On the method of solving the Cauchy problem for stochastic and deterministic generalized Burgers equations

Zotova Ekaterina Igorevna

Under the Guidance of
Prof. Nasyrov Farit Sagitovich

Cauchy problem for the generalized Burgers equation

$$u_t + (f(u))_x = u_{xx}, \quad u(x,0) = \varphi(x),$$
$$u = u(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

Cauchy problem for the Burgers equation [1]

$$u_t + uu_x = u_{xx}, \quad u(x,0) = \varphi(x),$$
$$u = u(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

- **Hydrodynamics and shock wave theory.** Used as a simplified model of viscous fluid flow. Describes the formation and attenuation of shock waves in gases and liquids.
- **Acoustics.** Models the propagation of large-amplitude sound waves.
- **In turbulence theory** the Burgers equation is used as a test model for studying the interaction of nonlinear and dissipative effects.

[1] Burgers J. M., The Nonlinear Diffusion Equation: Asymptotic Solutions and Statistical Problems. Dordrecht: D. Reidel, 1974.

Method for solving the Cauchy problem for the Burgers equation

The Cole-Hopf transform (see [2],[3])

$$u = -2\frac{\partial \ln \phi}{\partial x},$$

reduces the Burgers equation to the heat equation

$$\phi_t = \phi_{xx}, \quad \phi = \phi(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+.$$

The solution of the Cauchy problem for the Burgers equation in this case is determined from the relations

$$u(x,t) = \frac{\int_{-\infty}^{\infty} \frac{x-\eta}{t} \exp\left\{\frac{-G(\eta,x,t)}{2}\right\} d\eta}{\int_{-\infty}^{\infty} \exp\left\{\frac{-G(\eta,x,t)}{2}\right\} d\eta},$$

$$G(\eta,x,t) = \frac{(x-\eta)^2}{2t} + \int_0^x \varphi(\mu) d\mu.$$

[2] Cole J. D., On a quasi-linear parabolic equation occurring in aerodynamics // Quart. Appl. Math., V. 9, N. 3, P. 225–236, 1951.
[3] Hopf E., The partial differential equation $u_t + uu_x = u_{xx}$ // Comm. Pure and Appl. Math., V. 3, P. 201–230, 1950.

Stochastic Burgers equation with additive white noise in space and time [4]

$$u_t + uu_x = u_{xx} + \varepsilon\eta_t(x),$$

where ε is a constant, $\eta_t(x)$ is white noise in space and time, i.e.

$$\mathbf{E}(\eta_t(x)\eta_{t'}(x')) = \delta(x - x')\delta(t - t').$$

Cauchy problem for stochastic Burgers equation [5]

$$u_t + uu_x = u_{xx} + F(u) + W'(t),$$

where $W(t)$ is a Wiener process and $W'(t)$ is its formal derivative.

[4] Bertini L., Cancrini N., Jona-Lasinio G. The stochastic Burgers Equation // Commun. Math. Phys. V. 165, P. 211–232, 1994.
[5] Nasyrov F.S., Paramoshina I.G., Numerical analytical method of resolve of some classes of stochastic partial differential equations // Bulletin of the Ufa State Aviation Technical University. 2008. V. 11, № 1 (28), P. 175–180. (in Russian)

The goal of research is to construct a new method for solving the Cauchy problem for the generalized Burgers equation both with noise and without noise.

Formal notation of the Cauchy problem for the generalized Burgers equation with noise in the nonlinear part

$$u_t + (f(u))_x V'(t) = u_{xx}, \quad u(x,0) = \varphi(x),$$
$$u = u(x,t), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where $V'(t)$ is the formal derivative of a continuous deterministic function $V(t)$ or a random process $V(t)$ with continuous realizations that may not exist (for example, $V(t) = W(t)$ is a Wiener process).

Let a random process $V(t)$, $V(0) = 0$, $t \in [0, T]$, with continuous realizations with probability 1 be defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Cauchy problem for stochastic generalized Burgers equation in integral form

$$u(x,t) - u(x,0) + \int_0^t (f(u(x,s)))_x * dV(s) = \int_0^t u_{xx}(x,s)ds, \qquad (1)$$
$$u(x,0) = \varphi(x).$$

where the integral on the left side of the equality is **a symmetric integral** with respect to the process $V(t)$.

A symmetric integral with respect to a continuous function is a generalization of the Stratonovich stochastic integral and coincides with it in the case of a Wiener process [6].

[6] F.S. Nasyrov. Local Times, Symmetric Integrals, and Stochastic Analysis. Fizmatlit, Moscow, 2011 (in Russian).

Theorem.

Let the function $g(x,v)$ be determined from the relation

$$g(x,v) = \varphi(x - vf'(g(x,v))), \quad (x,v) \in \mathbf{R} \times \mathbf{R}. \tag{2}$$

Then the function

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\xi, V(t)) \exp\left\{\frac{-(x-\xi)^2}{4t}\right\} d\xi,$$

is a solution to the Cauchy problem for the stochastic generalized Burgers equation (1).

Remark 1. The implicit relation (2) is obtained by solving the Cauchy problem for the generalized Hopf equation

$$g_v + f'(g)g_x = 0, \quad g(x,0) = \varphi(x), \quad (x,v) \in \mathbf{R} \times \mathbf{R}.$$

Remark 2. Setting $f(u) = \frac{u^2}{2}$ in problem (1). We obtain the Cauchy problem for the stochastic Burgers equation. The solution of this problem is determined from the relations

$$u(x,t,V(t)) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\xi, V(t)) \exp\left[\frac{-(x-\xi)^2}{4t}\right] d\xi, \qquad (3)$$
$$g(x, V(t)) = \varphi(x - V(t)g(x, V(t))).$$

Remark 3. Let in problem (1) $f(u) = \frac{u^2}{2}$ and $V(t) = t$, we obtain the Cauchy problem for the deterministic Burgers equation

$$u_t + uu_x = u_{xx}, \quad u(x,0) = \varphi(x).$$

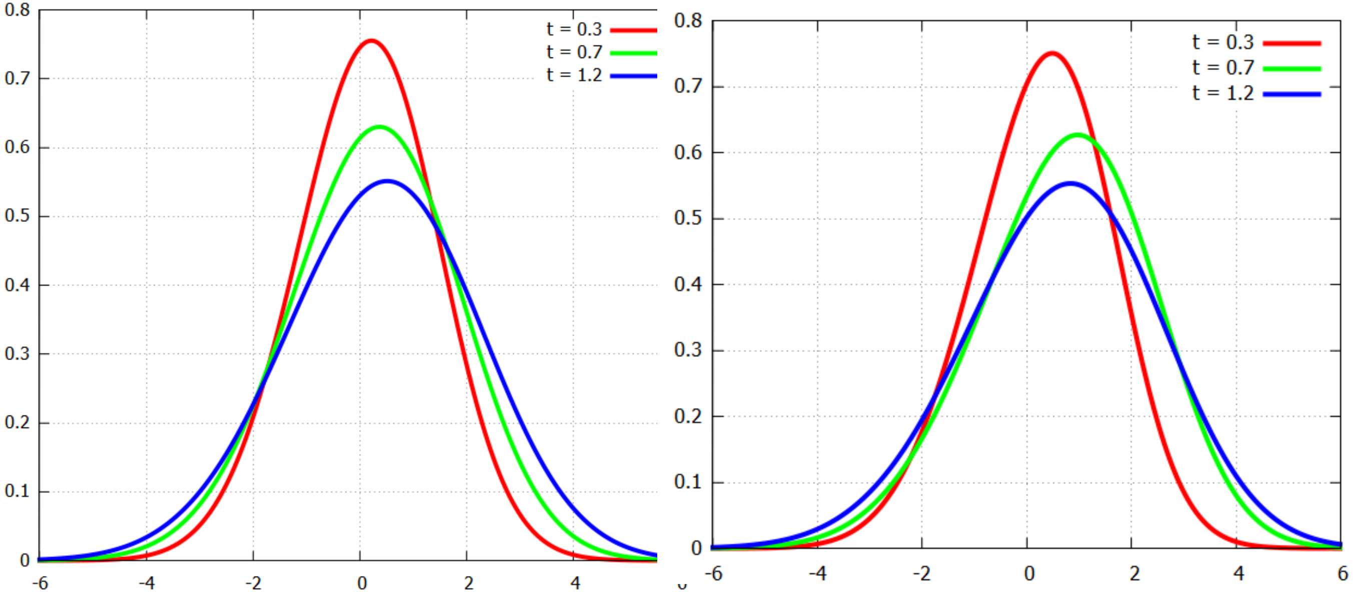
The solution to this problem is determined from the relations

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} g(\xi, t) \exp\left[\frac{-(x-\xi)^2}{4t}\right] d\xi,$$

$$g(x,t) = \varphi(x - tg(x,t))).$$

Example 1.

$$\varphi(x) = \exp\left\{\frac{-x^2}{2}\right\} \tag{4}$$

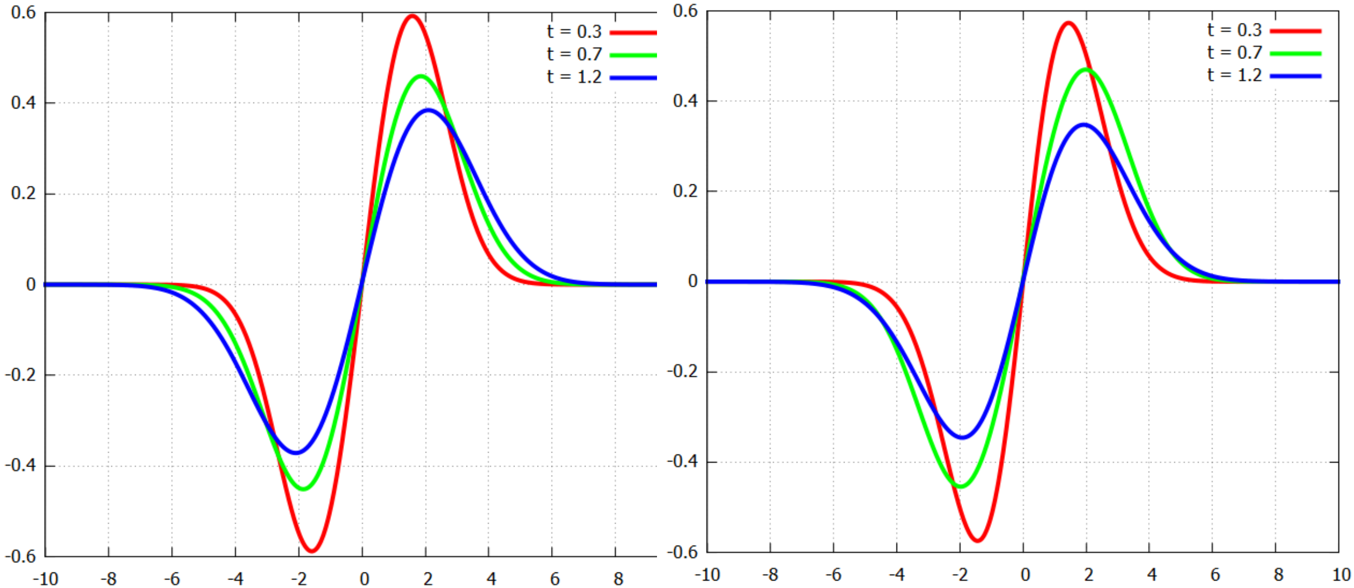


a. $V(t) = t$ b. $V(t) = t + W(t)$

Fig. 1: Results of modeling solution of Cauchy problem for stochastic Burgers equation with initial condition (4) for linear and random functions $V(t)$ at different times ($W(t)$ is a Wiener process).

Example 2.

$$\varphi(x) = \exp\left\{\frac{-(x-1)^2}{2}\right\} - \exp\left\{\frac{-(x+1)^2}{2}\right\} \quad (5)$$

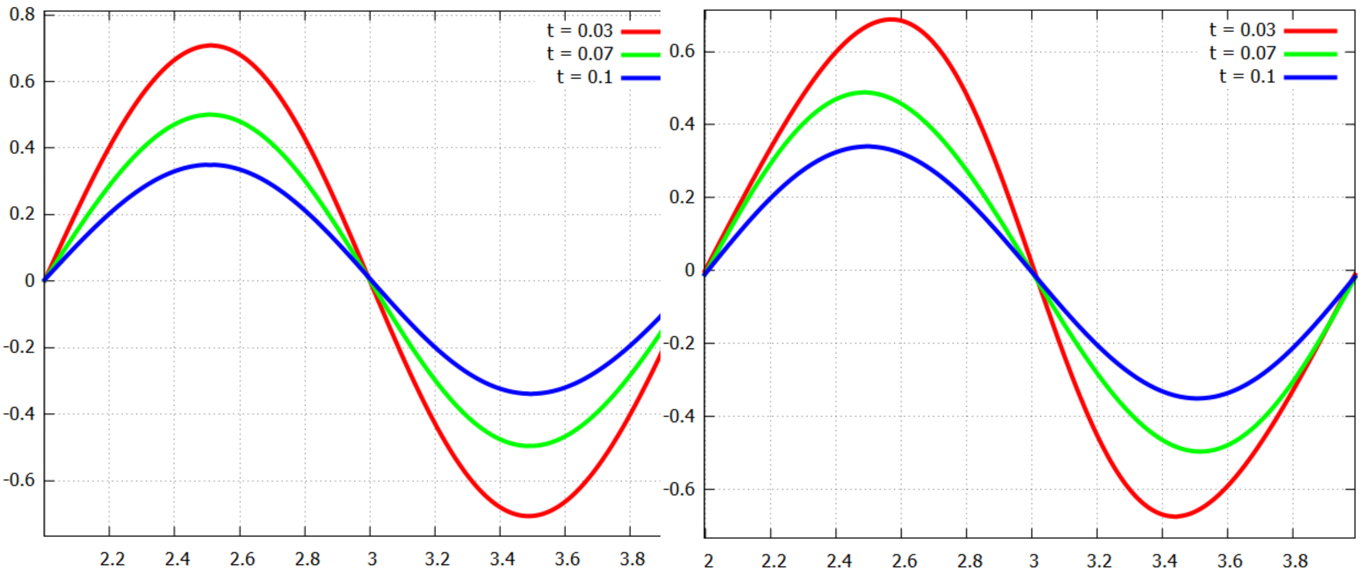


a. $V(t) = t$ b. $V(t) = t + W(t)$

Fig. 2: Results of modeling solution of Cauchy problem for stochastic Burgers equation with initial condition (5) for linear and random functions $V(t)$ at different times ($W(t)$ is a Wiener process).

Example 3.

$$\varphi(x) = \sin(2\pi x) \tag{6}$$



a. $V(t) = t$ b. $V(t) = t + W(t)$

Fig. 3: Results of modeling solution of Cauchy problem for stochastic Burgers equation with initial condition (6) for linear and random functions $V(t)$ at different times ($W(t)$ is a Wiener process).

[1] Burgers J. M., The Nonlinear Diffusion Equation: Asymptotic Solutions and Statistical Problems. Dordrecht: D. Reidel, 1974.

[2] Cole J. D., On a quasi-linear parabolic equation occurring in aerodynamics // Quart. Appl. Math., V. 9, N. 3, P. 225–236, 1951.

[3] Hopf E., The partial differential equation $u_t + uu_x = u_{xx}$ // Comm. Pure and Appl. Math., V. 3, P. 201–230, 1950.

[4] Bertini L., Cancrini N., Jona-Lasinio G. The stochastic Burgers Equation // Commun. Math. Phys. V. 165, P. 211–232, 1994.

[5] Nasyrov F.S., Paramoshina I.G., Numerical analytical method of resolve of some classes of stochastic partial differential equations // Bulletin of the Ufa State Aviation Technical University. 2008. V. 11, № 1 (28), P. 175–180. (in Russian)

[6] F.S. Nasyrov. Local Times, Symmetric Integrals, and Stochastic Analysis. Fizmatlit, Moscow, 2011 (in Russian).

Thank you for your attention!

Definition 1. Let $V(t)$, $t \in [0, +\infty)$, be an arbitrary continuous function, then the symmetric integral is called

$$\int_0^t f(s, V(s)) * dV(s) = \lim_{n\to\infty} \sum_k \frac{1}{\Delta t_k^{(n)}} \int_{[\Delta t_k^{(n)}]} f(s, V^{(n)}(s))ds\, \Delta V_k^{(n)} =$$

$$\lim_{n\to\infty} \int_0^t f(s, V^{(n)}(s))(V^{(n)})'(s)ds,$$

where $V^{(n)}(s)$ is a broken line constructed by the function $V(t)$ and the partition $\{t^{(n)}\}$ of the interval $[0, t]$ such that $\max_n(t_k^{(n)} - t_{k-1}^{(n)}) \to 0$ for $n \to \infty$.

Definition 2. For a pair of functions $(V(s), f(s,u))$, condition (S) is said to be satisfied if the following assumptions are satisfied:

(a) $V(s)$, $s \in [0,t]$, is a continuous function;

(b) For almost all u, the function $f(s,u)$, $s \in [0,t]$, is right-continuous and has bounded variation;

(c) The total variation $|f|(t,u)$ with respect to s of the function $f(s,u)$ on $[0,t]$ is locally summable with respect to the variable u;

(d) For almost all u $\int_0^t \mathbf{1}(s : V(s) = u)|f|(ds,u) = 0$, where $\mathbf{1}(A)$ is the indicator of the set A, i.e. a function equal to 1 on A and 0 outside A.

Some properties of the symmetric integral:
1. Let a pair of functions $(V(s), f(s,v))$ satisfy condition (S), then

$$\int_0^t f(s,V(s)) * dV(s) = \int_{V(0)}^{V(t)} f(t,v)dv - \int_{\mathbb{R}} \int_0^t \kappa(v,V(0),V(s)) f(ds,v)dv, \quad (3.1)$$

where $\kappa(v,a,b) = sgn(b-a)\mathbf{1}(a \wedge b < v < a \vee b)$. This means that the symmetric integral is a function of three variables.
2. Let the function $F(t,u)$ have continuous partial derivatives $F_t'(t,u)$ and $F_u'(t,u)$, then there exists a symmetric integral $\int_0^t F_u'(s,V(s)) * dV(s)$, and the formula

$$F(t,V(t)) - F(0,V(0)) = \int_0^t F_s'(s,V(s))ds + \int_0^t F_u'(s,V(s)) * dV(s) is valid. \quad (3.2)$$

If $V(s)$ is a Wiener process, then formula (3.2) coincides with the formula for the Stratonovich stochastic differential.

Lemma 1. (On the equality of two integrals) Let $V(s)$, $s \in [0, T]$, be a continuous nowhere differentiable function. Suppose that the continuous functions $f_1(s, v)$ and $f_2(s, v)$, $(s, v) \in [0, T] \times \mathbb{R}$ satisfy the following conditions:

(a) The function $f_2(s, V(s))$, $s \in [0, T]$, is summable;

(b) The function $f_1(s, v)$ for each s is summable over the variable $v \in R$ and has a continuous derivative $(f_1(s, v))'_s$ satisfying the condition $\int_{\mathbb{R}} \int_0^T |(f_1(s, v))'_s| ds dv < \infty$.

Then the condition

$$\int_0^t f_1(s, V(s)) * dV(s) = \int_0^t f_2(s, V(s)) ds, \quad t \in [0, T],$$

is equivalent to the condition

$$f_1(s, V(s)) = f_2(s, V(s)) = 0, \quad s \in [0, T].$$