On signed martingale interpolating measures

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Conference dedicated to the 90th anniversary

of the Department of Probability Theory at MSU

ICSM-10

5 июня 2025 г.

Signed deflators and signed martingale measures — 1

Consider:

$\Omega = \{\omega_1, \omega_2, \dots, \omega_m\}$ — finite set;

$F = (\mathcal{F}_k)_{k=0}^K$ — a filtration on Ω: $(\Omega, \emptyset) = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_K = \mathcal{F}$;

P — a prob. on $(\Omega, \mathcal{F})$, $p_1 := P(\omega_1) > 0, \dots, p_m := P(\omega_m) > 0$,

and we mean $P = (p_1, p_2, \dots, p_m) \in R^m$;

$Z = (Z_k, \mathcal{F}_k, P)_{k=0}^K$ — a process (discounted price of stock).

Signed deflators and signed martingale measures — 2

By the first FTAP: $(1, Z)$ is arbitrage free market $\Leftrightarrow$ on $\mathcal{F}$

there exists a martingale measure $Q \sim P$ of the process Z, i.e.

$\forall k : 0 \leq k < K$

$$E^Q[Z_{k+1}|\mathcal{F}_k] = Z_k. \tag{1}$$

Signed deflators and signed martingale measures — 3

Since $Q \sim P \Rightarrow Q|_{\mathcal{F}_k} \sim P|_{\mathcal{F}_k}$ $(k = 1, 2, \dots, K)$, we can calculate

Radon-Nikodym derivatives:

$$h_k(A) = \frac{dQ|_{\mathcal{F}_k}}{dP|_{\mathcal{F}_k}} > 0 \quad P - a.s.\ (k = 1, 2, \dots, K). \tag{2}$$

It is obvious that $h = (h_k, \mathcal{F}_k, P)_{k=0}^{K}$ is strictly positive

$(P - a.s.)$ martingale. Using generalized Bayes formula it is easy

to see that the process $hZ = (h_k Z_k, \mathcal{F}_k, P)_{k=0}^{K}$ is a martingale.

Processes like h are called deflators.

Signed deflators and signed martingale measures — 4

Definition

Let $Z = (Z_k, \mathcal{F}_k)_{k=0}^{K}$ be an adabted process that can take any real values. A martingale $D = (D_k, \mathcal{F}_k, P)_{k=0}^{K}$ is said a deflator of the process Z if D_0=1 and the process $DZ = (D_k Z_k, \mathcal{F}_k, P)_{k=0}^{K}$ is a martingale.

There is a one-to-one correspondence between the set $\mathcal{P}(Z, F)$ of all martingale measures of Z and the set of all strictly positive (P-a.s.) deflators of the process Z.

Signed deflators and signed martingale measures — 5

Conclusion: **We can not use strictly positive deflators to study arbitrage markets**.

Definition

A signed deflator $D = (D_k, \mathcal{F}_k, P)_{k=0}^{K}$ of the process Z is said admissible if for $D_K = \sum_{i=1}^{m} d_i I_{\{\omega_i\}}$ and for all non-empty subset $J \subset \{1, 2, \dots, m\}$ $\sum_{j \in J} d_j p_j \neq 0$.

We mean $(d_1 p_1, \dots, d_m p_m) \in R^m$ as a signed measure.

Signed deflators and signed martingale measures — 6

We obtained in the paper of Pavlov, I.V., Danekyants, A.G., Neumerzhitskaia, N.V., Tsvetkova, I.V.: Signed interpolating deflators and Haar Uniqueness Properties, *Global and Stochastic Analysis* **9**, N3 (2021) 67-75 some results on so called signed interpolating deflators, but now it is clear that a systematic investigation of such deflators is associated with the following definition.

Definition

A signed measure $\mu = (\mu_1, \dots, \mu_m)$ on the σ-algebra $\mathcal{F}$ is said admissible ($\mu \in ASM$) if

$$\sum_{i=1}^{m} \mu_i = 1 \tag{3}$$

and for all non-empty subsets $I \subset \{1, 2, \dots, m\}$

$$\sum_{i \in I} \mu_i \neq 0. \tag{4}$$

Signed deflators and signed martingale measures — 8

1) Signed measure, generated by admissible signed deflator, is ASM.

2) Every probability measure $P = (p_1, \ldots, p_m)$ on the σ-algebra $\mathcal{F}$ such that $p_i > 0$ $(i = 1, 2, \ldots, m)$ is admissible.

3) It follows from (3) and (4) that for any $\mu \in ASM$ we have $\mu_i \neq 0$ $(i = 1, 2, \ldots, m)$ and for all subsets $I \subset \{1, 2, \ldots, m\}$ (besides $I = \emptyset$ and $I = \{1, 2, \ldots, m\}$) $\sum_{i \in I} \mu_i \neq 1$.

Signed deflators and signed martingale measures — 9

Let f be $\mathcal{F}$-mesurable, $\mathcal{G} \subset \mathcal{F}$, and $\mu \in ASM$. Since (3) and (4) are satisfied, we can define $E^\mu[f|\mathcal{G}]$ in the usual way:

$$E^\mu[f|\mathcal{G}] = \sum_{j=1}^{n} \left(\frac{\sum\limits_{i \in G_j} f(i)\mu_i}{\sum\limits_{i \in G_j} \mu_i} \right) I_{G_j}, \tag{5}$$

where $\{G_1, G_2, \ldots, G_n\}$ is the partition of Ω into atoms of the σ-algebra $\mathcal{G}$. If $\mathcal{H}$ is a sub-σ-algebra of $\mathcal{G}$, the telescopic property $E^\mu\left[E^\mu[f|\mathcal{G}] \mid \mathcal{H}\right] = E^\mu[f|\mathcal{H}]$ is also valid. Martingale property $E^\mu[f_{n+1}|\mathcal{F}_n] = f_n$ for a process $(f_n, \mathcal{F}_n)_{n=0}^N$ is the same too.

Signed deflators and signed martingale measures — 10

Nelson, P.I.: A class of orthogonal series related to martingales, *The Annals of Mathematical Statistics*, **41**, N5 (1970) 1684-1694.

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Interpolation idea — 11

Now we will briefly describe the main ideas of interpolation through extensions of filtration.

Definition

We say that a filtration $\left(\Omega, G = (\mathcal{G}_n)_{n=0}^{N}\right)$ interpolates the filtration $\left(\Omega, F = (\mathcal{F}_k)_{k=0}^{K}\right)$ if there exist integers $0 = n_0 < n_1 < n_2 < ... < n_K = N$ such that $\mathcal{G}_{n_0} = \mathcal{F}_0, \mathcal{G}_{n_1} = \mathcal{F}_1, \ldots, \mathcal{G}_{n_K} = \mathcal{F}_K$.

Interpolation idea — 12

Let $\mu \in ASM$ be a martingale measure of an adapted process $Z = (Z_k, \mathcal{F}_k)_{k=0}^K$. Consider a filtration $\big(G = (\mathcal{G}_n)_{n=0}^N\big)$, that interpolates the filtration $F = (\mathcal{F}_k)_{k=0}^K$. Define the process $Z^{int} = \big(Z_n^{int}, \mathcal{G}_n\big)_{n=0}^N$ by putting $Z_n^{int} := E^\mu[Z_K|\mathcal{G}_n]$, $n = 0, 1, \ldots, N$. It is clear that $Z_{n_k}^{int} = Z_k$, $k = 0, 1, \ldots, K$. Thus, μ is a martingale measure of the process Z^{int}.

Interpolation idea — 13

Definition

If μ is the **unique** martingale measure of the process Z^{int}, we say that this measure satisfies G-uniqueness property. If μ satisfies this property for every interpolating filtration $G \in \mathbf{G}$ of some collection $\mathbf{G}$, we say that this measure satisfies $\mathbf{G}$-uniqueness property.

Interpolation idea — 14

In this work we consider two kinds of such collections. If **G** is the set of all interpolating Haar filtrations, we say instead of **G**-uniqueness property Universal Haar Uniqueness Property (UHUP). If **G** is the set of all interpolating filtrations so called special Haar filtrations, we say instead of **G**-uniqueness property Special Haar Uniqueness Property (SHUP).

Interpolation idea — 15

A filtration $H = (\mathcal{H}_n)_{n=0}^{N}$ is called Haar filtration if $\mathcal{H}_0 = \{\Omega, \emptyset\}$ and each σ-algebra $\mathcal{H}_n$ is generated by a partition of the set Ω into exactly $n+1$ atoms $H_0^n, H_1^n, ..., H_n^n$. Interpolating Haar filtration (IHF) is defined as on the slide 11 (namely $\mathcal{H}_{n_k} = \mathcal{F}_k$, $0 \le k \le K$).

Interpolation idea — 15a

An interpolating Haar filtration H of F is said special interpolating Haar filtration of F if $\forall k$ $(0 \leq k < K)$ and $\forall n$ $(n_k \leq n < n_{k+1})$ an atom A of $\mathcal{H}_{n_k} = \mathcal{F}_k$ (when moving from n_k to $n_k + 1$) is divided in such a way that at least one atom of σ-algebra $\mathcal{H}_{n_{k+1}} = \mathcal{F}_{k+1}$ is obtained.

Interpolation idea — 16

Consider the set $\mathcal{P}(Z, \mathbf{F})$. Suppose that $\mathcal{P}(Z, \mathbf{F}) \neq \emptyset$. Then the financial market $(1, Z)$ is arbitrage-free. If a measure $P \in \mathcal{P}(Z, \mathbf{F})$ satisfies G-uniqueness property, where G is a filtration interpolating the filtration F, then we can pass to the interpolating market with the stock $Z^{int} = \left(Z_n^{int}, \mathcal{G}_n\right)_{n=0}^{N}$ with one martingale measure, namely the measure P. By the second FTAP this market is complete and we can calculate fair price of each contingent claim and form corresponding hedging portfolio.

Interpolation idea — 17

Bogacheva, M.N., Pavlov, I.V.: Haar extensions of arbitrage-free financial markets to markets that are complete and arbitrage-free, *Russian Math. Surveys* **57**, N3 (2002) 581–583.

Bogacheva, M.N., Pavlov, I.V.: Haar extensions of arbitrage-free financial markets to markets that are complete and arbitrage-free, *Izvestiya VUZov, Severo-Kavkaz. Region, Estestvenn. Nauki*, N3 (2002) 16–24.

We will study here interpolation properties of ASMM.

Some analytic properties of admissible signed martingale measures for static processes — 18

A static process is: $Z_0 = a$, $Z_1 = \sum_{i=1}^{m} b_i I_{\{\omega_i\}}$, $m \geq 2$.

In this case a measure $\mu = (\mu_1, \ldots, \mu_m) \in ASM$ is a martingale measure ($\mu \in ASMM$) for the process Z if

$$\sum_{i=1}^{m} b_i \mu_i = a. \tag{6}$$

If $b_1 = \cdots = b_m = a$, it is clear that $ASMM = ASM$. If $b_1 = \cdots = b_m \neq a$, then $ASMM = \emptyset$. These cases will not be considered further.

IIf $m = 2$ and $b_1 \neq b_2$, then $ASMM \neq \emptyset$ iff $a \neq b_1$ and $a \neq b_2$. In this case $ASMM$ consists of only one measure.

Let $m = 3$. If numbers b_1, b_2, b_3 are different, then all measures

$\mu = (\mu_1, \mu_2, \mu_3) \in ASMM$ are given by the system:

$$\begin{cases} \mu_1 = \dfrac{b_2 - a + (b_3 - b_2)\mu_3}{b_2 - b_1} \\ \mu_2 = \dfrac{a - b_1 + (b_1 - b_3)\mu_3}{b_2 - b_1} \end{cases} \tag{7}$$

where $\mu_3 \neq 0, 1, -\dfrac{b_1 - a}{b_3 - b_2}, -\dfrac{b_2 - a}{b_3 - b_2}, -\dfrac{b_1 - a}{b_3 - b_1}, -\dfrac{b_2 - a}{b_3 - b_1}$.

If $b_1 \neq b_2$, but $b_1 = b_3$ or $b_2 = b_3$, then $ASMM \neq \emptyset$ iff $a \neq b_1$

and $a \neq b_2$; $ASMM$ is described by (7) with $\mu_3 \neq 0; 1$.

Let $m \geq 4$.

1) If numbers $b_1, \ldots, b_m$ are different, then $ASMM \neq \emptyset$.

2) If among the numbers $b_1, \ldots, b_m$ there are only two different ones (denote them b' and b''), then $ASMM \neq \emptyset$ iff $a \neq b'$ and $a \neq b''$.

3) If among the numbers $b_1, \ldots, b_m$ there are more than two different ones, then $ASMM \neq \emptyset$.

Definition

We say that a measure $\mu \in ASMM$ satisfies the noncoicidence barycenters condition ($\mu \in NBC$) if for any subsets $I = \{i_1, i_2, \ldots, i_\alpha\}$ and $J = \{j_1, j_2, \ldots, j_\beta\}$ of the set $\{1, 2, \ldots, m\}$ such that $I \cap J = \emptyset$ the following inequality is fulfilled:

$$\frac{\sum_{k=1}^{\alpha} b_{i_k}\mu_{i_k}}{\sum_{k=1}^{\alpha} \mu_{i_k}} \neq \frac{\sum_{k=1}^{\beta} b_{j_k}\mu_{j_k}}{\sum_{k=1}^{\beta} \mu_{j_k}}. \quad (8)$$

Definition

We say that a measure $\mu \in ASMM$ satisfies the weakened noncoicidence barycenters condition ($\mu \in WNBC$) if inequalities (8) are satisfied only for single point sets I, i.e. have the form:

$$b_i \neq \frac{\sum\limits_{k=1}^{\beta} b_{j_k}\mu_{j_k}}{\sum\limits_{k=1}^{\beta} \mu_{j_k}}, \tag{9}$$

where $i \in \{1, 2, \ldots, m\}$ and $i \notin J$.

It is clear that

$$NBC \subset WNBC \subset ASMM \subset ASM \subset R^m. \tag{10}$$

The main purpose of this section is to prove that $WNBC$ is dense in $ASMM$ and NBC is dense in $WNBC$ (according to the norm of R^m).

Lemma

If $WNBC \neq \emptyset$, then numbers $a, b_1, b_2, \ldots, b_m$ are different.

Main results for static model— 25

SM —the set of all measures (points) $\mu = (\mu_1, \ldots, \mu_m) \in R^m$.

ASM — the set of all admissible $\mu = (\mu_1, \ldots, \mu_m) \in SM$:

$\sum_{i=1}^m \mu_i = 1$ and for all non-empty subsets $I \subset \{1, 2, \ldots, m\}$

$\sum_{i \in I} \mu_i \neq 0$.

For a static process $Z_0 = a$, $Z_1 = \sum_{i=1}^m b_i I_{\{\omega_i\}}$, $m \geq 2$:

ASMM — the set of all martingale measures

$\mu = (\mu_1, \ldots, \mu_m) \in ASM$: $\sum_{i=1}^m b_i \mu_i = a$.

Main results for static model— 26

For a static process $Z_0 = a$, $Z_1 = \sum_{i=1}^{m} b_i I_{\{\omega_i\}}$, $m \geq 2$:

$WNBC$ — the set of all $\mu = (\mu_1, \dots, \mu_m) \in ASMM$:

$$b_i \neq \frac{\sum\limits_{k=1}^{\beta} b_{j_k} \mu_{j_k}}{\sum\limits_{k=1}^{\beta} \mu_{j_k}} \ \forall i \in \{1, 2, \dots, m\}, \ \forall J \subset \{1, 2, \dots, m\}, \text{ where}$$

$i \notin J$.

NBC — the set of all $\mu = (\mu_1, \dots, \mu_m) \in ASMM$:

$$\frac{\sum\limits_{k=1}^{\alpha} b_{i_k} \mu_{i_k}}{\sum\limits_{k=1}^{\alpha} \mu_{i_k}} \neq \frac{\sum\limits_{k=1}^{\beta} b_{j_k} \mu_{j_k}}{\sum\limits_{k=1}^{\beta} \mu_{j_k}}, \ \forall I = \{i_1, i_2, \dots, i_\alpha\}, \ \forall J = \{j_1, j_2, \dots, j_\beta\},$$

$I, J \subset \{1, 2, \dots, m\}$, $I \cap J = \emptyset$.

For a static process $Z_0 = a$, $Z_1 = \sum_{i=1}^{m} b_i I_{\{\omega_i\}}$, $m \geq 2$:

$SHUP$ (resp., $UHUP$) — the set of all

$\mu = (\mu_1, \ldots, \mu_m) \in ASMM$: μ is the unique martingale

measure for the process $Z_n^{int} = E^{\mu}[Z_1|\mathcal{H}_n]$, $n = 0, 1, \ldots, m-1$,

for any inteprolating special Haar filtration (resp., inteprolating

Haar filtration) $(\mathcal{H}_n)_{n=0}^{m-1}$, where $\mathcal{H}_0 = (\Omega, \emptyset)$,

$\mathcal{H}_{m-1} = \sigma\{\omega_1, \ldots, \omega_m\}$.

Main results — 28

Theorem

If the numbers $a, b_1, b_2, \ldots, b_m$ *are different, then* $WNBC$ *is dense in* $ASMM$ *and* NBC *is dense in* $WNBC$.

Theorem

f the numbers $a, b_1, b_2, \ldots, b_m$ *are different, then*

$SHUP = WNBC$ *and* $UHUP = NBC$.

Remark

Theorems 1 and 2 remain true for dynamic models $Z = (Z_k, \mathcal{F}_k, P)_{k=0}^{K}$. The proofs are carried out by induction by combining the results obtained for each atom $A \in \mathcal{F}_k$, decomposed into atoms $B_1, \ldots, B_r \in \mathcal{F}_{k+1}$.

References — 30

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THANK YOU VERY MUCH