Limit theorems for the spectrum of random block matrices

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Consider the following model of block random matrices. Let $\mathrm{X}^{(ij)}$, $i, j = 1, \ldots, k$ a family of $n \times n$ real symmetric random matrices. We will assume that for any fixed pair of indices (i, j), $i, j = 1, \ldots, k$, the elements of the matrix $\mathrm{X}^{(i,j)}$ are independent (with respect to symmetry), have mean zero and finite variance, i.e.

$$\mathrm{E}\, X_{pq}^{(i,j)} = 0 \text{ and } \mathrm{E}(X_{pq}^{(i,j)})^2 = (\sigma_{pq}^{(i,j)})^2 < \infty. \tag{1}$$

About the matrices $\mathrm{X}^{(i,j)}$ we will assume that they either coincide at $(i, j) \neq (i_1, j_1)$ or are independent. Let the symbol $\otimes$ denote the Kronecker product.

Consider a matrix of the form

$$\mathrm{W}_{\boldsymbol{X}} = \sum_{i,j=1}^{k} \mathrm{E}^{(i,j)} \otimes \mathrm{X}^{(i,j)}. \tag{2}$$

We find it more convenient to change the representation of (2). We define the so-called generating matrices $\mathrm{Z}^{(1)}, \ldots, \mathrm{Z}^{(m)}$ $(m \leq k^2)$ that are random independent matrices such that for any pair of indices (i,j): $i,j = 1, \ldots, k$ there exists an index $l = l(i,j) : 1 \leq l \leq m$ such that $\mathrm{X}^{(i,j)} = \mathrm{Z}^{(l)}$. Let, for $l = 1, \ldots, m$,

$$\mathcal{A}_l := \{(i,j)\ i,j = 1, \ldots, k : \mathrm{X}^{(i,j)} = \mathrm{Z}^{(l)}\}. \tag{3}$$

Then the matrix $\mathrm{W}_{\boldsymbol{X}}$ can be rewritten as

$$\mathrm{W}_{\boldsymbol{X}} = \sum_{l=1}^{m} \Big(\sum_{(i,j)\in\mathcal{A}_l} \mathrm{E}^{(i,j)} \Big) \otimes \mathrm{Z}^{(l)}.$$

Let the generating matrices be $Z^{(1)}, \ldots, Z^{(k)}$.

We define the multiplicities sets as $\mathcal{A}_l = \{(i,j) : |i-j| = l+1\}$, for $l = 1, \ldots, k$. The symmetric Töplitz matrix has the form

$$W = \sum_{l=1}^{k} Z^{(l)} \otimes \Big(\sum_{i,j \in \mathcal{A}_l} E^{(ij)} \Big) = \begin{bmatrix} Z^{(1)}, & Z^{(2)} & \ldots & Z^{(k-1)} & Z^{(k)} \\ Z^{(2)} & Z^{(1)} & \ldots & Z^{(k-2)} & Z^{(k-1)} \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ \ldots & \ldots & \ldots & \ldots & \\ Z^{(k)} & Z^{(k-1)} & \ldots & Z^{(2)} & Z^{(1)} \end{bmatrix} \tag{4}$$

In a Hankel matrix, the number of distinct blocks is $m = 2k - 1$, and there are equal blocks on all diagonals perpendicular to the main diagonal: $\mathcal{A}_l = \{(i,j) : i + j = l + 1\}, \quad l = 1, \ldots, 2k-1$:

$$\boldsymbol{W_X} = \begin{bmatrix} Z^{(1)} & Z^{(2)} & Z^{(3)} & \ldots & Z^{(k-1)} & Z^{(k)} \\ Z^{(2)} & Z^{(3)} & Z^{(4)} & \ldots & Z^{(k)} & Z^{(k+1)} \\ Z^{(3)} & Z^{(4)} & \ddots & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ Z^{(k-1)} & Z^{(k)} & \ldots & \ldots & Z^{(2k-3)} & Z^{(2k-2)} \\ Z^{(k)} & Z^{(k+1)} & \ldots & \ldots & Z^{(2k-2)} & Z^{(2k-1)} \end{bmatrix}.$$

Let $\mathrm{V}^{(1)}, \mathrm{V}^{(2)}, \ldots, \mathrm{V}^{(m)}$ be Hermite random matrices of size $n \times n$ whose elements are independent. Using the previously introduced index sets (3), we define the following matrices:

$$\mathrm{Y}^{(ij)} = \mathrm{V}^{(l)}, \text{ для } (i,j) \in \mathcal{A}_l, \quad l = 0, \ldots, m,$$

$$\boldsymbol{W}_{\mathrm{Y}} = \sum_{i,j=1}^{k} \mathrm{E}^{(ij)} \otimes \mathrm{Y}^{(ij)} = \sum_{l=0}^{m} \sum_{(i,j)\in\mathcal{A}_l} \mathrm{E}^{(ij)} \otimes \mathrm{V}^{(l)}.$$

We shall assume that random variables $Y_{pq}^{(ij)}$ for any fixed $i,j = 1 < \ldots k, i \leq j$ are independent for $p,q = 1, \ldots, n, p \leq q$, and have the same first two moments as random variables $X_{pq}^{(ij)}$, i.e.

$$\mathrm{E}\, Y_{pq}^{(ij)} = 0, \text{ and } \mathrm{E}(Y_{pq}^{(ij)})^2 = (\sigma_{pq}^{(ij)})^2. \tag{5}$$

We denote $\lambda_1, \dots, \lambda_{nk}$ the eigenvalues of the matrix $\frac{1}{\sqrt{nk}} \boldsymbol{W}_{\boldsymbol{X}}$, and $\mu_1, \dots, \mu_{nk}$ the eigenvalues of the matrix $\frac{1}{\sqrt{nk}} \boldsymbol{W}_{\mathrm{Y}}$.

Let us define the empirical spectral distribution functions

$$F_{n\boldsymbol{X}}(x) = \frac{1}{nk} \sum_{j=1}^{nk} \mathbb{1}\{\lambda_j < x\}$$

и

$$F_{n\mathrm{Y}}(x) = \frac{1}{nk} \sum_{j=1}^{nk} \mathbb{1}\{\mu_j < x\},$$

where $\mathbb{1}\{\cdot\}$ stands for the event indicator.

In what follows, for $(i,j) \in \mathcal{A}_l$, we shall write

$$\sigma_{pq}^{(ij)} = \sigma_{pq}^{(l)}.$$

Theorem 1

We assume that there exists a constant C_0 such that for all $n \geq 1$

$$\frac{1}{(nk)^2}\sum_{l=1}^{m}\sum_{p,q=1}^{n}(\sigma_{pq}^{(l)})^2 \leq C_0. \qquad (6)$$

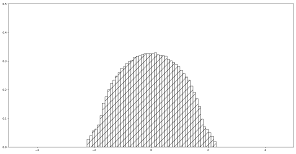
Suppose that for all $i,j = 1,\ldots,k$ for random matrices $\boldsymbol{X}^{(ij)}$ and $\mathrm{Y}^{(ij)}$ the Lindeberg condition is satisfied, i.e., for any $\tau > 0$

$$L_{n\boldsymbol{X}}(\tau) := \frac{1}{n^2}\sum_{r,s=1}^{n}\mathrm{E}\left(X_{rs}^{(ij)}\right)^2\mathbb{I}\{|X_{rs}^{(ij)}| > \tau\sqrt{n}\} \xrightarrow[n\to\infty]{} 0$$

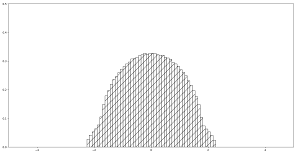
$$L_{n\mathrm{Y}}(\tau) := \frac{1}{n^2}\sum_{r,s=1}^{n}\mathrm{E}\left(Y_{rs}^{(ij)}\right)^2\mathbb{1}\{|Y_{rs}^{ij}| > \tau\sqrt{n}\},\} \xrightarrow[n\to\infty]{} 0. \qquad (7)$$

Then

$$F_{n\boldsymbol{X}}(x) - F_{n\mathrm{Y}}(x) \xrightarrow[n\to\infty]{} 0 \text{ in probability.}$$

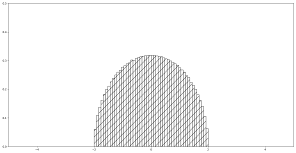


(a) Student df=5

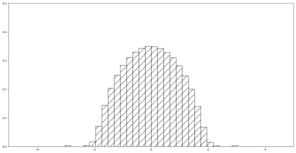


(b) Normal

Figure: The histograms of 5×5 Töplitz matrices with blocks entries distributed according Student distribution with $df = 5$ degrees of freedom (left) and standard normal distribution (right).

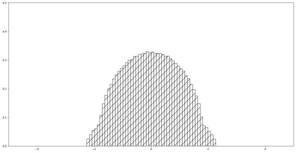


(a) Polynomial100

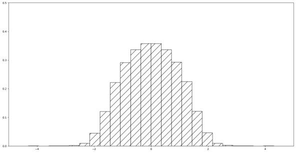


(b) Polynomial3

Figure: The histograms of 5×5 Hankel matrices with blocks entries distributed according distribution with polynomial density $p(x) = \frac{C}{(1+|x|)^{100}}$ (left) and $p(x) = \frac{C}{(1+|x|)^{\frac{7}{2}}}$ (right).



(a) Polynomial100



(b) Polynomial3

Figure: The histograms of 5×5 Töplitz matrices with blocks entries distributed according Student distribution with $df = 5$ degrees of freedom (left) and standard normal distribution (right).

Proof sketch

To prove the theorem, we only need to show the convergence of the corresponding Stieltjes transforms in any subset of the upper complex half-plane with non-empty interior. Let

$$\boldsymbol{R}_{\boldsymbol{A}}(z) := (\boldsymbol{A} - z\boldsymbol{I})^{-1},$$

is the resolvent matrix of the symmetric matrix $\boldsymbol{A}$. Symbol $\boldsymbol{I}$ denotes a unit matrix of the appropriate dimension. The matrix $\boldsymbol{R}_{\boldsymbol{A}}(z)$ is defined for all $z = u + iv, v > 0$. We only need to prove that in some region $\mathcal{G} \subset \mathbb{C}_+$ with a non-empty interior there is the convergence

$$\frac{1}{nk}\mathrm{Tr}\,\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{X}}}(z) - \frac{1}{nk}\mathrm{Tr}\,\boldsymbol{R}_{\boldsymbol{W}_{\mathrm{Y}}} \to 0 \text{ when } n \to \infty \text{ in probability.} \quad (8)$$

We will divide the proof into three parts. First we show that we can replace random variables by so-called truncated random variables. Then we prove that for truncated quantities, the difference between the expectation of the Stiltjes transformations of the matrices $\boldsymbol{W}_{\boldsymbol{X}}$ and $\boldsymbol{W}_{\mathrm{Y}}$ tends to 0. Finally, we show that the variance of the Stiltjes transformations tends to 0 (Girko's lemma). The convergence of expectations and the convergence of variances to zero entails the convergence of probabilities.

For any $\alpha \in [0, \pi/2]$, we introduce random matrices

$$\mathrm{Z}^{(l)}(\alpha) = \boldsymbol{X}_l \cos\alpha + \mathrm{Y}_l \sin\alpha.$$

Accordingly, let's denote

$$\boldsymbol{W}_{\boldsymbol{X}}(\alpha) := \frac{1}{\sqrt{nk}} \sum_{l=1}^{m} \Big(\sum_{(i,j)\in\mathcal{A}_l} \mathrm{E}^{(i,j)} \Big) \otimes \mathrm{Z}^{(l)}(\alpha),$$

$$\boldsymbol{R}_{\boldsymbol{X}}(z, \alpha) = (\boldsymbol{W}(\alpha) - z\boldsymbol{I})^{-1}.$$

In what follows, we will denote the elements of the matrices $\boldsymbol{W}_{\boldsymbol{X}}(\alpha) = (W_{jk})$, and $\mathrm{Z}^{(l)}(\alpha) = (Z_{jk}^{(l)})$ by $\boldsymbol{W}$ and $\mathrm{Z}^{(l)}$, omitting α and $\boldsymbol{X}$ in the notations unless this is controversial.

It is obvious that

$$
\begin{aligned}
\frac{1}{nk}\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\mathrm{Y}}}(z)-\frac{1}{nk}\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}(z) & \qquad (9)\\
=\frac{1}{nk}\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}(z,\frac{\pi}{2})-\frac{1}{nk}\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}(z,0) & \\
=\frac{1}{nk}\int_0^{\frac{\pi}{2}}\frac{\partial\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}(z,\alpha)}{\partial\alpha}d\alpha & \qquad (10)\\
=\frac{1}{nk}\sum_{j=1}^{nk}\int_0^{\frac{\pi}{2}}\frac{\partial[\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}]_{jj}(z,\alpha)}{\partial\alpha}d\alpha. & \qquad (11)
\end{aligned}
$$

We can write down

$$\frac{\partial [\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}]_{jj}(z,\alpha)}{\partial \alpha} = \sum_{l=1}^{m} \sum_{p,q=1}^{n} \frac{\partial [\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{x}}}]_{jj}(z,\alpha)}{\partial Z_{pq}^{(l)}} \frac{\partial Z_{pq}^{(l)}}{\partial \alpha}.$$

Note that for any invertible matrix $\boldsymbol{A}$, the following differentiation formulae are valid

$$\frac{\partial \boldsymbol{A}_{jk}^{-1}}{\partial A_{pq}} = -[\boldsymbol{A}^{-1}]_{jp}[\boldsymbol{A}^{-1}]_{qk}. \tag{12}$$

See, for example, Alexei M. Khorunzhy, Boris A. Khoruzhenko, Leonid A. Pastur; *On asymptotic propertiies of large random matrices with independnet entries.* June 1996, Journal of Mathematical Physics 37(10).

Applying the formula (12) to the resolvent matrix $\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{X}}}(z,\alpha)$, we get

$$\frac{\partial[\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{X}}}]_{jj}(z,\alpha)}{\partial Z_{pq}^{(l)}} = -\sum_{(r,s)\in\mathcal{A}_l}(2-\delta_{pq})[\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{X}}}(z,\alpha)]_{j,(r-1)n+p} \times [\boldsymbol{R}_{\boldsymbol{W}_{\boldsymbol{X}}}(z,\alpha)]_{j,(s-1)n+q},$$

where δ_{pq} stands for the Kronecker symbol. Note also that

$$\frac{\partial Z_{pq}^{(l)}}{\partial\alpha} = -X_{pq}^{(l)}\sin\alpha + Y_{pq}^{(l)}\cos\alpha =: \widetilde{Z}_{pq}^{(l)}.$$

Summing up the above formulas, we can write down

$$\mathsf{E}\Big(\frac{1}{nk}\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\mathbf{Y}}}(z)-\frac{1}{nk}\operatorname{Tr}\boldsymbol{R}_{\boldsymbol{W}_{\mathbf{X}}}(z)\Big)$$

$$=-\frac{1}{nk\sqrt{nk}}\sum_{j=1}^{nk}\sum_{l=1}^{m}\sum_{p,q=1}^{n}\sum_{(r,s)\in\mathcal{A}_l}(2-\delta_{pq})$$

$$\times\int_0^{\frac{\pi}{2}}\mathsf{E}\,\widetilde{Z}_{pq}^{(l)}(\alpha)[\boldsymbol{R}_{\boldsymbol{W}_{\mathbf{X}}}(z,\alpha)]_{j,(r-1)n+p}[\boldsymbol{R}_{\boldsymbol{W}_{\mathbf{X}}}(z,\alpha)]_{j,(s-1)n+q}d\alpha. \tag{13}$$

In the following, we will estimate the values

$$D_{pq}^{(l,r,s)}=\mathsf{E}\,\widetilde{Z}_{pq}^{(l)}[\boldsymbol{R}_{\boldsymbol{W}_{\mathbf{X}}}(z,\alpha)]_{j,(r-1)n+p}[\boldsymbol{R}_{\boldsymbol{W}_{\mathbf{X}}}(z,\alpha)]_{j,(s-1)n+q}$$

for $l=1,\ldots,m$; $r,s\in\mathcal{A}_l$ and $p,q=1,\ldots,n$.

Let us introduce the matrices $\mathrm{Z}^{(l,p,q)} = \mathrm{Z}^{(l)} - \frac{1}{\sqrt{nk}} Z_{pq}^{(l)} (\mathrm{E}^{(p,q)} + \mathrm{E}^{(q,p)})$, i.e. the matrix $\mathrm{Z}^{(l,p,q)}$ has zeros instead of the element $Z_{pq}^{(l)}$. Accordingly, we define the matrices $\boldsymbol{W}^{(l,p,q)}(z,\alpha)$, $\boldsymbol{R}_{\boldsymbol{W_x}}^{(l,p,q)}(z,\alpha)$. The simple resolvent equality $\boldsymbol{R}_{\boldsymbol{A}_1} - \boldsymbol{R}_{\boldsymbol{A}_2} = \boldsymbol{R}_{\boldsymbol{A}_1}(\boldsymbol{A}_2 - \boldsymbol{A}_1)\boldsymbol{R}_{\boldsymbol{A}_2}$ entails

$$\boldsymbol{R}_{\boldsymbol{W_x}}(z,\alpha) = \boldsymbol{R}_{\boldsymbol{W_x}}^{(l,p,q)}(z,\alpha) \tag{14}$$

$$- \boldsymbol{R}_{\boldsymbol{W_x}}(z,\alpha)(\boldsymbol{W} - \boldsymbol{W}^{(l,p,q)}(z,\alpha))\boldsymbol{R}_{\boldsymbol{W_x}}^{(l,p,q)}(z,\alpha). \tag{15}$$

Palindromic Töplitz matrizes

We will study the limiting behaviour of the spectrum distribution of a random symmetric palindromic palindromic teplitz matrix. A palindromic matrix is called palindromic if its first row is a palindrome. Let $X_1, X_2, \ldots, X_n$ be independent random variables with $\mathrm{E}\, X_j = 0$ and $\mathrm{E}\, X_j^2 = 1$.

For any $l = 1, \ldots, n$ we define the sets

$$\mathcal{E}^{(l)} = \begin{cases} \sum_{j=1}^{2n} \varepsilon_j \varepsilon_j^T + \varepsilon_1 \varepsilon_{2n}^T + \varepsilon_1 \varepsilon_{2n}^T, \text{ если } l = 1, \\ \sum_{j=1}^{2n-l+1} (\varepsilon_j \varepsilon_{j+l-1}^T + \varepsilon_{j+l-1} \varepsilon_j^T) \\ + \sum_{j=2n-l+1}^{2n} (\varepsilon_j \varepsilon_{j-2n+l}^T + \varepsilon_{j-2n+l} \varepsilon_j^T), \text{ for } l = 2, \ldots, n. \end{cases} \tag{16}$$

Hereafter ε_j, $j = 1, \ldots, 2n$ means the jth unit orth in $\mathbb{R}^{2n}$, i.e., the vector that has all coordinates except jth equal to 0 and jth equal to 1. The matrix $\boldsymbol{W}$ can be represented as

$$\boldsymbol{W} = \frac{1}{\sqrt{2n}} \sum_{l=1}^{n} X_l \mathcal{E}^{(l)}. \tag{17}$$

$$
\begin{bmatrix}
X_1 & X_2 & X_3 \cdots & X_{n-1} & X_n & X_n \cdots & X_3 & X_2 & X_1 \\
X_2 & X_1 & X_2 \cdots & X_{n-2} & X_{n-1} & X_n \cdots & X_4 & X_3 & X_2 \\
X_3 & X_2 & X_1 \cdots & X_{n-3} & X_{n-2} & X_{n-1} \cdots & X_5 & X_4 & X_3 \\
\cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\
X_4 & X_5 & X_6 \cdots & X_{n-1} & X_{n-2} & X_{n-3} \cdots & X_2 & X_3 & X_4 \\
X_3 & X_4 & X_5 \cdots & X_n & X_{n-1} & X_{n-2} \cdots & X_1 & X_2 & X_3 \\
X_2 & X_3 & X_4 \cdots & X_n & X_n & X_{n-1} \cdots & X_2 & X_1 & X_2 \\
X_1 & X_2 & X_3 \cdots & X_{n-1} & X_n & X_n \cdots & X_3 & X_2 & X_1 .
\end{bmatrix}
\tag{18}
$$

Now let $X_1, X_2, \ldots, X_n$ be independent random variables with $\mathrm{E}\, X_k = 0$ and $\mathrm{E}\, |X_k|^2 = 1$. We will assume that there exists a constant μ_3 such that

$$\sup_{k \geq 1} \mathrm{E}\, |X_k|^3 \leq \mu_3 < \infty. \tag{19}$$

Theorem 2

Let $X_1, X_2, \ldots$ be a sequence of independent random variables with $\mathrm{E}\, X_k = 0$ and $\mathrm{E}\, |X_k|^2 = 1$ for $k \geq 1$. We will assume that the (19) condition is satisfied. Let $F_n(x)$ be a sequence of empirical spectral distribution functions of palindromic random heptic matrices $\boldsymbol{W}_n$ constructed by the values $X_1, X_2, \ldots, X_n$. then there is a constant C such that

$$\sup_x |\, \mathrm{E}\, F_n(x) - \Phi(x)| \leq C \mu_3 n^{-\frac{1}{4}} \tag{20}$$

To prove the theorem, we will use the method proposed by the author in 1976, 1980. Consider the characteristic function of the expected spectral distribution function $\mathsf{E}\, F_n(x)$

$$f_n(t) = \frac{1}{2n}\sum_{j=1}^{2n} \mathsf{E}\exp\{it\lambda_j\} = \frac{1}{2n}\,\mathsf{E}\,\mathrm{Tr}\,\exp\{it\boldsymbol{W}_n\}. \qquad (21)$$

We show that function $f_n(t)$ satisfies the equation

$$f_n'(t) = -tf_n(t) + \theta(t)C\mu_3\frac{t^2+|t|}{\sqrt{n}}, \qquad (22)$$

where $|\theta(t)| \le 1$.

To analyse the function $f_n(t)$ we will use Duhamel's vormula. For any symmetric matrices $\mathrm{M}_1, \mathrm{M}_2$ the following is true

$$\exp\{it(\mathrm{M}_1 + \mathrm{M}_2)\} = \exp\{it\mathrm{M}_1\} + i\int_0^t \exp\{is\mathrm{M}_1\}\mathrm{M}_2 \exp\{i(t-s)(\mathrm{M}_1 + \mathrm{M}_2)\}ds.$$

We denote $\mathrm{U} := \mathrm{U}(t) = \exp\{it\boldsymbol{W}\}$, $\boldsymbol{W}^{(l)} = \frac{1}{\sqrt{2n-1}} \sum_{j \neq l} X_l \mathcal{E}^{(l)}$, $\mathrm{U}^{(l)} := \mathrm{U}^{(l)} := \mathrm{U}^{(l)}(t) = \exp\{it\boldsymbol{W}^{(l)}\}$. Applying Duhamel's formula, we get

$$\mathrm{U}(t) = \mathrm{U}^{(l)}(t) + iX_l \int_0^t \mathrm{U}^{(l)}(s)\mathcal{E}^{(l)}\mathrm{U}(t-s)ds. \tag{23}$$

It is not difficult to obtain the following representation

$$f_n'(t) = i\frac{1}{(2n)^{\frac{3}{2}}} \sum_{l=1}^{n} \mathrm{E}\, X_l \operatorname{Tr} \mathcal{E}^{(l)}\mathrm{U}(t) = T_1 + T_2 + T_3, \tag{24}$$

where

$$T_1 = i\frac{1}{(2n)^{\frac{3}{2}}} \sum_{l=1}^{n} \mathrm{E}\, X_l \operatorname{Tr} \mathcal{E}^{(l)}\mathrm{U}^{(l)}(t),$$

$$T_2 = -\frac{1}{(2n)^2} \sum_{l=1}^{n} \mathrm{E}\, X_l^2 \operatorname{Tr} \mathcal{E}^{(l)} \int_0^t \mathrm{U}^{(l)}(s)\mathcal{E}^{(l)}\mathrm{U}^{(l)}(t-s)ds,$$

$$T_3 = -\, i \frac{1}{(2n)^{\frac{5}{2}}} \sum_{l=1}^{n} \mathsf{E}\, X_l^3 \mathrm{Tr}\, \mathcal{E}^{(l)} \int_0^t \int_0^{t-s} \mathsf{U}^{(l)}(s_1) \mathcal{E}^{(l)} \mathsf{U}^{(l)}(s_2)$$

$$\times\, \mathcal{E}^{(l)} \mathsf{U}(t - s_1 - s_2) ds_1 ds_2.$$

Lemma 1

Under the conditions of theorem 2 , there exists an absolute constant $C > 0$ *such that the following is true*

$$|T_3| \le \frac{C\mu_3 t^2}{\sqrt{n}}. \tag{25}$$

Continuing, we can write down

$$T_1(t) = G_{11}(t) + G_{12}(t) + G_{13}(t), \tag{26}$$

where

$$T_{11}(t) = \frac{i}{(2n)^2}\sum_{l=1}^{n}\int_0^t \mathsf{E}\,\mathrm{Tr}\,\exp\{is\boldsymbol{W}\}\mathcal{E}^{(l)}\exp\{i(t-s)\boldsymbol{W}\})\mathcal{E}^{(l)}ds,$$

$$T_{12}(t) = -\frac{i}{(2n)^2\sqrt{2n}}\sum_{l=1}^{n}\mathsf{E}\,X_l\int_0^t\int_0^s \exp\{is_1\boldsymbol{W}\}\mathcal{E}^{(l)}\exp\{i(s-s_1)\boldsymbol{W}^{(l)}\}$$
$$\times\,\mathcal{E}^{(l)}\exp\{i(t-s)\boldsymbol{W}^{(l)}\}\mathcal{E}^{(l)}ds_1\,ds,$$

$$T_{13}(t) = -\frac{i}{(2n)^2\sqrt{2n}}\sum_{l=1}^{n}\mathsf{E}\,X_l\int_0^t\int_0^{t-s} \exp\{is\boldsymbol{W}^{(l)}\}\mathcal{E}^{(l)}\exp\{is_1\boldsymbol{W}\}$$
$$\times\,\mathcal{E}^{(l)}\exp\{i(t-s-s_1)\boldsymbol{W}^{(l)}\}\mathcal{E}^{(l)}dsds_1$$

Lemma 2

Under the conditions of the 2 theorem, there exists an absolute constant $C > 0$ *such that the following is true*

$$\max\{|T_{12}(t)|, |T_{13}(t)|\} \leq C\mu_3 \frac{t^2}{\sqrt{n}}. \tag{27}$$

We represent $T_{11}(t)$ in the form

$$T_{11} = \frac{-1}{(2n)^2} \sum_{l=1}^{n} \int_0^t \mathsf{E}\,\mathrm{Tr}\,\mathsf{U}(s)\mathcal{E}^{(l)}\mathsf{U}(t-s)\mathcal{E}^{(l)}ds \qquad (28)$$

Furthermore, let $[\mathsf{A},\mathsf{B}] := \mathsf{A}\mathsf{B} - \mathsf{B}\mathsf{A}$ denote the commutator of matrices A и B. We proof the following lemma.

Lemma 3

$$\mathsf{E}\,\mathrm{Tr}\,\mathsf{U}(s)\mathcal{E}^{(l)}\mathsf{U}(t-s)\mathcal{E}^{(l)} \qquad (29)$$

$$= \mathrm{Tr}\,\mathsf{U}(t)\mathcal{E}_l^2 + i\int_0^s \mathrm{Tr}\,\mathsf{U}(r)[\mathsf{W},\mathcal{E}^{(l)}]\mathsf{U}(-r)\mathsf{U}(t)\mathcal{E}^{(l)}dr. \qquad (30)$$

Applying the 3 lemma to the function $G_{11}(t)$, we get

$$G_{11}(t) = -t\frac{1}{(2n)^2}\sum_{l=1}^{n} \operatorname{Tr} \mathsf{U}(\mathcal{E}^{(l)})^2 + R(t), \tag{31}$$

where

$$R(t) = -\frac{i}{(2n)^2}\sum_{l=1}^{n}\int_0^t\int_0^s \mathsf{E}\operatorname{Tr}\mathsf{U}(r)\mathcal{E}^{(l)}[\boldsymbol{W},\mathcal{E}^{(l)}]\mathsf{U}(t-r)dr\mathcal{E}^{(l)}ds. \tag{32}$$

Lemma 4

There is an equality

$$\sum_{l=1}^{n}(\mathcal{E}^{(l)}\mathcal{E}^{(l)}) = \begin{bmatrix} 2n-1 & 1 & 1 & \cdots & 1 & 1 & 2n \\ 1 & 2n-1 & 1 & \cdots & 1 & 1 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ \dots & \dots & \dots & \dots & \dots & \dots & \\ 2n & 1 & 1 & \cdots & 1 & 1 & 2n-1 \end{bmatrix} \tag{33}$$

The lemma 4 implies equality

$$f_n'(t) = -tf_n(t) + \frac{1}{4n^2}\sum_{j\neq k} U_{jk} + \frac{C\mu_3 t^2}{\sqrt{n}} + R(t). \qquad (34)$$

Thank you for your attention!