} + 2c\sum_{n=1}^{N(t)}(-1)^{\varepsilon(\tau_n)}(t-\tau_n)^{\alpha}\right], \qquad 2c = c_0 - c_1.$$

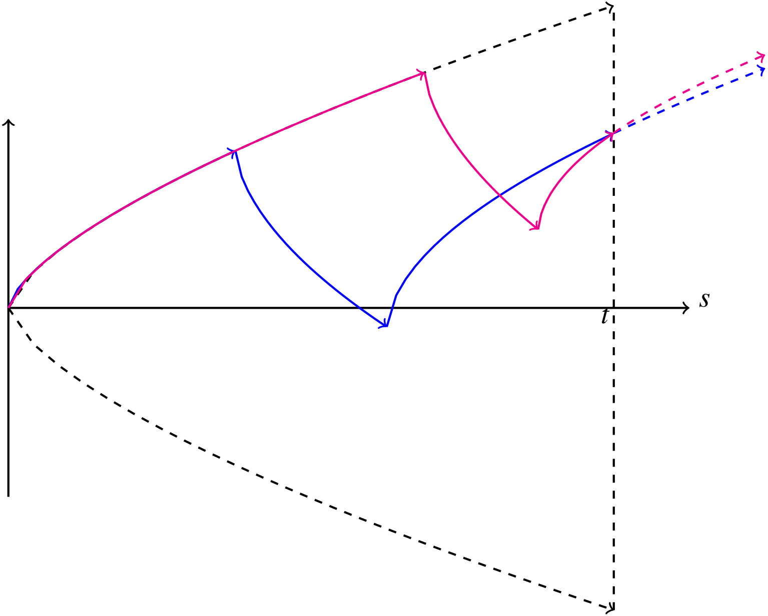

Figure: Two sample paths of $\mathscr{F}^{\alpha}(s,t)$, $\alpha = 2/3$.

Note that for any $t > 0$, the distribution of $\mathscr{F}^{\alpha}(t)$ is determined by the complete switching history $\{\tau_n\}_{1 \le n \le N(t)}$. Therefore, the fractional telegraph process $\mathscr{F}^{\alpha}(t)$ as well as the pair $(\mathscr{F}^{\alpha}(t), \varepsilon(t))$ are not Markov processes.

Theorem

For $0 < \alpha \leq 1$, *the expectations* $\mathfrak{M}_0^{(1)}(t) = \mathbb{E}[\mathscr{F}^\alpha(t) \mid \varepsilon(0) = 0]$ *and* $\mathfrak{M}_1^{(1)}(t) = \mathbb{E}[\mathscr{F}^\alpha(t) \mid \varepsilon(0) = 1]$, $t > 0$, *are explicitly given by*

$$\mathfrak{M}_0^{(1)}(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} \left[(p_0^* c_0 + p_1^* c_1) + 2c p_1^* \mathrm{e}^{-2\lambda t} \Phi(\alpha, 1+\alpha; 2\lambda t) \right],$$

$$\mathfrak{M}_1^{(1)}(t) = \frac{t^\alpha}{\Gamma(1+\alpha)} \left[(p_0^* c_0 + p_1^* c_1) - 2c p_0^* \mathrm{e}^{-2\lambda t} \Phi(\alpha, 1+\alpha; 2\lambda t) \right].$$

Here $p_0^* = \lambda_1/(\lambda_0 + \lambda_1)$, $p_1^* = \lambda_0/(\lambda_0 + \lambda_1)$, $\Phi = {}_1F_1$ *is the confluent hypergeometric (Kummer) function.*

If $\alpha = 1$, then

$$\begin{aligned}\mathfrak{M}_0^{(1)}(t) =& (p_0^* c_0 + p_1^* c_1)t + p_1^* c \frac{1 - \mathrm{e}^{-2\lambda t}}{\lambda},\\ \mathfrak{M}_1^{(1)}(t) =& (p_0^* c_0 + p_1^* c_1)t - p_0^* c \frac{1 - \mathrm{e}^{-2\lambda t}}{\lambda}.\end{aligned}$$

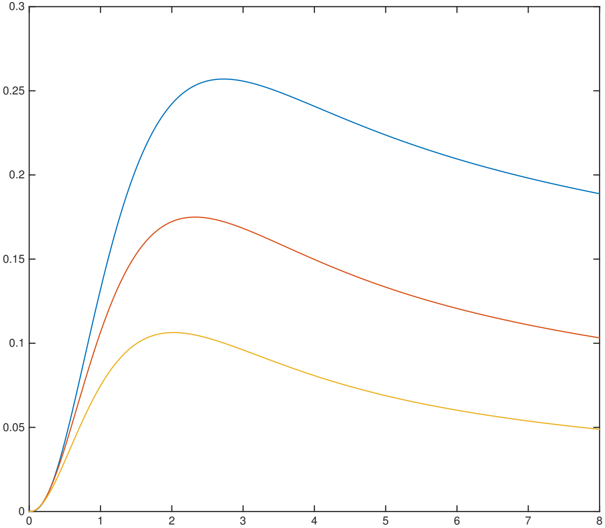

Figure: $\mathfrak{M}_0^{(1)}(t;\alpha)$ under $c_0 = -c_1 = 1$, $\lambda_0 = \lambda_1 = 1$ with (from bottom to top) $\alpha = 1/3,\ 1/2,\ 2/3$.

Theorem

The second moments $\mathfrak{M}_0^{(2)}(t) = \mathbb{E}[\mathscr{F}^{\alpha}(t)^2 \mid \varepsilon(0) = 0]$ *and* $\mathfrak{M}_1^{(2)}(t) = \mathbb{E}[\mathscr{F}^{\alpha}(t)^2 \mid \varepsilon(0) = 1]$ *are given by*

$$
\begin{aligned}
\mathfrak{M}_0^{(2)}(t) =& \frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}\Big[(p_0^* c_0 + p_1^* c_1)^2 + 4c^2 p_1^*(p_0^* - p_1^*)\mathrm{e}^{-2\lambda t}\Phi(2\alpha, 1+2\alpha; 2\lambda t) \\
&+4cp_1^*(p_1^* c_0 + p_0^* c_1)\mathrm{e}^{-2\lambda t}\Phi(\alpha, 1+\alpha; 2\lambda t) \\
&+8c^2\alpha p_0^* p_1^* \cdot (\lambda t)^{-2\alpha}\int_0^{\lambda t} u^{2\alpha-1}\mathrm{e}^{-2u}\Phi(\alpha, 1+\alpha; 2u)\mathrm{d}u\Big]
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{M}_1^{(2)}(t) =& \frac{t^{2\alpha}}{\Gamma(1+\alpha)^2}\Big[(p_0^* c_0 + p_1^* c_1)^2 + 4c^2 p_0^*(p_1^* - p_0^*)\mathrm{e}^{-2\lambda t}\Phi(2\alpha, 1+2\alpha; 2\lambda t) \\
&-4cp_0^*(p_1^* c_0 + p_0^* c_1)\mathrm{e}^{-2\lambda t}\Phi(\alpha, 1+\alpha; 2\lambda t) \\
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\end{aligned}
$$

If $\lambda_0 = \lambda_1 = \lambda$, $c_0 = -c_1 = c$, and $\alpha = 1$, then

$$\mathfrak{M}_2(t) = c^2 \frac{e^{-2\lambda t} - 1 + 2\lambda t}{2\lambda^2}.$$

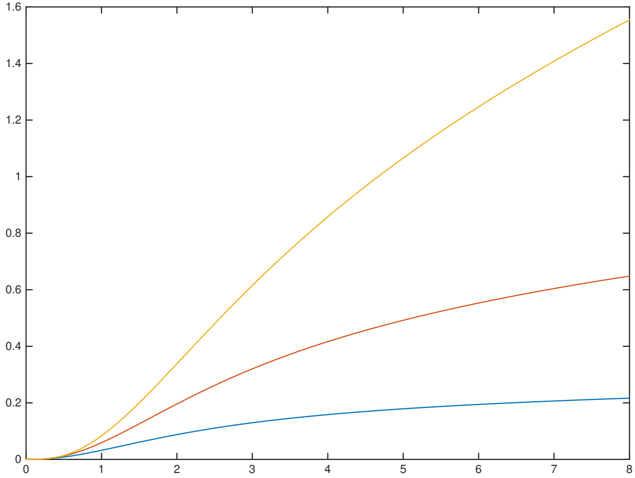

Figure: $\mathfrak{M}_2(t;\alpha)$ under $c_0 = -c_1 = 1$, $\lambda_0 = \lambda_1 = 1$ with (from bottom to top) $\alpha = 1/3,\ 1/2,\ 2/3$.

For the symmetric case, $c_0 = -c_1 = c > 0$ and $\lambda_0 = \lambda_1 = \lambda$, the results look simpler:

$$\mathfrak{M}_0^{(1)}(t) = \frac{ct^{\alpha}\mathrm{e}^{-2\lambda t}}{\Gamma(1+\alpha)}\Phi(\alpha, 1+\alpha; 2\lambda t), \quad \mathfrak{M}_1^{(1)}(t) = -\frac{ct^{\alpha}\mathrm{e}^{-2\lambda t}}{\Gamma(1+\alpha)}\Phi(\alpha, 1+\alpha; 2\lambda t),$$

and

$$\mathfrak{M}_0^{(2)}(t) = \mathfrak{M}_1^{(2)}(t) = \frac{2c^2}{\Gamma(\alpha)\Gamma(1+\alpha)\lambda^{2\alpha}} \int_0^{\lambda t} u^{2\alpha-1}\mathrm{e}^{-2u}\Phi(\alpha, 1+\alpha; 2u)\mathrm{d}u,$$

Theorem

Let $\mathscr{F}^{\alpha}(t)$ be symmetric, i.e. $c_0 = -c_1 = c > 0$ *and* $\lambda_0 = \lambda_1 = \lambda > 0$. *For $\lambda t \to \infty$ the following asymptotic relations are valid:*

$$\mathfrak{M}_0^{(1)}(t) \sim \frac{t^{\alpha-1}}{2\Gamma(1+\alpha)\Gamma(\alpha)} \cdot \frac{c}{\lambda}, \qquad \mathfrak{M}_1^{(1)}(t) \sim -\frac{t^{\alpha-1}}{2\Gamma(1+\alpha)\Gamma(\alpha)} \cdot \frac{c}{\lambda}$$

and

$$\mathfrak{M}_0^{(2)}(t),\, \mathfrak{M}_1^{(2)}(t) \sim \begin{cases} \frac{t^{2\alpha-1}}{(2\alpha-1)\Gamma(1+\alpha)\Gamma(\alpha)^2} \cdot \frac{c^2}{\lambda}, & \text{if } \alpha > 1/2, \\ \frac{2\ln(2\lambda t)}{\pi^{3/2}} \cdot \frac{c^2}{\lambda}, & \text{if } \alpha = 1/2, \\ \frac{2K_\alpha}{\Gamma(1+\alpha)\Gamma(\alpha)} \cdot \frac{c^2}{\lambda^{2\alpha}}, & \text{if } 0 < \alpha < 1/2, \end{cases}$$

where

$$K_\alpha = \int_0^\infty u^{2\alpha-1} \mathrm{e}^{-2u} \Phi(\alpha, 1+\alpha; 2u) \mathrm{d}u.$$

Kac-Ornstein-Uhlenbeck

Consider a random process $Z = Z_\beta(t)$ defined by the equation

$$\frac{\mathrm{d}Z}{\mathrm{d}t}(t) = -\beta Z(t) + c_{\varepsilon(t)}, \qquad t > 0,$$

with $Z(0) = 0$. The solution is given by

$$Z_\beta(t) = \int_0^t \mathrm{e}^{-\beta(t-s)} c_{\varepsilon(s)} \mathrm{d}s, \qquad 0 \le t < \infty.$$

Kac-Ornstein-Uhlenbeck

The piecewise deterministic continuous process $Z_\beta(t)$ sequentially follows two deterministic patterns ψ_0 and ψ_1, alternating at random times τ_n.
Patterns ψ_0 and ψ_1 are defined as

$$\psi_0(t,x) = \rho_0 + (x - \rho_0)\mathrm{e}^{-\beta t}, \qquad \psi_1(t,x) = \rho_1 + (x - \rho_1)\mathrm{e}^{-\beta t},$$

where $\rho_0 = c_0/\beta, \rho_1 = c_1/\beta$.
Let $\beta > 0$. Then both levels ρ_0 and ρ_1 are attractive, and the interval $[\rho_1, \rho_0]$ serves as an attractor for the process Z_β: if the process starts at a point x outside this interval, $x \notin [\rho_1, \rho_0]$, it falls into $[\rho_1, \rho_0]$ in a finite time. Moreover, once caught, the process remains there forever.

Kac-Ornstein-Uhlenbeck

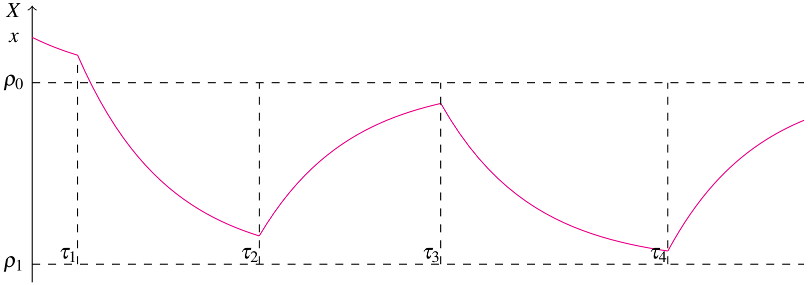

Figure: A sample path of $Z_\beta(t)$, $\beta > 0$.

The process $(Z_\beta,\, \varepsilon)$ is Markov with an infinitesimal generator $\Lambda + \mathrm{diag}(L_0, L_1)$, where Λ is the infinitesimal generator of $\varepsilon(t)$, and L_0, L_1 are the differential operators,

$$L_0 = (c_0 - \beta x)\frac{\mathrm{d}}{\mathrm{d}x}, \qquad L_1 = (c_1 - \beta x)\frac{\mathrm{d}}{\mathrm{d}x}.$$

Theorem

The fractionally integrated telegraph process $\mathscr{F}^{\alpha}$, $0 < \alpha < 1$, *is represented through the infinite dimensional Kac-Ornstein-Uhlenbeck process* Z_{β}, $\beta \in (0, \infty)$,

$$\mathscr{F}^{\alpha}(t) = \frac{1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^{\infty} \beta^{-\alpha} Z_{\beta}(t) \mathrm{d}\beta.$$

Cf. Muravlev, 2011.

Time-changed telegraph process

Consider the time-changed telegraph processes $\mathscr{T}^{\alpha}(t)$ and $\mathscr{T}_{\alpha}(t)$ which are defined by

$$\begin{aligned} \mathscr{T}^{\alpha}(t) = \mathscr{T}(E_{\alpha}(t)) &= \int_0^{E_{\alpha}(t)} c_{\varepsilon(s)} \mathrm{d}s, \\ \mathscr{T}_{\alpha}(t) &= \int_0^t c_{\varepsilon(E_{\alpha}(s))} \mathrm{d}s, \end{aligned} \qquad t > 0,$$

where $E_{\alpha} = E_{\alpha}(t)$ is the inverse α-stable subordinator.

Time-changed telegraph process

The first two moments of $\mathscr{T}^{\alpha}(t)$ and $\mathscr{T}_{\alpha}(t)$ can be represented explicitly.

Theorem

For $0 < \alpha \leq 1$, $t \geq 0$,

$$\mathbb{E}[\mathscr{T}^{\alpha}(t) \mid \varepsilon(0) = 0] = \frac{p_0^* c_0 + p_1^* c_1}{\Gamma(1+\alpha)} t^{\alpha} + \frac{p_1^* c}{\lambda} [1 - \mathscr{E}_{\alpha}(-2\lambda t^{\alpha})],$$
$$\mathbb{E}[\mathscr{T}^{\alpha}(t) \mid \varepsilon(0) = 1] = \frac{p_0^* c_0 + p_1^* c_1}{\Gamma(1+\alpha)} t^{\alpha} - \frac{p_0^* c}{\lambda} [1 - \mathscr{E}_{\alpha}(-2\lambda t^{\alpha})]$$

and

$$\mathbb{E}[\mathscr{T}_{\alpha}(t) \mid \varepsilon(0) = 0] = (p_0^* c_0 + p_1^* c_1) t + 2 p_1^* c \delta_0(t;\alpha),$$
$$\mathbb{E}[\mathscr{T}_{\alpha}(t) \mid \varepsilon(0) = 1] = (p_0^* c_0 + p_1^* c_1) t - 2 p_0^* c \delta_0(t;\alpha),$$

where $\delta_0(t;\alpha) = \int_0^t \mathscr{E}_{\alpha}(-2\lambda s^{\alpha}) \mathrm{d}s$.

Theorem

For $0 < \alpha \leq 1$, $t \geq 0$,

$$\begin{aligned}
\mathbb{E}_0\left[\mathscr{T}^{\alpha}(t)\right]^2 =& \frac{2(p_0^* c_0 + p_1^* c_1)^2}{\Gamma(1+2\alpha)} t^{2\alpha} \\
&+ \frac{2p_1^*(p_1^* c_0 + p_0^* c_1)c}{\lambda}\left[\frac{1}{\Gamma(1+\alpha)} - \frac{1}{\alpha}\mathscr{E}_{\alpha,\alpha}(-2\lambda t^{\alpha})\right] t^{\alpha} \\
&+ \frac{2p_1^*(2p_0^* - p_1^*)c^2}{\lambda^2}\left[\mathscr{E}_{\alpha}(-2\lambda t^{\alpha}) - 1 + \frac{2\lambda}{\Gamma(1+\alpha)} t^{\alpha}\right],
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}_1\left[\mathscr{T}^{\alpha}(t)\right]^2 =& \frac{2(p_0^* c_0 + p_1^* c_1)^2}{\Gamma(1+2\alpha)} t^{2\alpha} \\
&- \frac{2p_0^*(p_1^* c_0 + p_0^* c_1)c}{\lambda}\left[\frac{1}{\Gamma(1+\alpha)} - \frac{1}{\alpha}\mathscr{E}_{\alpha,\alpha}(-2\lambda t^{\alpha})\right] t^{\alpha} \\
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\end{aligned}$$

Theorem

For $0 < \alpha \leq 1$, $t \geq 0$,

$$\mathbb{E}\left[\mathscr{T}_\alpha(t)^2 \mid \varepsilon(0) = 0\right]$$
$$= (p_0^* c_0 + p_1^* c_1)^2 t^2 + 2p_1^*(p_0^* c_0 + p_1^* c_1) c t \delta_0(t;\alpha)$$
$$+4p_1^*(p_1^* - p_0^*) c \delta_1(t;\alpha) + 8 p_0^* p_1^* c^2 \delta_2(t;\alpha);$$

$$\mathbb{E}\left[\mathscr{T}_\alpha(t)^2 \mid \varepsilon(0) = 1\right]$$
$$= (p_0^* c_0 + p_1^* c_1)^2 t^2 - 2p_0^*(p_0^* c_0 + p_1^* c_1) c t \delta_0(t;\alpha)$$
$$-4p_1^*(p_0^* - p_1^*) c \delta_1(t;\alpha) + 8 p_0^* p_1^* c^2 \delta_2(t;\alpha).$$

Here

$$\delta_0(t;\alpha) = \int_0^t \mathscr{E}_\alpha(-2\lambda s^\alpha) \mathrm{d}s, \quad \delta_1(t;\alpha) = \int_0^t s \mathscr{E}_\alpha(-2\lambda s^\alpha) \mathrm{d}s,$$
$$\delta_2(t;\alpha) = \int_0^t \delta_0(s;\alpha) \mathrm{d}s.$$

Thank you for your kind attention!