# A new numerical algorithm to solve the portfolio optimal control problem in the Heston model

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# Outline

*Quasilinear* (QL) equation is PDE with the coefficients depend on $u$, $\nabla u$.

*Fully non-linear* (FnL) equation is PDE with the coefficients depend on $u$, $\nabla u$, $\nabla^2 u$.

FnL parabolic equations arise in the optimal control of portfolio investments in Heston market [БелЧуб23][1]

$$V_\tau - \frac{1}{2}\frac{((\mu - r)V_x + \rho\nu V_{x\nu})^2}{\nu V_{xx}} + \frac{1}{2}\sigma^2 \nu V_{\nu\nu} + rxV_x + \kappa(\theta - \nu)V_\nu = 0,$$

$$V(T,x,\nu) = h(x). \quad (1.1)$$

[1] Я.И. Белопольская и А.А. Чубатов. “Оптимизация инвестиционного портфеля в модели Хестона”. В: Зап. научн. сем. ПОМИ 526 (2023), с. 29—51.

*Backward Stochastic Differential Equations (BSDEs)* were introduced in the linear case by [Bis73][2]. Connections between *quasilinear* PDEs and the theory of *BSDEs* are well known since pioneer papers [PP90][3], [PP92][4].

FBSDEs may be weaky **coupled** and strongly (fully) **coupled**

$$\begin{aligned} dX_t &= a(t, X_t, Y_t, Z_t)dt + A(t, X_t, Y_t, Z_t)dW_t, \quad X_0 = x, \\ dY_t &= f(t, X_t, Y_t, Z_t)dt + Z_t^{+} dW_t, \quad Y_T = \xi. \end{aligned}$$

If the parabolic PDE is **quasi**linear then the FBSDE is **strongly coupled**.
However, `PDEs corresponding to standard FBSDEs cannot be` *`fully non-linear`*.

[2] J.-M. Bismut. "Conjugate Convex Function in Optimal Stochastic Control". In: Journal of Mathematical Analysis and Applications 44 (1973), pp. 384–404.

[3] E. Pardoux and S. Peng. "Adapted solution of a backward stochastic differential equation". In: Systems Control Letters 14 (1990), pp. 55–61.

[4] E. Pardoux and S. Peng. "Backward stochastic differential equations and quasilinear parabolic partial differential equations". In: Lecture Notes in CIS 176 (1992), pp. 200–217.

Probabilistic approch for *fully non-linear* parabolic PDEs was constructed in paper [БелДал80][5] on the base SDE theory, [Che+07][6] on the base $2^{nd}$ order *BSDEs* (*2BSDEs*) and to systems of *fully non-linear* parabolic PDEs in [BW13][7], [Bel14][8].

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[5] Я.И. Белопольская и Ю.Л. Далецкий. "Марковские процессы, связанные с нелинейными параболическими системами". В: Докл. Акад. Наук СССР 250.3 (1980), с. 521—524.

[6] P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir. "Second order backward stochastic differential equations and fully non-linear parabolic PDEs". In: Comm. Pure Appl. Math. 60.7 (2007), pp. 1081–1110.

[7] Ya. I. Belopolskaya and W. A. Woyczynski. "SDEs, FBSDEs and fully nonlinear parabolic systems". In: Rendiconti del Seminario Matematico Univ. Politec. Torino 71.2 (2013), pp. 209–217.

[8] Ya. I. Belopolskaya. "Probabilistic counterparts of nonlinear parabolic PDE systems". In: Springer Optimization and Its Applications, 2014. Chap. A chapter in Modern Stochastics and Applications, pp. 71–94.

Reducing the FBSDE solution to a solution of a *stochastic optimimal control (SOC)* problem [MY07][9], [Ji+20][10]

[9] J. Ma and J. Yong. Forward-Backward stochastic differential equations and their applications. Springer, 2007.

[10] S. Ji, S. Peng, Y. Peng, and X. Zhang. "Three algorithms for solving high-dimensional fully-coupled FBSDEs through deep learning". In: IEEE Intelligent Systems 35.3 (2020), pp. 71–84.

In paper [EHJ17][11] it was shown that one can develop an effective algorithm to obtain a numerical solution of the Cauchy problem (1.2) combining the 2BSDE approach from [Che+07] and *neural networks (NN)*.
In [Rai18a][12] was solved Cauchy problem for *quasilinear* PDE using original *Forward-backward stochastic neural networks (FBSNN)* approach based on *Reinforcement learning (RL)*, namely *Physics-informed neural networks (PINN)* [RPK17][13].

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[11] W. E, J. Han, and A. Jentzen. "Deep learning-based numerical methods for high-dimensional parabolic partial differential equations and backward stochastic differential equations". In: Commun. Math. Stat 5.4 (2017), pp. 349–380.

[12] M. Raissi. "Forward-backward stochastic neural networks: deep learning of high-dimensional partial differential equations". In: arXiv preprint, arXiv:1804.07010 (2018).

[13] Maziar Raissi, Paris Perdikaris, and George Em Karniadakis. "Physics Informed Deep Learning (Part I): Data-driven Solutions of Nonlinear Partial Differential Equations". In: arXiv preprint arXiv:1711.10561 (2017).

We consider the Cauchy problem for a *fully non-linear (FnL)* parabolic equation

$$\frac{\partial u}{\partial \tau} + f(\tau, x, u, \nabla u, \nabla^2 u) = 0, \;\; u(T,x) = h(x), \quad (\tau,x) \in [0,T] \times \mathbb{R}^d, \tag{1.2}$$

where $u = u(\tau,x)$, $\nabla_k u = \frac{\partial u}{\partial x_k}$, $\nabla^2_{kj} u = \frac{\partial^2 u}{\partial x_k \partial x_j}$, $k,j = 1,\ldots,d$.

**Our aim is to develop a new probabilistic approach** to solution of (1.2) based on

– *differential prolongation* [БелДал80],
– FBSDE theory,
– reducing of FBSDE to *SOC* problem,
– **new extention** of original *FBSNN* approch by Raissi [Rai18a].

Main steps of **new (w.r.t. 2BSDE) FBSDE** approach for *fully non-linear (FnL)* PDE consist

1. Reducing the Cauchy problem for *FnL* PDE (1.2) to a Cauchy problem for *quasilinear* parabolic PDEs system using *differential prolongation* [БелДал80].
2. Reducing solution of the Cauchy problem for *quasilinear* parabolic PDEs system to a solution of *stochastic problem* (FBSDE).

For numerical solution of FBSDE we used

1. Reducing solution of the FBSDE to a solution of problem of a *SOC* [Ji+20].
2. We solve FBSDE using the extention on a case of *quasilinear* PDEs system of *FBSNNs* methodology suggested by Raissi for *quasilinear* PDE [Rai18a].
3. We produced **new FnLPDEs_FBSNNs framework** is a extension of the original FBSNN framework (by M. Raissi)[Rai18b][14] for FnL PDEs.

[14] Miazar Raissi. https://github.com/maziarraissi/FBSNNs. 2018.

Set notation $v = \nabla u \in \mathbb{R}^d$, $\quad \gamma = \nabla^2 u = \nabla v \in \mathbb{R}^d \otimes \mathbb{R}^d$, $\; v_k = \nabla_{x_k} u$, $\; \gamma_{kj} = \nabla_{x_j} v_k$.

**Assumption 1.** Assume that $f(\tau,x,u,v,\gamma)$ is smooth and bounded in $\tau$,$x$,$u$,$v$,$\gamma$.

By formal differentiation the equation (1.2) in new (quasilinear) notation

$$\frac{\partial u}{\partial \tau} + f(\tau, x,u,v,\nabla v) = 0, \; u(T,x) = h(x), \tag{2.1}$$

we derive an *quasilinear parabolic* equation for the function $v \in \mathbb{R}^d$

$$\frac{\partial v_k}{\partial \tau} + \mathrm{Tr}[\nabla_\gamma f \nabla^2 v_k] + \psi_k(\tau, x, u, v, \nabla v) = 0, \; k = 1, \ldots ,d, \; v(T,x) = \nabla h(x), \tag{2.2}$$

$$\psi(\tau, x, u, v, \nabla v) = \nabla f + \frac{\partial f}{\partial u} v + \nabla v \nabla_v f. \tag{2.3}$$

**Assumption 2.** Assume that $\nabla_\gamma f > 0$.

Then exists the decomposition ($^+$ denote transpose operation)

$$AA^+ = 2\nabla_\gamma f, \quad \text{where } A = A(\tau, x, u, v, \nabla v) \in \mathbb{R}^{d\times d}. \tag{2.4}$$

We rewrite the *quasilinear system* (2.1), (2.2) in the form

$$\frac{\partial u}{\partial \tau} + \frac{1}{2}\,\mathrm{Tr}[AA^+\nabla^2 u] + \varphi(\tau, x, u, v, \nabla v) = 0, \quad u(T,x) = h(x), \tag{2.5}$$

$$\frac{\partial v_k}{\partial \tau} + \frac{1}{2}\,\mathrm{Tr}[AA^+\nabla^2 v_k] + \psi_k(\tau, x, u, v, \nabla v) = 0, \;\; k = 1, \dots, d, \;\; v(T,x) = g(x), \tag{2.6}$$

where $\varphi(\tau,x, u, v, \nabla v) = f(\tau,x,u,v,\nabla v) - \frac{1}{2}\,\mathrm{Tr}[AA^+\,\nabla v], \quad g(x) = \nabla h(x).$

$$\text{Set } U(t) = u(t,X(t)),\; Z(t) = \nabla u(t,X(t)),\; V(t) = v(t,X(t)),\; G(t) = \nabla v(t,X(t)). \tag{2.7}$$

We write a *FBSDE system* (assotiated with (2.5)–(2.6)) in the form

$$dX(t) = A(t,X,U,V,G)dW(t),\;\; X(\tau) = x, \quad \tau \in [0,T],\;\; t \in [\tau, T], \tag{2.8}$$

$$dU(t) = -\varphi(t, X,U,V,G)dt + Z(t)^+ A(t,X,U,V,G)dW(t),\;\; U(T) = h(X(T)), \tag{2.9}$$

$$dV(t) = -\psi(t, X,U,V,G)dt + G(t)^+ A(t,X,U,V,G)dW(t),\;\; V(T) = g(X(T)), \tag{2.10}$$

where $W(t) \in \mathbb{R}^d$ – standard Wiener process.

**The solution of FBSDEs** (2.8)–(2.10) **give us** $u(\tau,x)$**, namely** $U(\tau) = u(\tau,x)$**.**

As a result we have obtained a *strongly coupled* forward-backward system of SDEs. We roughly state the conditions from [Ji+20, Assumption 1–2] as **condition C 1**.

**Condition C 1** holds $\iff$ $A(t,x,u,v,\nabla v)$, $\varphi(t,x,u,v,\nabla v)$, $\psi(t,x,u,v,\nabla v)$, $h(x)$, $g(x)$ are bounded and uniformly Lipschitz continuous and some conditions of coefficient monotonicity are fulfilled.

**Theorem.** **If condition C 1 holds then**

1. $\exists!$ **solution of the FBSDEs** (2.8)-(2.10);
2. $U(t) = u(t,x)$ **is a viscosity solution of** (1.2).

The proof of this statement one can find for example in [Bel14], [AR06][15].
To obtain a *closed system* we can apply the *Ito martingale theorem*, but we apply other way using *reduction to SOC problem*.

[15] R. Abraham and O. Riviére. “Forward-backward stochastic differential equations and PDE with gradient dependent second order coefficients”. In: ESAIM: Probability and Statistics 10 (2006), pp. 184–205.

We write a *FBSDE system* (2.8)–(2.10) in the following (Pardoux-Peng) form

$$dX(t) = \tilde{A}(t,X,Y,\Gamma)dW(t),\ \ X(\tau) = x,\quad \tau \in [0,T],\ \ t \in [\tau,T], \tag{2.11}$$

$$dY(t) = -\tilde{\phi}(t,X,Y,\Gamma)\,dt + \Gamma(t)^{+}\,dW(t),\ \ Y(T) = q(X(T)), \tag{2.12}$$

where $Y = \begin{pmatrix} U \\ V \end{pmatrix} \in R^{1+d}$, $q = \begin{pmatrix} h \\ g \end{pmatrix}$, $\tilde{\phi}(\cdot) = \begin{pmatrix} \tilde{\varphi}(\cdot) \\ \tilde{\psi}(\cdot) \end{pmatrix} \in R^{1+d}$, $\Gamma = \begin{pmatrix} A^{+}Z & A^{+}G \end{pmatrix} \in R^{d} \otimes R^{1+d}$,
$\tilde{\varphi}(t,X,Y,\Gamma) = \varphi(t,X,U,V,G)$, $\tilde{\psi}(t,X,Y,\Gamma) = \psi(t,X,U,V,G)$, $\tilde{A}(t,X,Y,\Gamma) = A(t,X,U,V,G)$.

We reduce (2.11)–(2.12) to the **SOC problem** [Ji+20, p. 5, case 2]

$$\inf_{\{\mathcal{A}_t\},\tau\le t\le T} L = \inf_{\{\mathcal{A}_t\},\tau\le t\le T} \mathbb{E}\left[ |Y^{\mathcal{A}}(T) - q(X^{\mathcal{A}}(T))|^2 + \int_{\tau}^{T} |Y^{\mathcal{A}}(t) - y_t|^2 dt \right], \tag{2.13}$$

where $\mathcal{A}_t = (y_t, g_t)$ is control, $y_t, g_t$ is $\mathcal{F}_t$-adapted and square-integrable processes,

$X^{\mathcal{A}}(t) = x + \int_{\tau}^{t} \tilde{A}(s,X^{\mathcal{A}}(s),y_s,g_s)dW(s)$, $Y^{\mathcal{A}}(t) = y_\tau - \int_{\tau}^{t} \tilde{\phi}(s,X^{\mathcal{A}}(s),Y^{\mathcal{A}}(s),g_s)\,ds + \int_{\tau}^{t} g_s^{+}\,dW(s)$.

**Note [Ji+20] that under condition C 1 we have** $\inf_{\{\mathcal{A}_t\},t\in[\tau;T]} L = 0$,
$\big(X(t),(U(t),V(t)),(Z(t),G(t))\big) = \big(X(t),Y(t),\Gamma(t)\big)$ **is the** ***unique solution*** **of** (2.11)–(2.12).

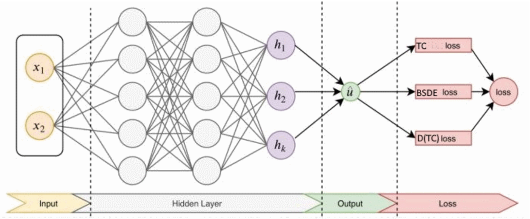
The *Euler-Maruyama scheme* applied to solve (2.8)–(2.10) rewritten the $[t_n, t_{n+1}]$ of the given partition $\pi : \tau = t_0 < \cdots < t_n < \cdots < t_N = T$ leads to

$$X_{n+1} \approx X_n + A(X_n, U_n, V_n, G_n)\Delta W_n, \;\; X_0 = x, \tag{2.14}$$

$$U_{n+1} \approx U_n - \varphi(X_n, U_n, V_n, G_n)\Delta t + Z_n^{+} A(X_n, U_n, V_n, G_n)\Delta W_n, \;\; U_N = h(X_N), \tag{2.15}$$

$$V_{n+1} \approx V_n - \psi(X_n, U_n, V_n, G_n)\Delta t + G_n^{+} A(X_n, U_n, V_n, G_n)\Delta W_n, \;\; V_N = g(X_N). \tag{2.16}$$

where
$\Delta W_n = W_{n+1} - W_n, X_n = X(t_n), U_n = U(t_n), V_n = V(t_n), Z_n = Z(t_n), G_n = G(t_n)$.
To obtain the FBSDE solution we solve the SOC problem using the extention on a case of *quasilinear* PDE system of FBSNNs suggested by Raissi for *quasilinear* PDE.




NumPy
TensorFlow
PyTorch

$$\begin{bmatrix} u^\theta(t,X(t)) \\ v^\theta(t,X(t)) \end{bmatrix} = \mathcal{NN}(t,X(t);\theta):$$

$$\tilde{U}(t) = u^\theta(t,X(t)) \in \mathbb{R},$$

$$\tilde{V}(t) = v^\theta(t,X(t)) \in \mathbb{R}^d.$$

$\mathcal{D}$ – **AutoDiff**:

$$\tilde{Z}(t) = \mathcal{D}u^\theta(t,X(t)),$$

$$\tilde{G}(t) = \mathcal{D}v^\theta(t,X(t)).$$

*Fully connected NN* $\supset L+2$ layers: input & output layer (IL, OL) $\supset (1+d)$ neurons, each of the *L hidden layers* (HL) $\supset S$ neurons.

$dim(\theta) = (2+d)S + (L-1)(S+1)S + (S+1)(1+d)$.

The *fully connected feedforward network* obtains $\theta$ using *Adam method*.

As a *loss function*, we choose a function that ensures the fulfillment (in the sense of *mathematical expectation*) of equalities (2.15), (2.16) along with *terminal conditions*

$$L(\theta; N) = \mathbb{E}\left[\Delta t \sum_{n=0}^{N-1} \left|\tilde{U}_{n+1} - \bar{U}_{n+1}\right|^2 + \Delta t \sum_{n=0}^{N-1} \left\|\tilde{V}_{n+1} - \bar{V}_{n+1}\right\|^2 + \right.$$
$$\left. + \left|\tilde{U}_N - h(X_N)\right|^2 + \left\|\tilde{V}_N - g(X_N)\right\|^2\right], \quad (2.17)$$

where

$$\tilde{U}_n = u^\theta(t_n,X_n), \quad \tilde{V}_n = v^\theta(t_n,X_n), \qquad \tilde{Z}_n = \mathcal{D}u^\theta(t_n,X_n), \quad \tilde{G}_n = \mathcal{D}v^\theta(t_n,X_n),$$

$\bar{U}_{n+1}$ and $\bar{V}_{n+1}$ calculate using *Euler-Maruyama scheme*

$$\bar{U}_{n+1} = \tilde{U}_n - \varphi(t_n, X_n, \tilde{U}_n, \tilde{V}_n, \tilde{G}_n)\,\Delta t + \tilde{Z}_n^+ A(t_n, X_n, \tilde{U}_n, \tilde{V}_n, \tilde{G}_n)\,\Delta W_n,$$
$$\bar{V}_{n+1} = \tilde{V}_n - \psi(t_n, X_n, \tilde{U}_n, \tilde{V}_n, \tilde{G}_n)\,\Delta t + \tilde{G}_n^+ A(t_n, X_n, \tilde{U}_n, \tilde{V}_n, \tilde{G}_n)\,\Delta W_n.$$

To find the *mathematical expectation* in (2.17) we use the *Monte Carlo method* with batches by $M$ realizations of the Wiener process $W^m(t)$

$$X_{n+1}^m = X_n^m + A(t_n, X_n^m, \tilde{U}_n^m, \tilde{V}_n^m, \tilde{G}_n^m)\,\Delta W_n^m, \quad X_0^m = x. \tag{2.18}$$

Then the *estimation* $\hat{L}(\theta; N, M)$ for the *loss functions* $L(\theta; N)$ takes the form

$$\hat{L}(\theta; N, M) = \frac{1}{M}\left(\Delta t \sum_{n=0}^{N-1}\sum_{m=1}^{M}\left|\tilde{U}_{n+1}^m - \bar{U}_{n+1}^m\right|^2 + \Delta t \sum_{n=0}^{N-1}\sum_{m=1}^{M}\left\|\tilde{V}_{n+1}^m - \bar{V}_{n+1}^m\right\|^2 + \right.$$
$$\left. + \sum_{m=1}^{M}\left|\tilde{U}_N^m - h(X_N^m)\right|^2 + \sum_{m=1}^{M}\left\|\tilde{V}_N^m - g(X_N^m)\right\|^2\right), \tag{2.19}$$

$$\bar{U}_{n+1}^m = \tilde{U}_n^m - \varphi(t_n, X_n^m, \tilde{U}_n^m, \tilde{V}_n^m, \tilde{G}_n^m)\,\Delta t + \tilde{Z}_n^{m+} A(t_n, X_n^m, \tilde{U}_n^m, \tilde{V}_n^m, \tilde{G}_n^m)\,\Delta W_n^m, \tag{2.20}$$

$$\bar{V}_{n+1}^m = \tilde{V}_n^m - \psi(t_n, X_n^m, \tilde{U}_n^m, \tilde{V}_n^m, \tilde{G}_n^m)\,\Delta t + \tilde{G}_n^{m+} A(t_n, X_n^m, \tilde{U}_n^m, \tilde{V}_n^m, \tilde{G}_n^m)\,\Delta W_n^m. \tag{2.21}$$

**Algorithm 1** The sequence of calculations $\hat{L}(\theta; N, M)$

**Require:** $x$; $\theta$; $N$, $M$

$X_0^m = x$

$\hat{L}_\Sigma = 0$

**for** $n = 0, \dots, N-1$ **do**

$\quad \tilde{U}_n^m = u^\theta(t_n, X_n^m)$; $\tilde{V}_n^m = v^\theta(t_n, X_n^m)$; $\quad \tilde{Z}_n^m = \mathcal{D}u(t_n, X_n^m)$; $\tilde{G}_n^m = \mathcal{D}v(t_n, X_n^m)$

$\quad \bar{U}_{n+1}^m \leftarrow (2.20)$; $\bar{V}_{n+1}^m \leftarrow (2.21)$; $\quad X_{n+1}^m \leftarrow (2.18)$

$\quad \tilde{U}_{n+1}^m = u^\theta(t_{n+1}, X_{n+1}^m)$; $\quad \tilde{V}_{n+1}^m = v^\theta(t_{n+1}, X_{n+1}^m)$

$\quad \hat{L_n} = \sum_{m=1}^{M} \left( \left| \tilde{U}_{n+1}^m - \bar{U}_{n+1}^m \right|^2 + \left\| \tilde{V}_{n+1}^m - \bar{V}_{n+1}^m \right\|^2 \right)$; $\hat{L}_\Sigma = \hat{L}_\Sigma + \hat{L_n}$

**end for**

$\hat{L}_N = \sum_{m=1}^{M} \left( \left| \tilde{U}_N^m - h(X_N^m) \right|^2 + \left\| \tilde{V}_N^m - g(X_N^m) \right\|^2 \right)$; $\quad \hat{L}_{TC} = \hat{L}_N$

$\hat{L}_{(1)} = \frac{1}{M}(\Delta t \hat{L}_\Sigma + \hat{L}_{TC})$

**Ensure:** **return** $\hat{L}_{(1)}$

$=0$

## 2.6. Differences (comparison) from Raissi's method & FBSNN framework

$$\begin{cases} \partial u/\partial\tau + \frac{1}{2}\operatorname{Tr}[AA^+\nabla^2 u] + \varphi(\tau,x,u,\nabla u = v, \nabla v) = 0, \quad u(T,x) = h(x), \\ \partial v_k/\partial\tau + \frac{1}{2}\operatorname{Tr}[AA^+\nabla^2 v_k] + \psi_k(\tau,x,u,v,\nabla v) = 0, \;\; k = \overline{1,d}, \;\; v(T,x) = g(x). \end{cases}$$

$$\begin{cases} dX(t) = A(t,X,U,V,G)dW(t), \;\; X(\tau) = x, \\ dU(t) = -\varphi(t,X,U,V,G)dt + Z(t)^+ A(t,X,U,V,G)dW(t), \;\; U(T) = h(X(T)), \\ dV(t) = -\psi(t,X,U,V,G)dt + G(t)^+ A(t,X,U,V,G)dW(t), \;\; V(T) = g(X(T)). \end{cases} \qquad \begin{cases} U(t) = u(t,X(t)), \\ Z(t) = \nabla u(t,X(t)), \\ V(t) = v(t,X(t)), \\ G(t) = \nabla v(t,X(t)). \end{cases}$$

$$\hat{L} = \frac{1}{M}\sum_{m=1}^{M}\left(\left|\tilde{U}_N^m - h(X_N^m)\right|^2 + \left\|\tilde{V}_N^m - g(X_N^m)\right\|^2 + \Delta t\sum_{n=0}^{N-1}\left(\left|\tilde{U}_{n+1}^m - \bar{U}_{n+1}^m\right|^2 + \left\|\tilde{V}_{n+1}^m - \bar{V}_{n+1}^m\right\|^2\right)\right).$$

| framework | FBSNNs (by M. Raissi) | FnLPDEs_FBSNNs (this) |
|---|---|---|
| quasiLinear equ. | 1 equ | system of $(1+d)$ equ |
| $d,\ x \in \mathbb{R}^d$ | $d = 20{,}50{,}100$ | $d = 1$ |
| $A$ | $A = \sigma E$ or $A = \sigma\, diag(x)$ | $A(v,v_x)$ |
| loss $\rightarrow$ | $\rightarrow$ "0" | $\rightarrow$ "0" or $\rightarrow$ "1" (reInit $\theta$) |
| loss normalization | $L = L_\Sigma + L_{TC}$ | $L = \frac{1}{M}\left(\Delta t L_\Sigma + L_{TC}\right)$ |
| LR selection | – | auto |

The *FnL* PDE for the optimal portfolio investments in *Heston market* has the form

$$u_\tau - \frac{1}{2}\frac{((\mu-r)u_x+\rho\nu u_{x\nu})^2}{\nu u_{xx}} + \frac{1}{2}\sigma^2\nu u_{\nu\nu} + rxu_x + \kappa(\theta-\nu)u_\nu = 0,\ u(T,x,\nu) = h(x). \tag{3.1}$$

For some $h(x)$ $(h \in \{\ln, \exp, pow\})$ the exact solution has the form $u(\tau,x,\nu) \in \{h(x)+p(\tau,\nu), h(x)p^\delta(\tau,\nu)\}$, where $p(\tau,\nu)$ is solution of some *linear* PDE

$$\frac{\partial p}{\partial \tau} + \frac{1}{2}A(\tau,\nu)A^+(\tau,\nu)\,\nabla^2 p + a(\tau,\nu)\nabla p - f(\tau,\nu)p = g(\tau,\nu),\ p(T,\nu) = q(\nu),$$

obtained using *Feynmann-Kac formula* [Kac51][16]

$$p(\tau,\nu) = \mathbb{E}\left[E(\tau,T)\,q(\xi_{\tau,\nu}(T)) - \int_\tau^T E(\tau,s)\,g(s,\xi_{\tau,\nu}(s))\,ds\right],$$

$E(\tau,t) = \exp\left(-\int_\tau^t f(t,\xi_{\tau,\nu}(t))\,dt\right)$, $\quad d\xi(t) = a(t,\xi(t))\,dt + A(t,\xi(t))\,dW_t,\ \xi(\tau) = \nu.$

[16] Mark Kac. "On Some Connections between Probability Theory and Differential and Integral Equations". In: Proceedings of the second Berkeley symposium on mathematical statistics and probability,2 (1951). 189–216.

**Logarithm case.** If $h(x) = \ln x, x > 0$ then $u(\tau, x,\nu) = \ln x + f^1(\tau, \nu)$, where $f^1(\tau,\nu)$ is solution of *linear* PDE

$$f^1_\tau + \frac{1}{2}\sigma^2\nu f^1_{\nu\nu} + \kappa(\theta - \nu)f^1_\nu + r + \frac{1}{2}(\mu - r)^2\frac{1}{\nu} = 0, \;\; f^1(T,\nu) = 0.$$

**Exp case.** If $r = 0$, $h(x) = -e^{-\gamma x}, x \in R, \gamma > 0$ then $u(\tau, x,\nu) = -e^{-\gamma x}\left(f^2(\tau, \nu)\right)^{\delta_2}$, where $\delta_2 = \frac{1}{1-\rho^2}$, $f^2(\tau,\nu)$ is solution of *linear* PDE

$$f^2_\tau + \frac{1}{2}\sigma^2\nu f^2_{\nu\nu} + \kappa(\theta_2 - \nu)f^2_\nu + \frac{1}{2}\mu^2(1 - \rho^2)\frac{1}{\nu}f^2 = 0, \;\; f^2(T,\nu) = 1, \quad \theta_2 = \theta - \frac{\mu\rho\sigma}{\kappa}.$$

**Power case.** If $h(x) = \frac{1}{\gamma}x^\gamma, x > 0, \gamma \in (0,1)$ then $u(\tau, x,\nu) = \frac{1}{\gamma}x^\gamma\left(f^3(\tau, \nu)\right)^{\delta_3}$, where $\delta_3 = \frac{1-\gamma}{1-\gamma+\rho^2\gamma}$, $f^3(\tau,\nu)$ is solution of *linear* PDE

$$f^3_\tau + \frac{1}{2}\sigma^2\nu f^3_{\nu\nu} + \kappa(\theta_3 - \nu)f^3_\nu + k(\nu)\frac{1}{\nu}f^3 = 0, \;\; f^3(T,\nu) = 1, \quad \theta_3 = \theta - \frac{\gamma\rho\sigma(\mu - r)}{(1 - \gamma)\kappa},$$

where $k(\nu) = \frac{1}{2}\frac{\gamma((\mu-r)^2}{(1-\gamma)\delta_3} - \frac{\gamma((1-\gamma)r)}{(1-\gamma)\delta_3}\nu$.

Let us transform *FnL* PDE (3.1) to the *quasilinear* PDEs system $\supset \cancel{1+2=3}\ 2\ PDEs$

$$\frac{\partial u}{\partial \tau} + \frac{1}{2}\,\mathrm{Tr}[B\ \nabla^2 u] + \langle c, \nabla u\rangle + \psi_0(\nu, v_1, v_{1x}, v_{1\nu}) = 0, \quad u(T,x,\nu) = h(x), \quad (3.2)$$

$$\frac{\partial v_1}{\partial \tau} + \frac{1}{2}\,\mathrm{Tr}[B\ \nabla^2 v_1] + \langle c, \nabla v_1\rangle + \psi_1(\nu, v_1, v_{1\nu}) = 0, \quad v_1(T,x,\nu) = h_x(x), \quad (3.3)$$

where $v_1 = v_1(\tau,x,\nu) = u_x, \quad \nabla v_1 = (v_{1x}, v_{1\nu}),\ \alpha(\nu, v_1, v_{1\nu}) = (\mu - r)v_1 + \rho\sigma\nu v_{1\nu},$

$$B = B(\nu, v_1, \nabla v_1) = \begin{pmatrix} \frac{\alpha^2(\nu, v_1, v_{1\nu})}{\nu (v_{1x})^2} & -\frac{\rho\sigma\alpha(\nu, v_1, v_{1\nu})}{v_{1x}} \\ -\frac{\rho\sigma\alpha(\nu, v_1, v_{1\nu})}{v_{1x}} & \nu\sigma^2 \end{pmatrix} > 0, \quad c = c(x,\nu) = \begin{pmatrix} rx \\ \kappa(\theta - \nu) \end{pmatrix},$$

$$\psi_0(\nu, v_1, v_{1x}, v_{1\nu}) = -(\mu - r)\frac{\alpha(\nu, v_1, v_{1\nu})}{\nu}\frac{v_1}{v_{1x}}, \quad \psi_1(\nu, v_1, v_{1\nu}) = r v_1 - (\mu - r)\frac{\alpha(\nu, v_1, v_{1\nu})}{\nu}.$$

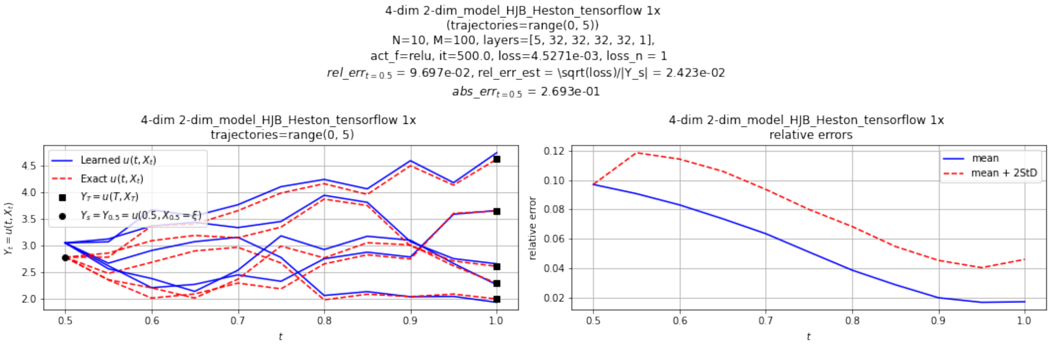



Figure 1 – Solution of (3.1)(500 iter)

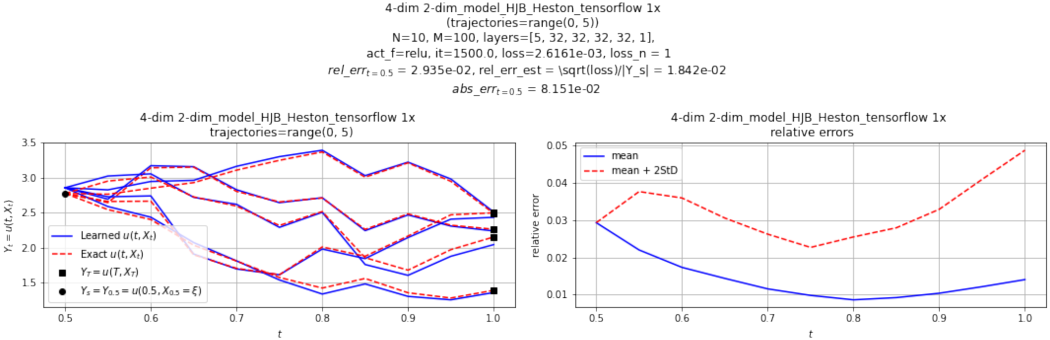



Figure 2 – Solution of (3.1)(1500 iter)

`We present` a *probabilistic approach* and a numerical method based on it for solving the Cauchy problem for *fully non-linear* parabolic equations.
Considered equations are reduced to systems of *quasilinear* parabolic PDEs and a *probabilistic representation* for *viscosity solution* of PDE systems `is constructed`.
To construct this representation we use the *Forward-Backward SDE (FBSDE)* theory and `we apply` an `alternative` *probabilistic representation* with respect to *2BSDE* (suggested in [Che+07]).
We follow the *FBSNN* type methodology suggested by Raissi [Rai18a] for and `we modify` FBSNN methodology to apply it to *fully non-linear* parabolic PDEs.
As an example, `we construct a numerical solution` of an equation, which describes the optimal control of portfolio investments in the Black-Scholes and the Heston markets.

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*Thank you for your attention!*