Rolfsen's Conjecture and wild knots that pierce wild disks

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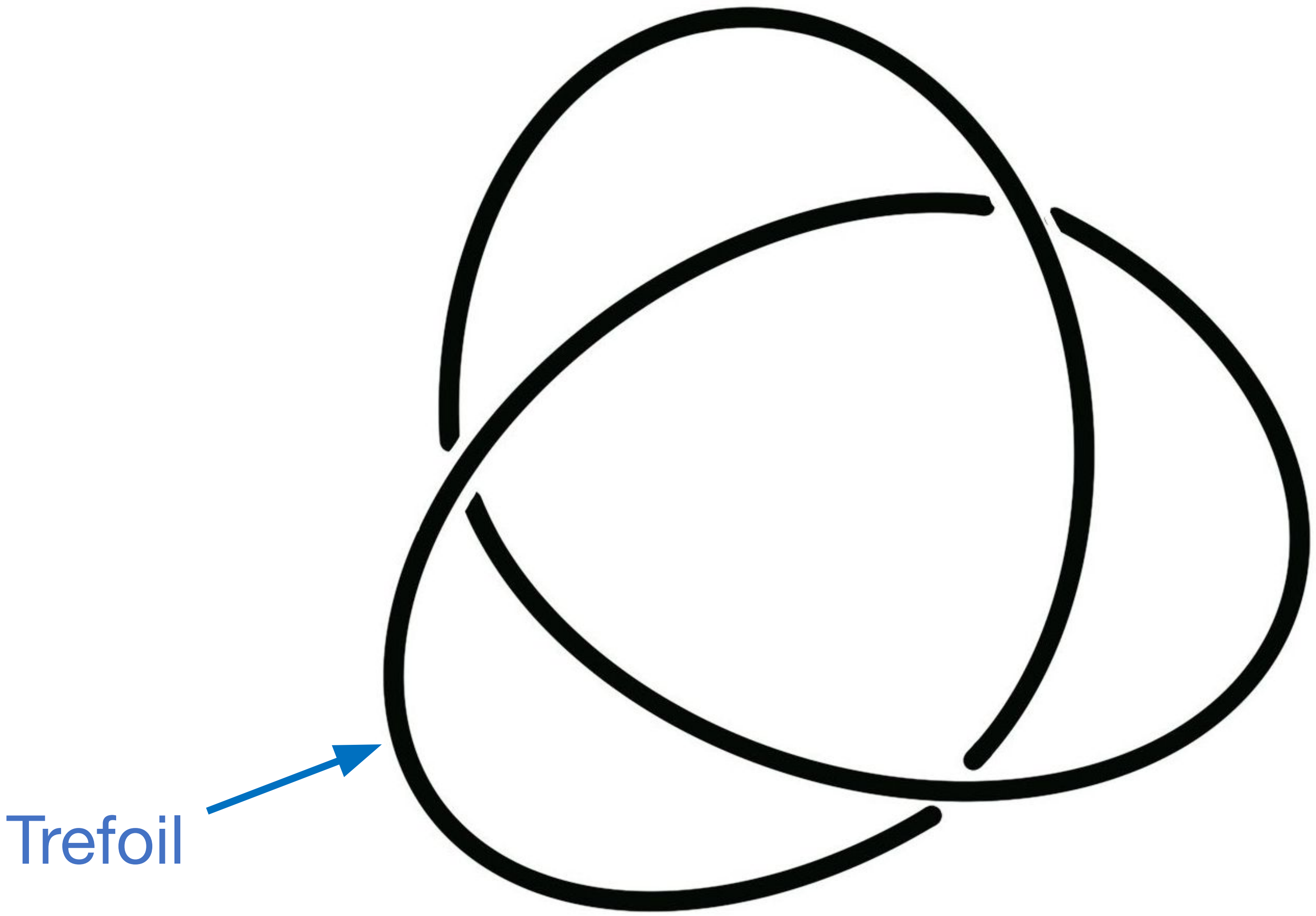
Example: Non-ambient isotopy from trefoil to unknot:



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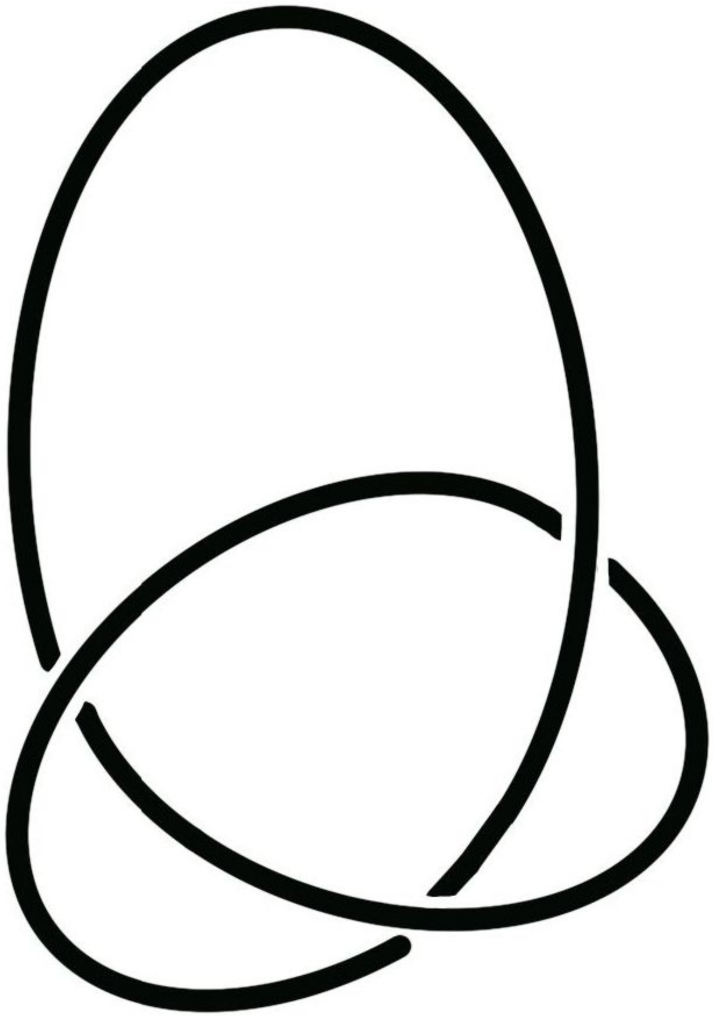

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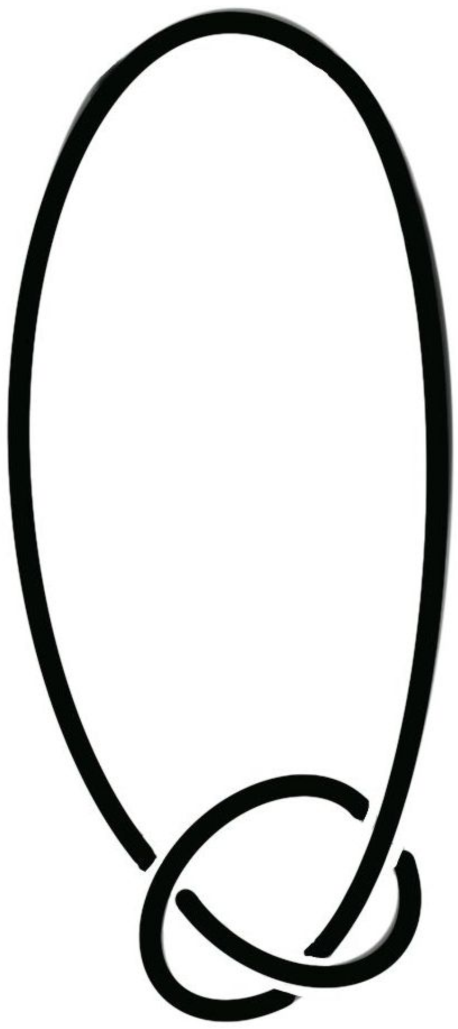

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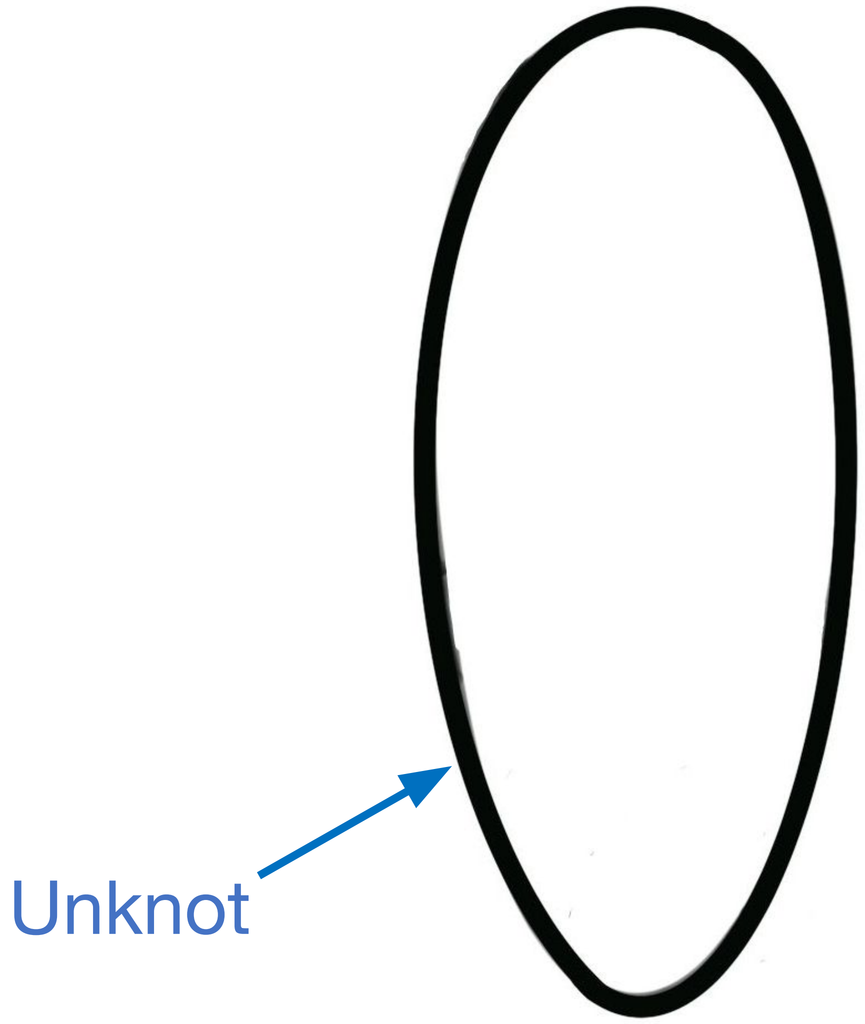
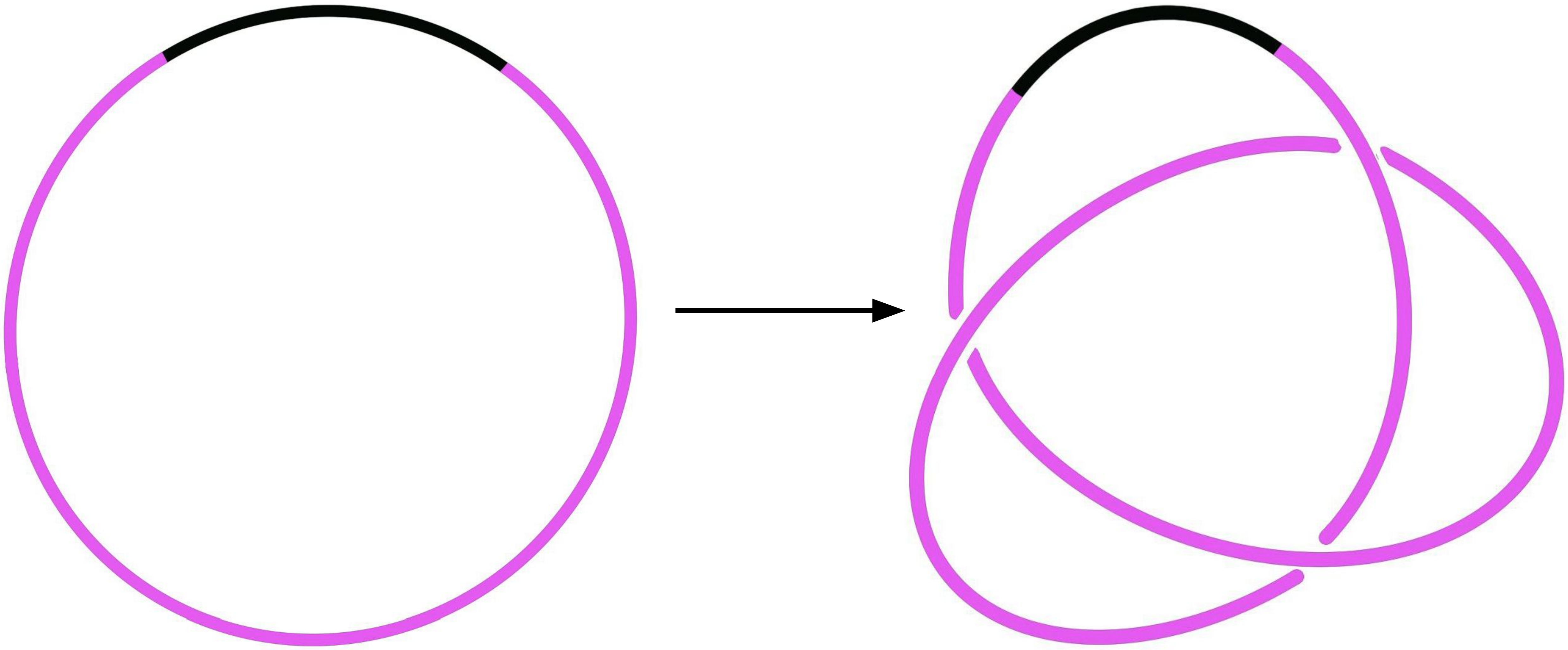


Note: Non-ambient isotopy requires ***reparametrization*** in domain.



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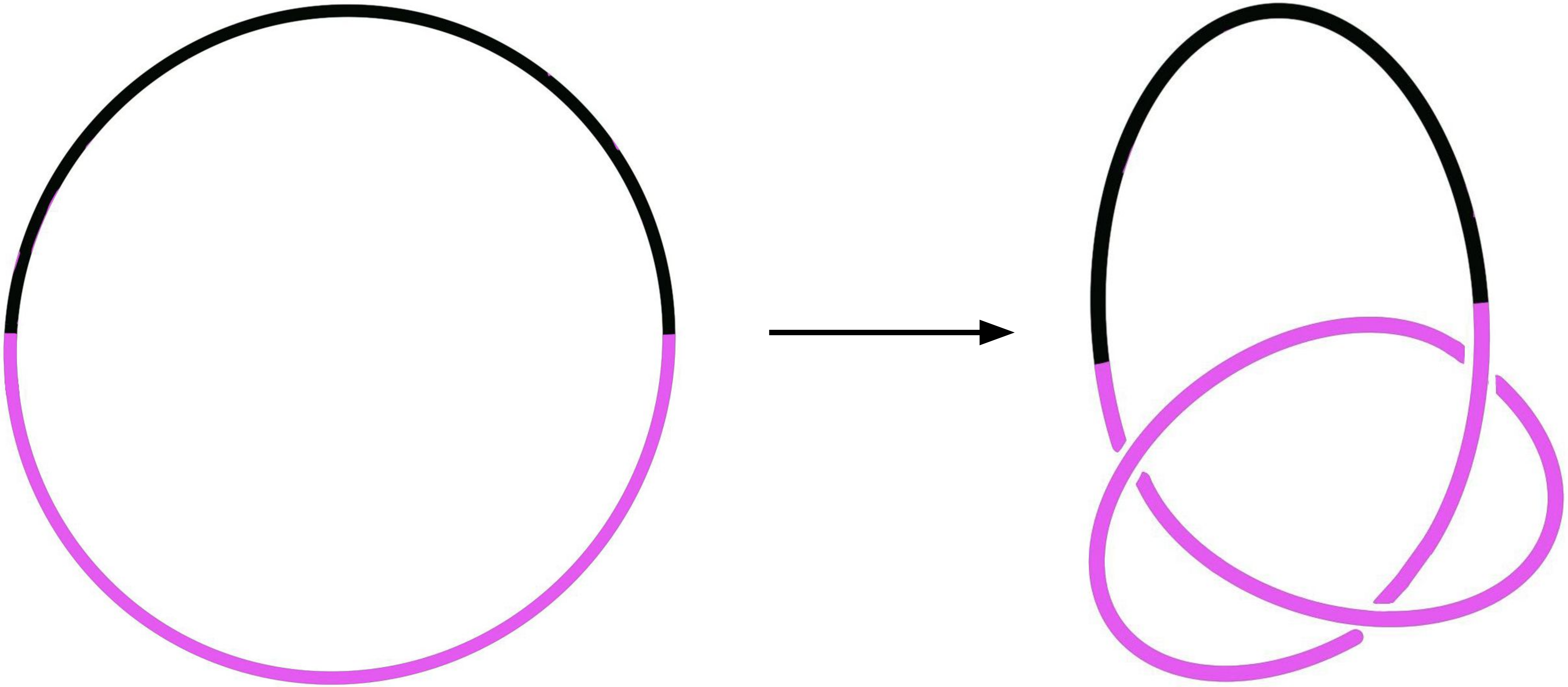

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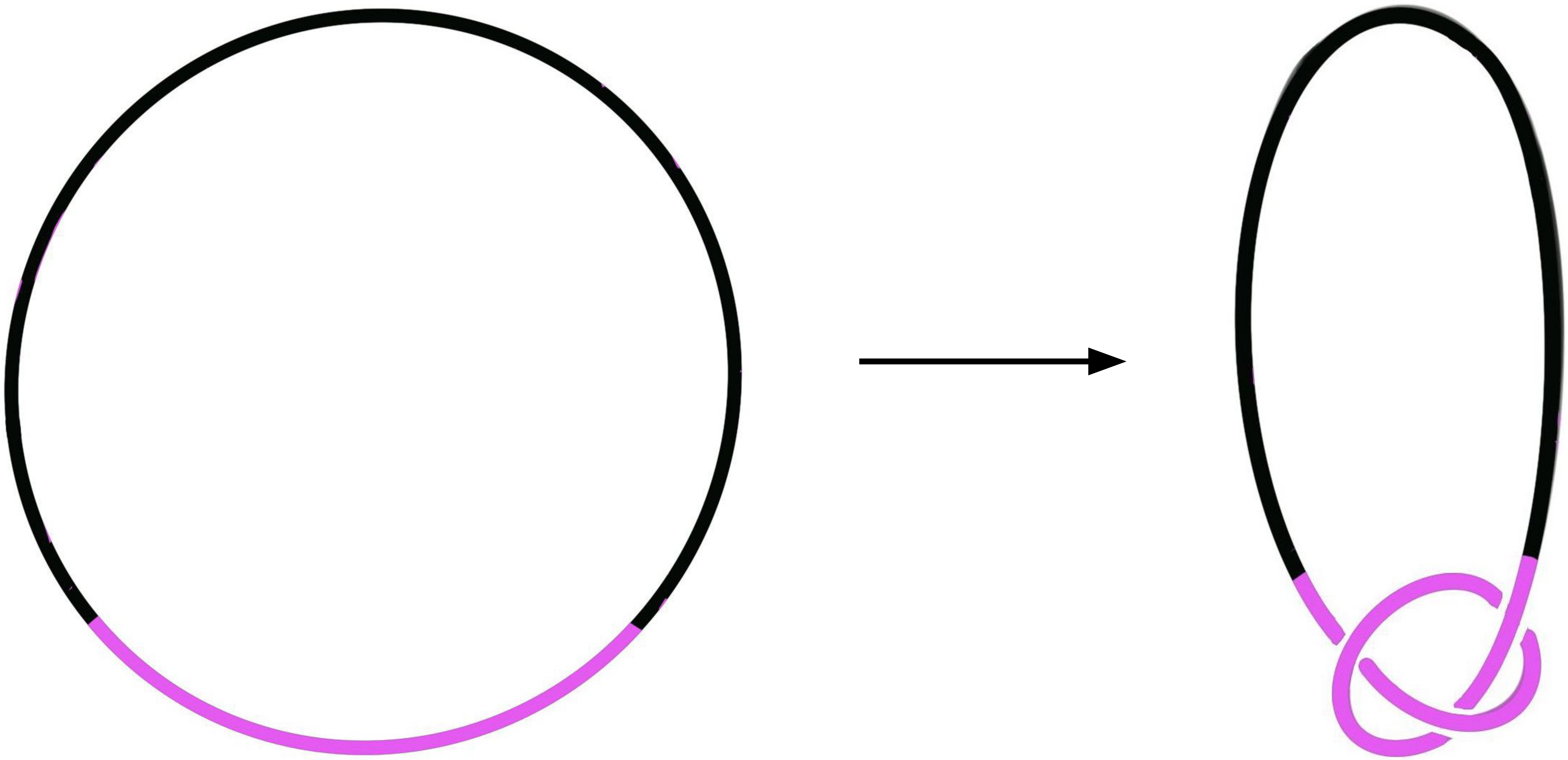

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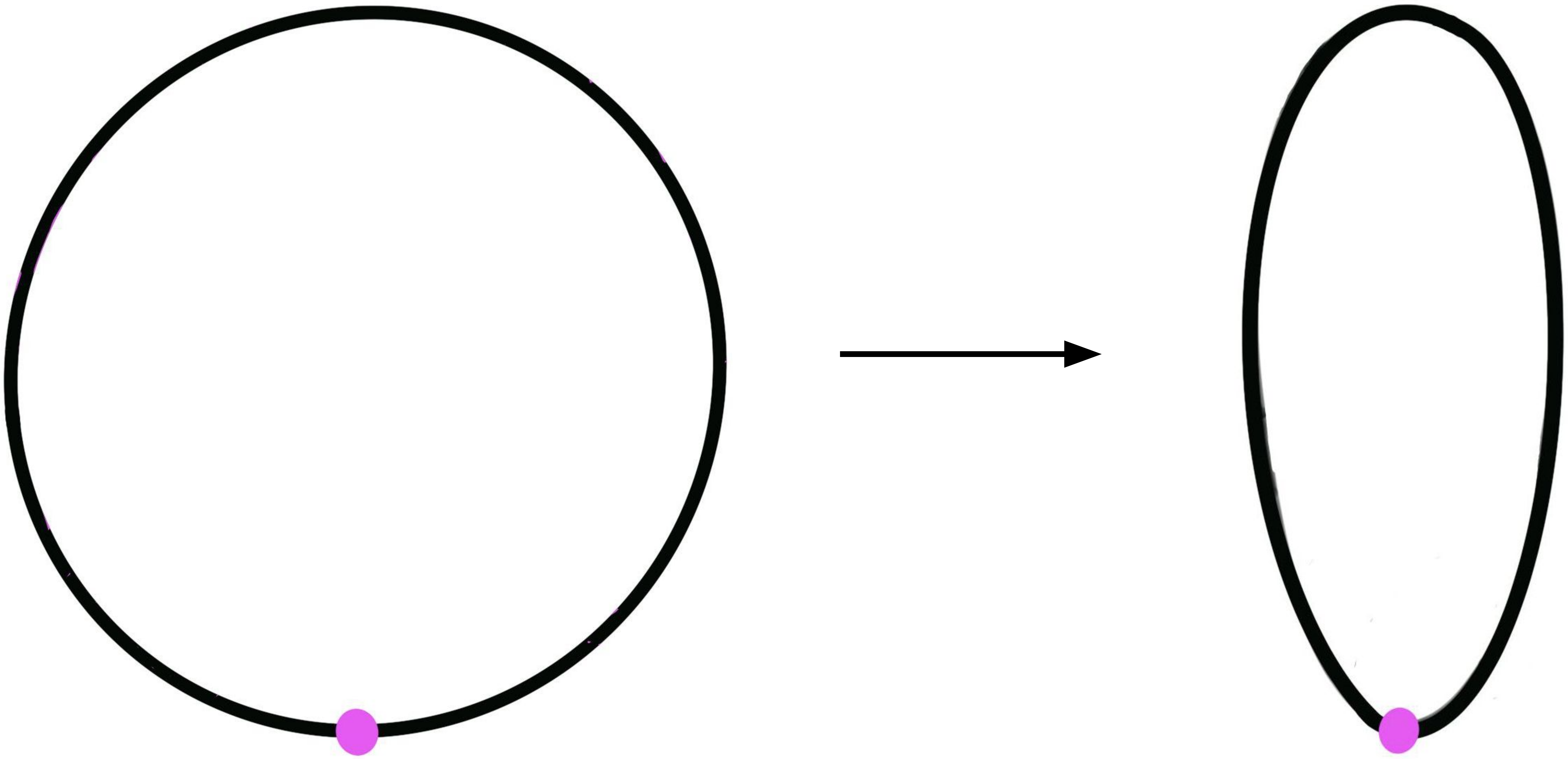
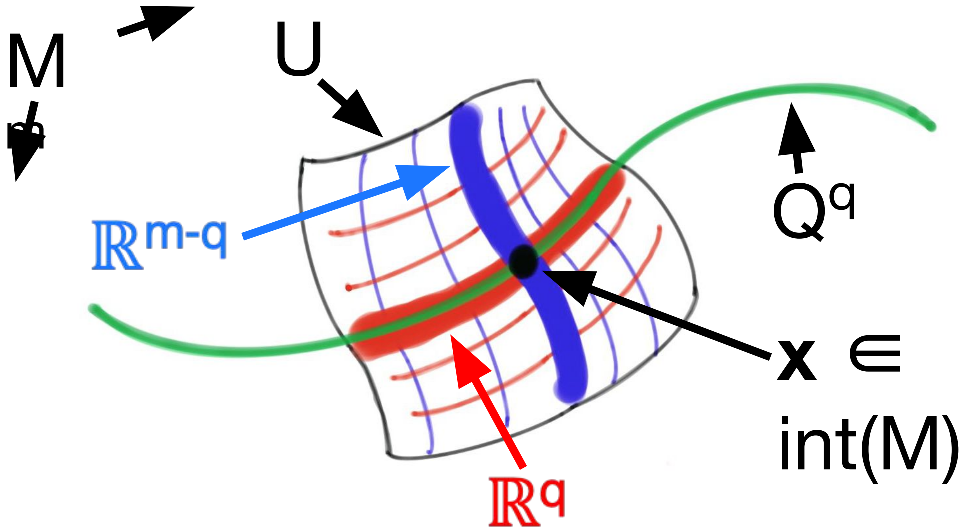


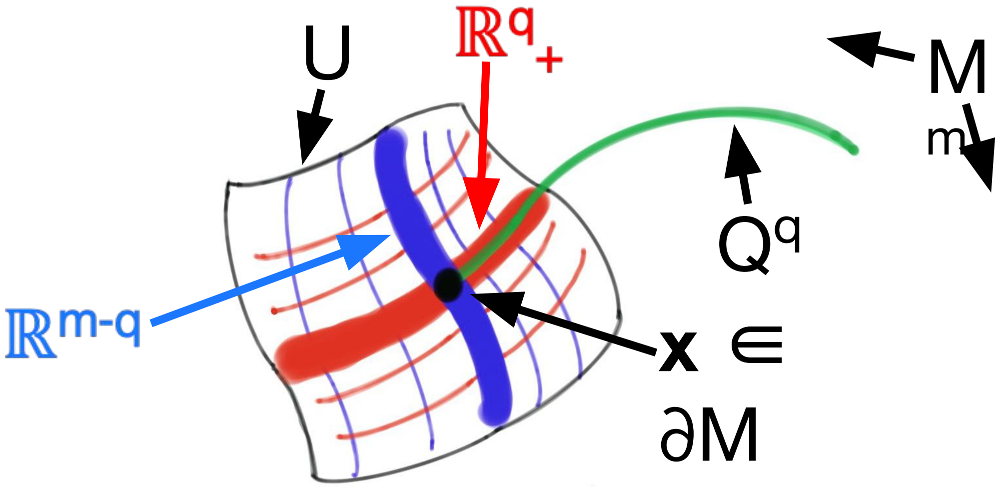
Let $Q^q \subset \mathrm{int}(M^m)$ where Q^q and M^m are manifolds. Let $\mathbf{x} \in Q^q$.

Q^q is ***tame at*** $\mathbf{x}$ if

$\exists$ an open nbhd U of $\mathbf{x}$ in M such that

$(U, U \cap Q^q) \approx (\mathbb{R}^q \times \mathbb{R}^{m-q}, \mathbb{R}^q \times \{\mathbf{0}\})$





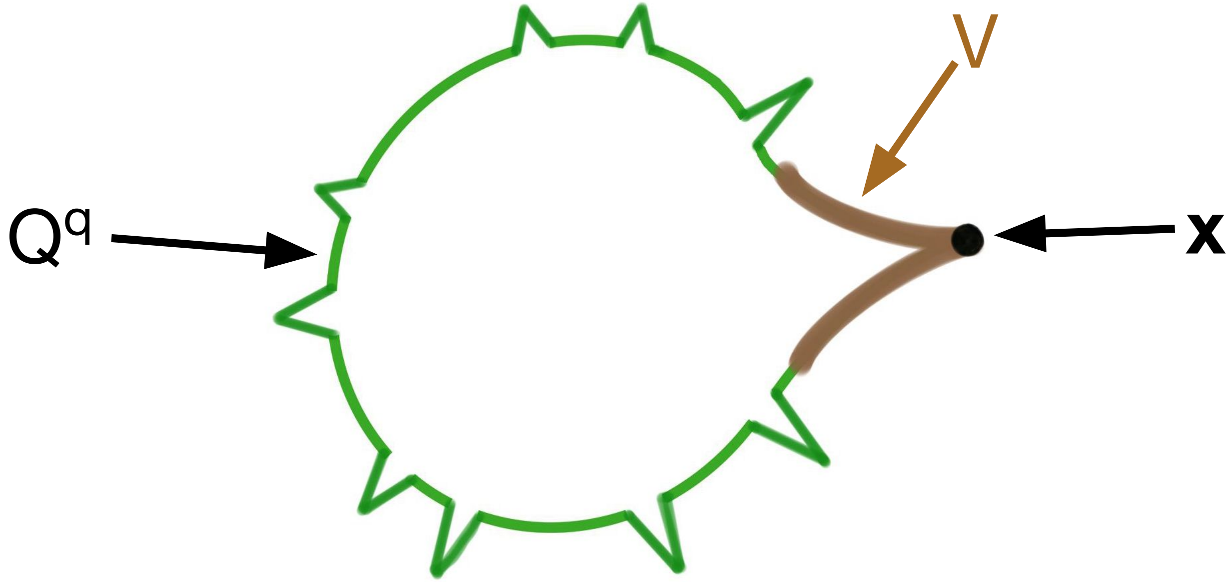
If not, Q^q is ***wild at*** $\mathbf{x}$.

Q^q is ***tame in*** M^m if

x is an *isolated* ***wild*** *point* of Q^q if

Q^q is **wild** at **x** and

∃ an open nbhd V of **x** in Q^q such that Q^q is **tame** at each point of V – {**x**}.



Let M be a 3-manifold.

A *knot* in M is a (tame or wild) embedded S^1 in int(M).

A *link* in M is a disjoint union of finitely many knots.

Let K, L $\subset$ M.

K is *ambiently isotopic* to L if

$\exists$ a level-preserving **homeomorphism** $h : M \times [0,1] \to M \times [0,1]$ such that $h = id$ on $M \times \{0\}$ and $h(K \times \{1\}) = L \times \{1\}$.

K is ***non***-*ambiently isotopic* to L if

$\exists$ a level-preserving **embedding** $e : K \times [0,1] \to M \times [0,1]$ such that $e(K \times \{0\}) = K \times \{0\}$ and $e(K \times \{1\}) = L \times \{1\}$.

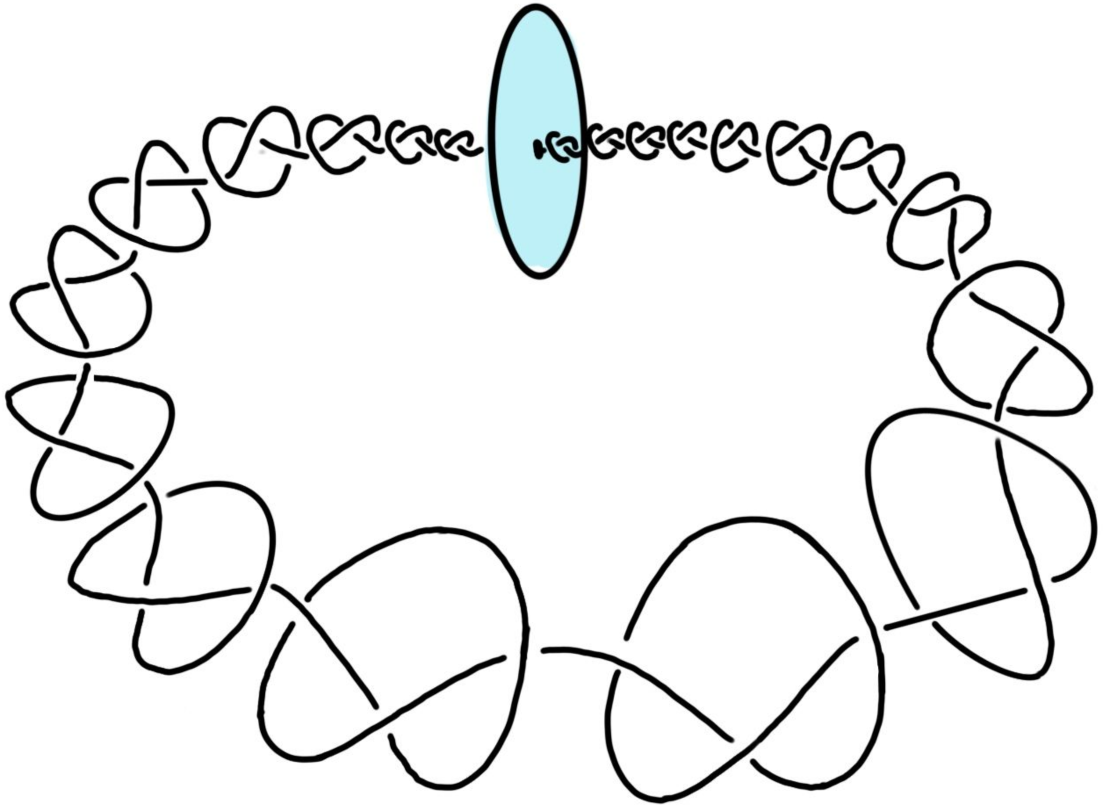
Example. A trefoil knot is: ***non***-ambiently isotopic to an unknot.

not ambiently isotopic to an unknot.

Rolfsen's Conjecture (1974)**:**

Every knot in S^3 is non-ambiently isotopic to an unknot.

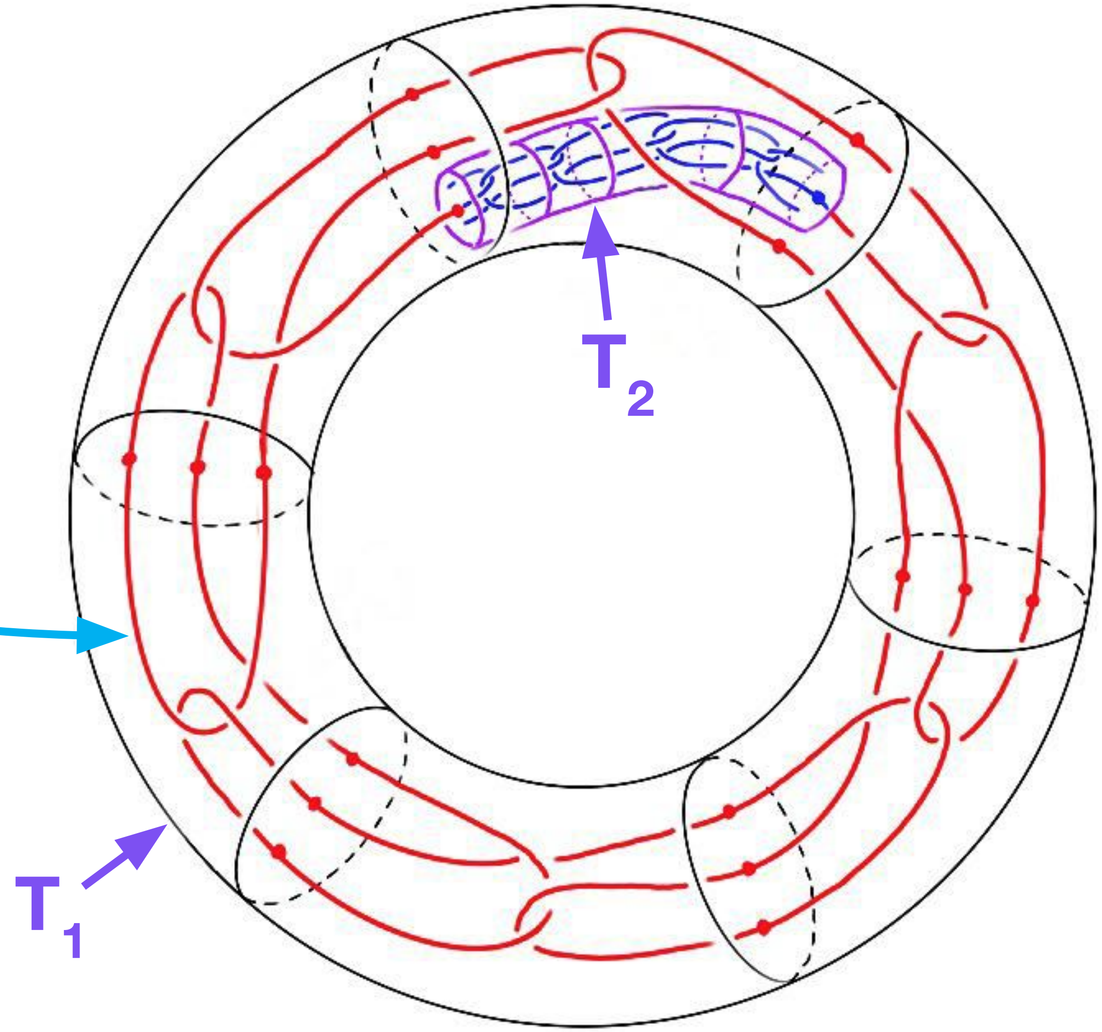
Example: A wild knot that is non-ambiently isotopic to an unknot.



Rolfsen's Conjecture is probably ***false***.

Main candidate for a counterexample:

The Bing sling: $\mathcal{B} = \bigcap_{i \geq 1} T_i$



Partial results concerning Rolfsen's conjecture:

A *thickening* of a link K in a 3-manifold M is:
a compact 3-manifold T in M such that $K \hookrightarrow T$ is a homotopy equivalence and each component of T is a solid torus or a solid Klein bottle.

K is *thickenable* if it has a thickening.

Let T be a solid torus or solid Klein bottle.

$\therefore T = D^2 \times [0,1]/\sim$ where $(x,1) \sim (h(x),0)$ and
either $h : D^2 \to D^2$ = identity

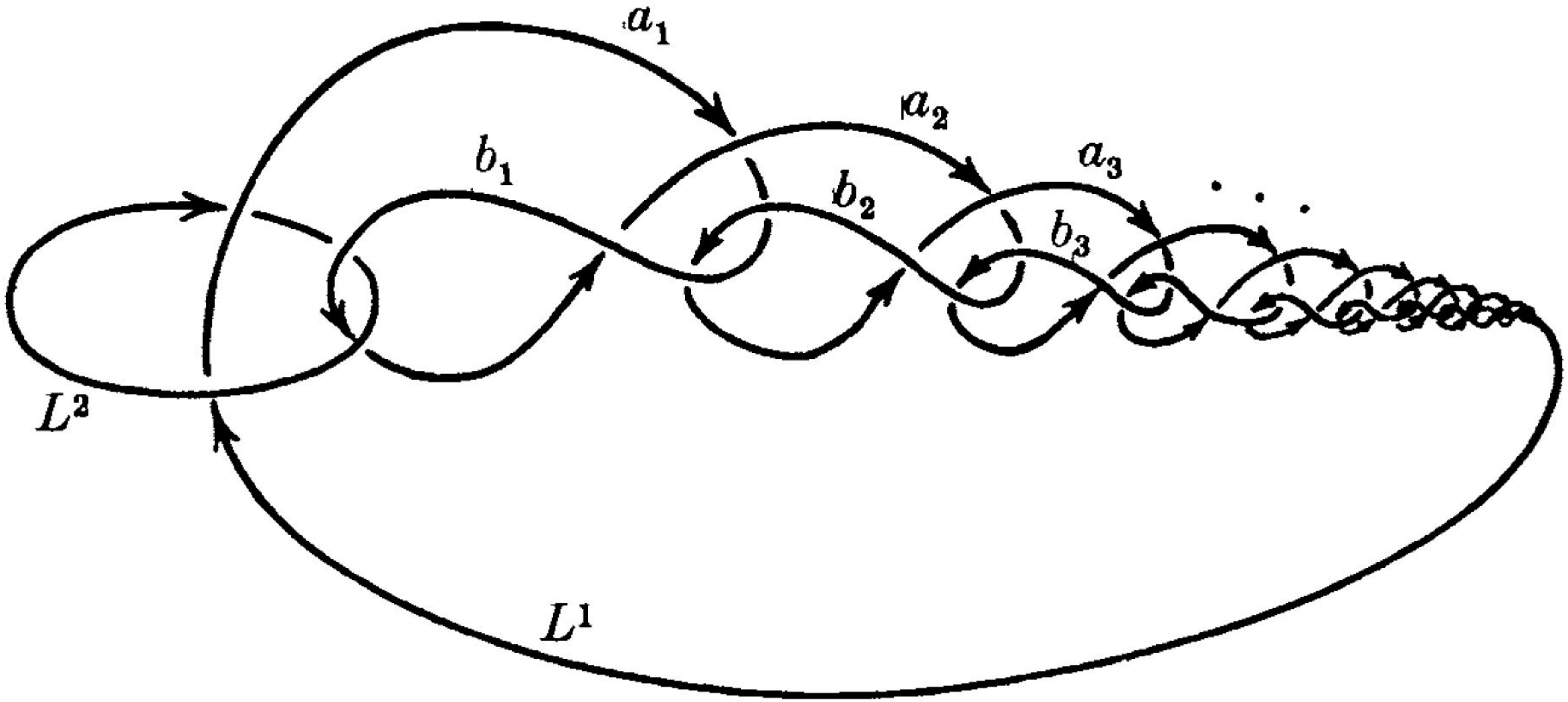
The simple closed curve $\{(0,0)\} \times [0,1]/\sim$ is a *core* of T.

Let T be a thickening.
$J \cap t$ is a core of t for every component t of T.

Theorem: Every knot in S^3 has a thickening.
(??,

Example: ∃ a 2-component link in S^3 that has **no** thickening.

The **Milnor link** (1957):



Exercise: The Milnor link is non-ambiently isotopic to an unlink.

Let $K, L \subset M$.

K is *semi-isotopic* to L if

$\exists$ an embedding $e : K \times [0,1] \to M \times [0,1]$ such that
$e(K \times \{0\}) = K \times \{0\}$, $e(K \times \{1\}) = L \times \{1\}$ and
$\exists$ a level-preserving homeo $h : K \times [0,1) \to e(K \times [0,1)) \subset M \times [0,1)$.

In other words:
but this reparametrization may not extend to $K \times \{1\}$.

Updated and generalized Giffen Theorem:
If a link K in a 3-manifold M has a thickening with core J,
then K is semi-isotopic to J.
(Giffen, 1976 – unpublished,

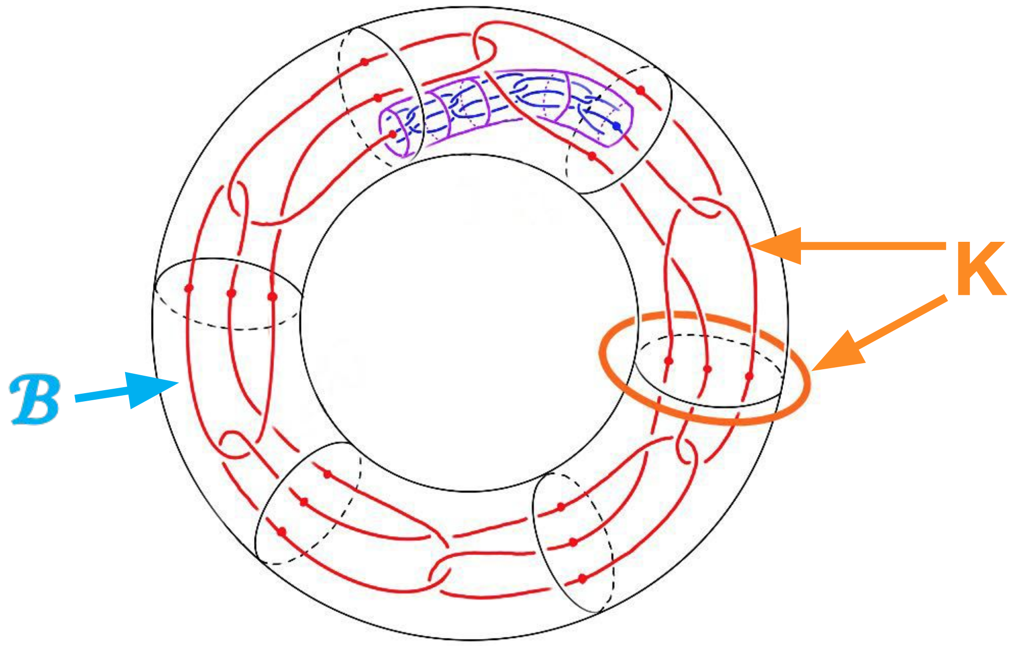
(**Proof:** *Shift spinning*.)

Rolfsen's Conjecture is **false** for 2-component links:

Two theorems of Melikhov (2024):

1) ∃ a 2-component link in S^3 that is **not** non-ambiently isotopic to a tame link.

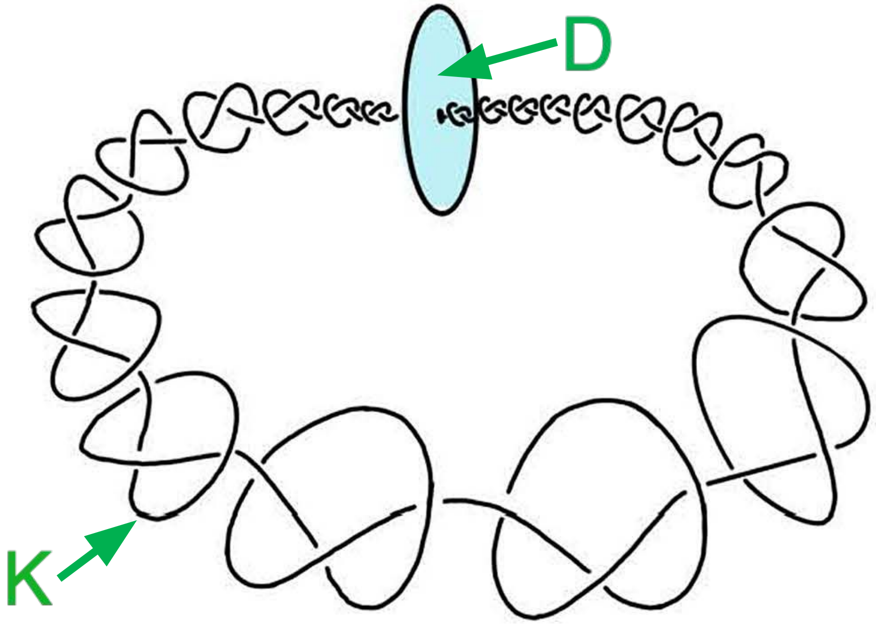
2) If **K** is a 2-component link with linking number 1 in which one component is the **Bing sling** $\mathcal{B}$, then **K** is **not** non-ambiently isotopic to a tame link.



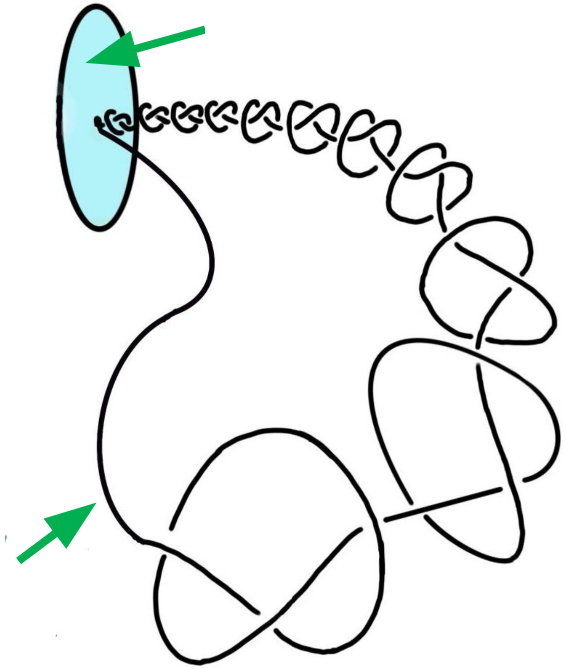
The knot invariants used in the proofs apply to **2**-component links, but **not** to knots.

A knot K in S^3 *pierces a disk* D if:

$K \cap D$ = a single interior point of D and



K pierces D

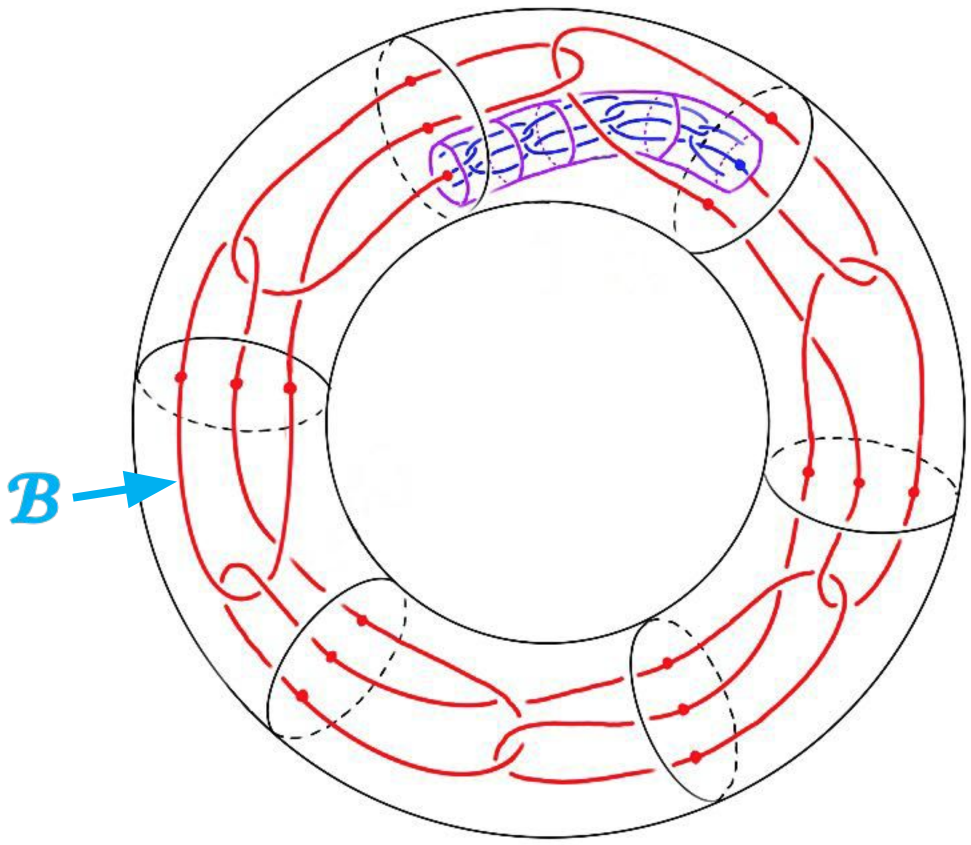


K′ does **not** pierce D

Theorem (Bing, 1956): The **Bing sling $\mathcal{B}$** pierces no disk.

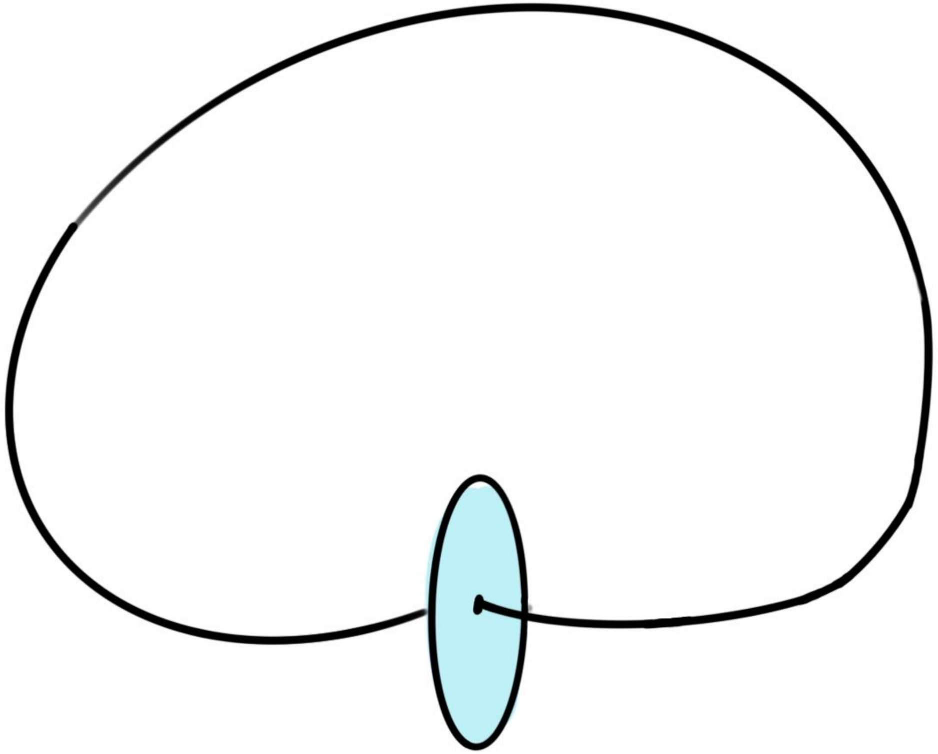
More precisely:

If the boundary of a disk D links $\mathcal{B}$,
then D ∩ $\mathcal{B}$ ⊃ a Cantor set.



Folk Theorem: Any knot in S^3 that pierces a ***tame*** disk is non-ambiently isotopic to an unknot.

You already know the ***proof:***

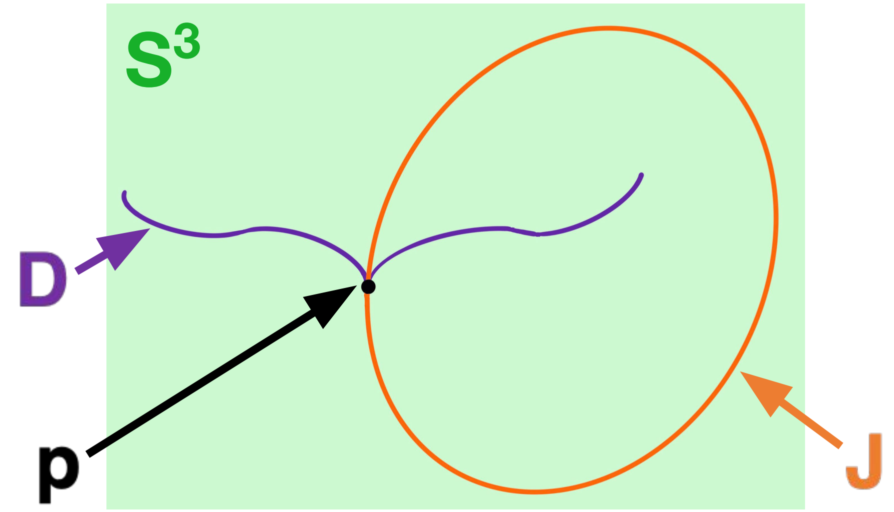


New Theorem:

non-ambiently isotopic to an unknot.

Proof:

Given: a knot **J** in $\mathbf{S^3}$ pierces a wild disk **D** at a point **p**.

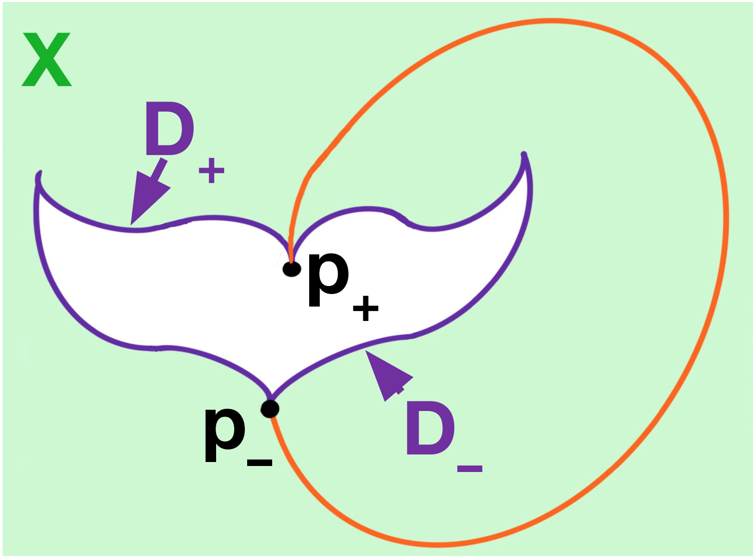


1) Make D tame mod p (Bing, 1957).

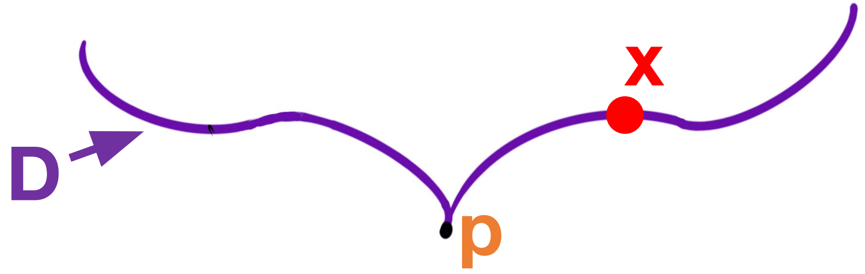
2) "Slice open" S^3 along **D**. Get **X**.

$\partial X = \mathbf{D_+} \cup \mathbf{D_-}$.

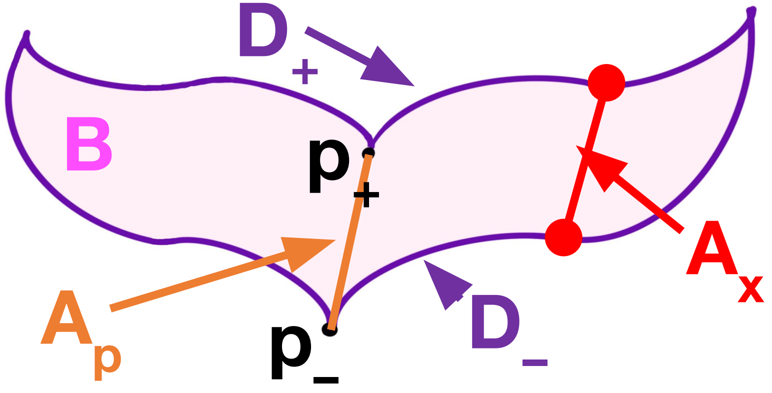
X is a 3-mfld with bdry except at $\mathbf{p_+}$, $\mathbf{p_-}$.

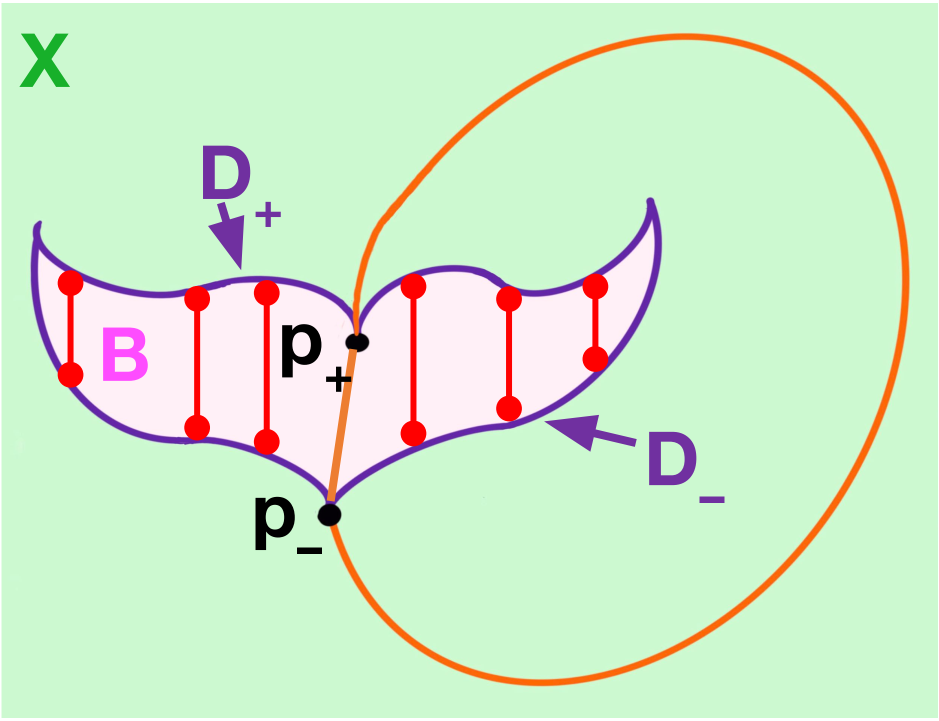


"Inflate" D to 3-ball B –
expand each x ∈ int(D) to arc A_x.



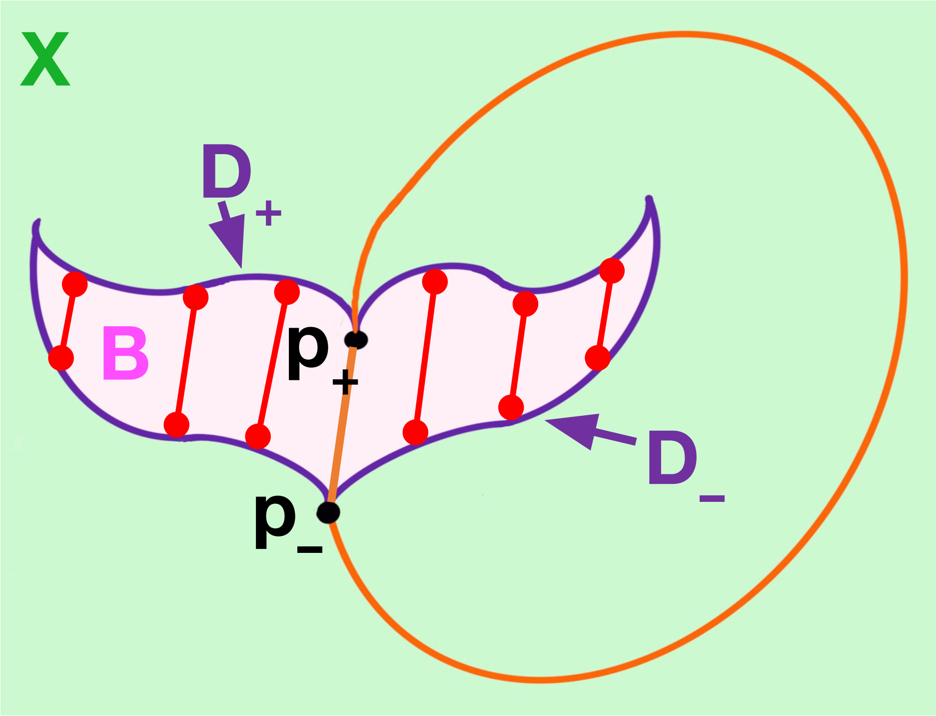
Attach B to X by identifying
$\partial X = D_+ \cup D_-$ with $\partial B = D_+ \cup D_-$.





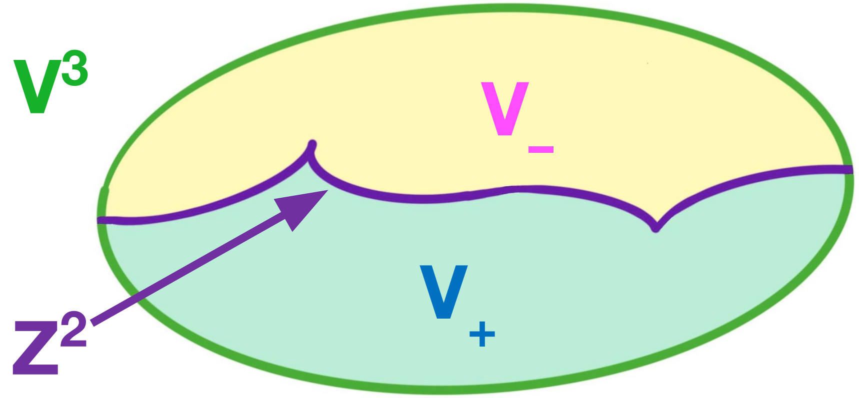
X ∪ B is a 3-manifold
except possibly at $\mathbf{p_+}$, $\mathbf{p_-}$.

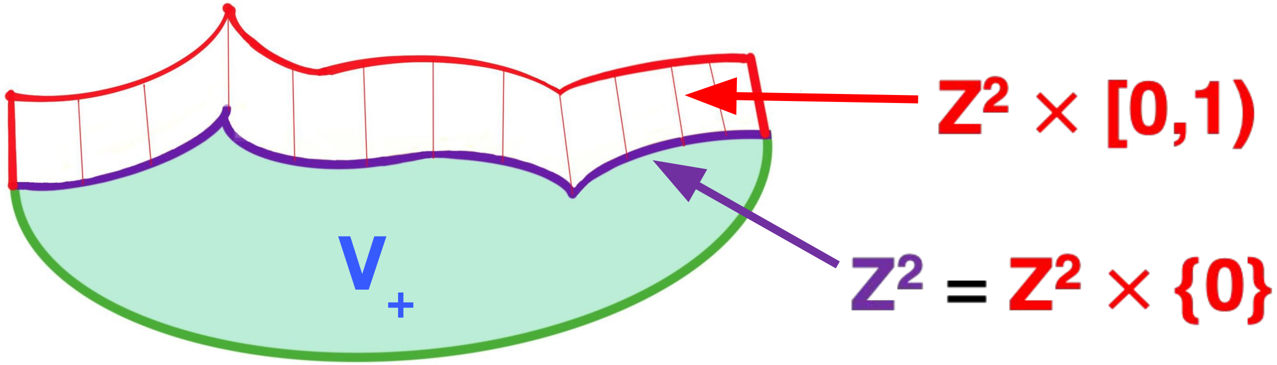
Claim: $X \cup B$ is a 3-manifold.



This follows from:

∗) If Z^2 (conn 2-mfld) $\subset$ (wild, closed) V^3 (conn 3-mfld) and Z^2 separates V^3 into V_+ and V_-, then $(V_+ \cup Z^2) \cup_{Z = Z \times \{0\}} (Z^2 \times [0,1))$ is a 3-manifold.





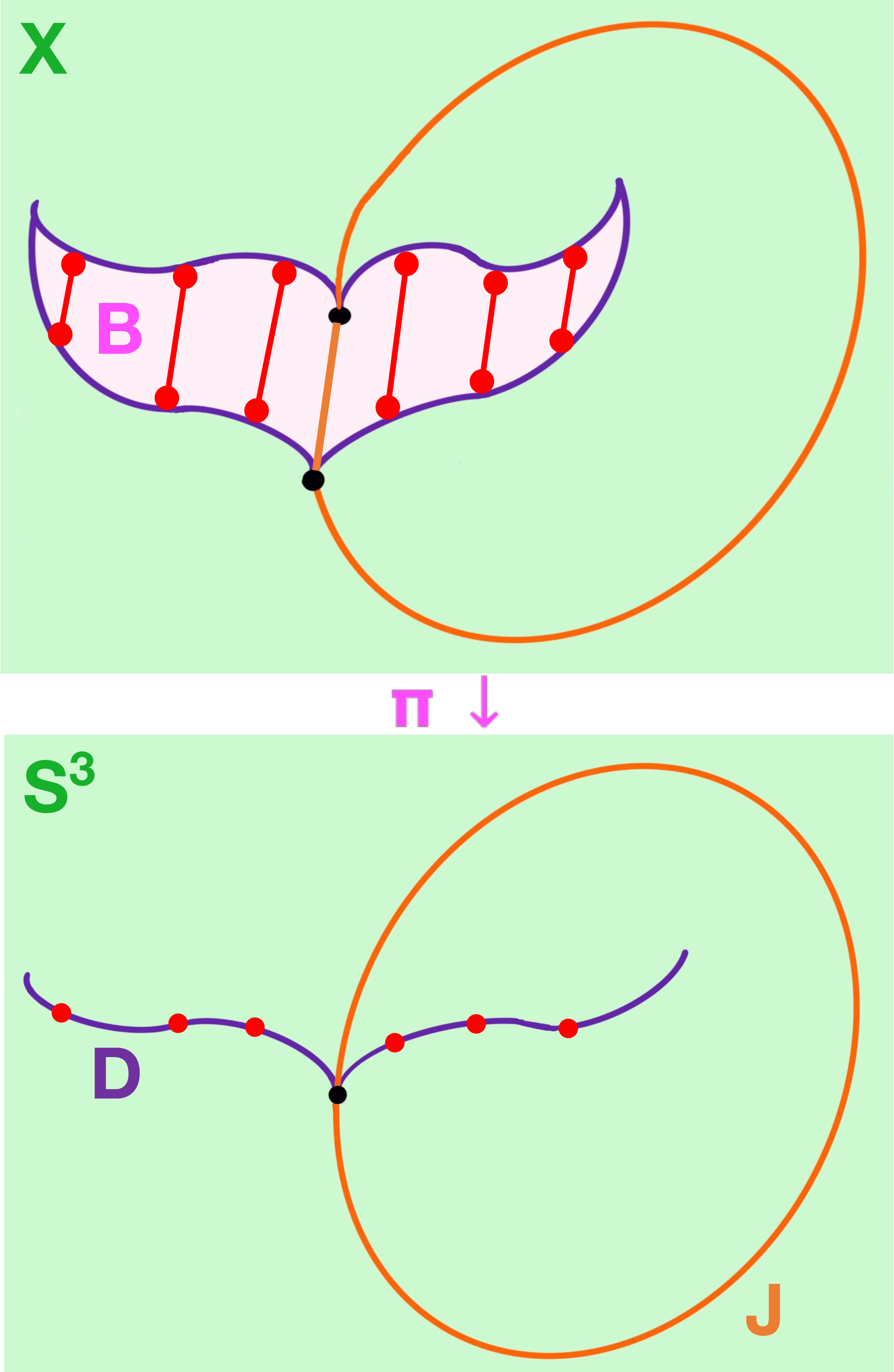
∗) proved by Hosay (1964), Lininger (1965) and later generalized by "mismatch theorems" of Eaton (1972) and Cannon-Daverman (1981).

Define $\pi : X \cup B \to S^3$ by
$\pi(A_x) = x$ for $x \in D$.

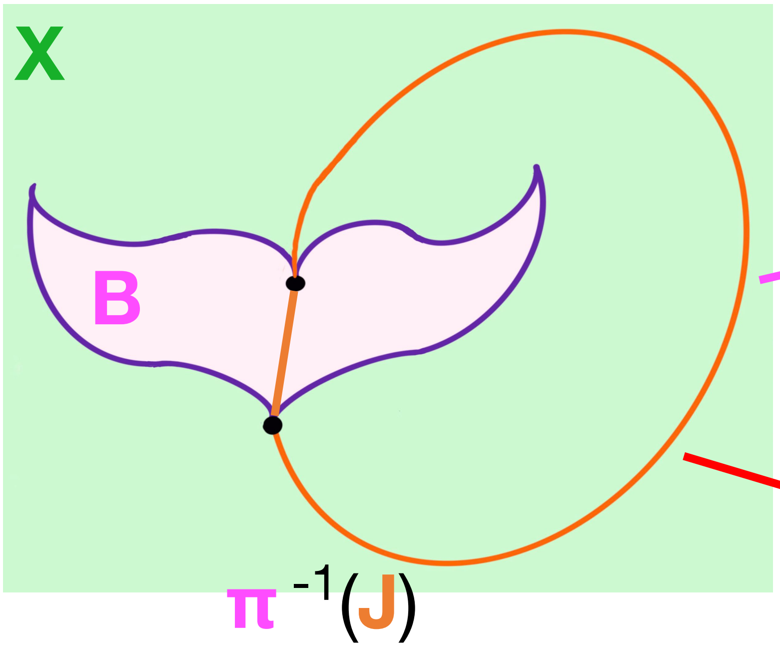
$\pi : X \cup B \to S^3$ is a
cell-like map between 3-manifolds.

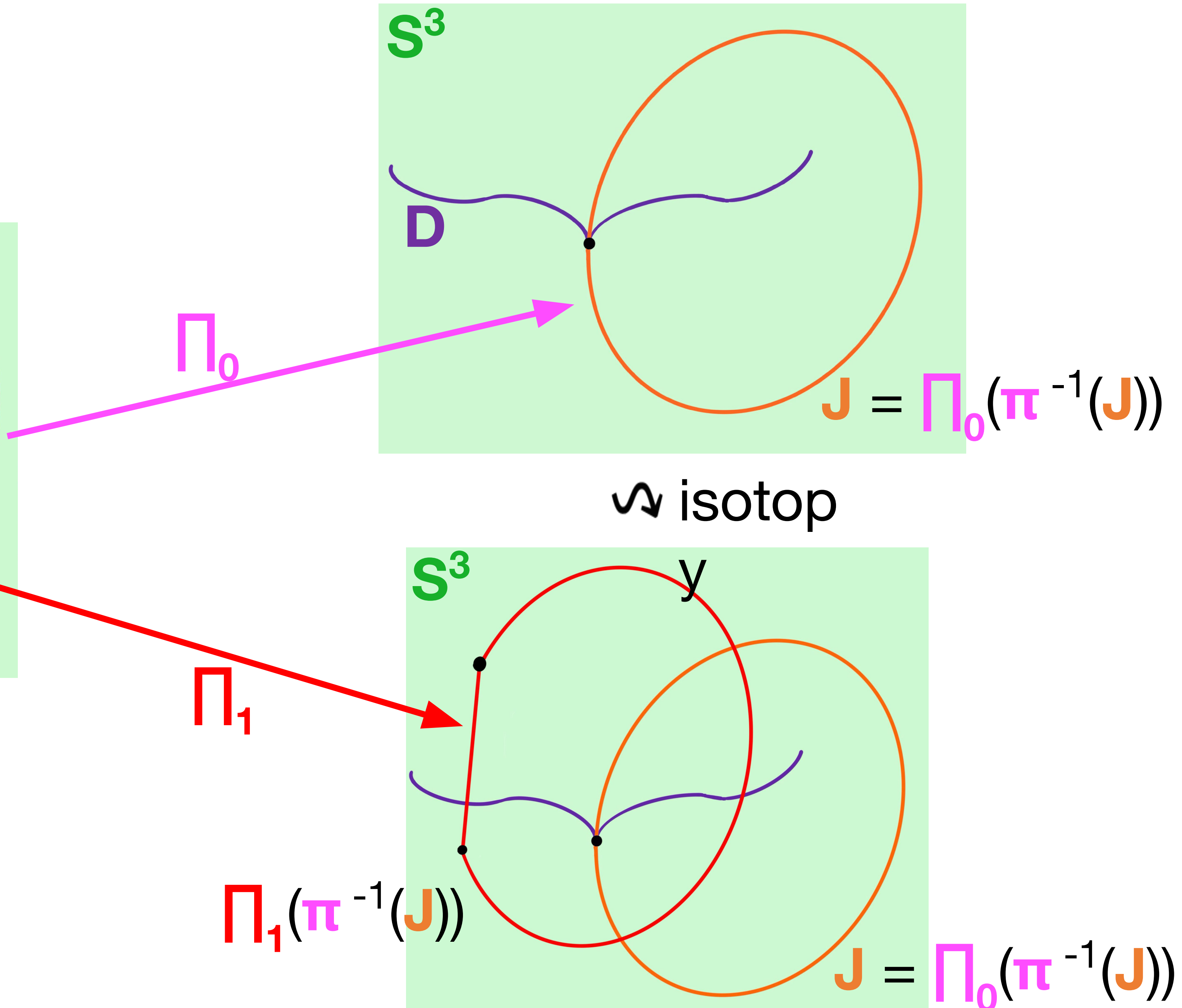
$\therefore$ $\pi : X \cup B \to S^3$ is a near-homeomorphism
(Armentrout (1971), Siebenmann (1972)).

$\therefore$ $\pi : X \cup B \to S^3$ is the **time-0 map** of a
pseudo-isotopy $\Pi : (X \cup B) \times [0,1] \to S^3$;
i.e., Π is a homotopy such that
$\Pi_t : X \cup B \to S^3$ is a homeo for $t \in (0,1]$
and $\Pi_0 = \pi$
(Edwards-Kirby (1971)).



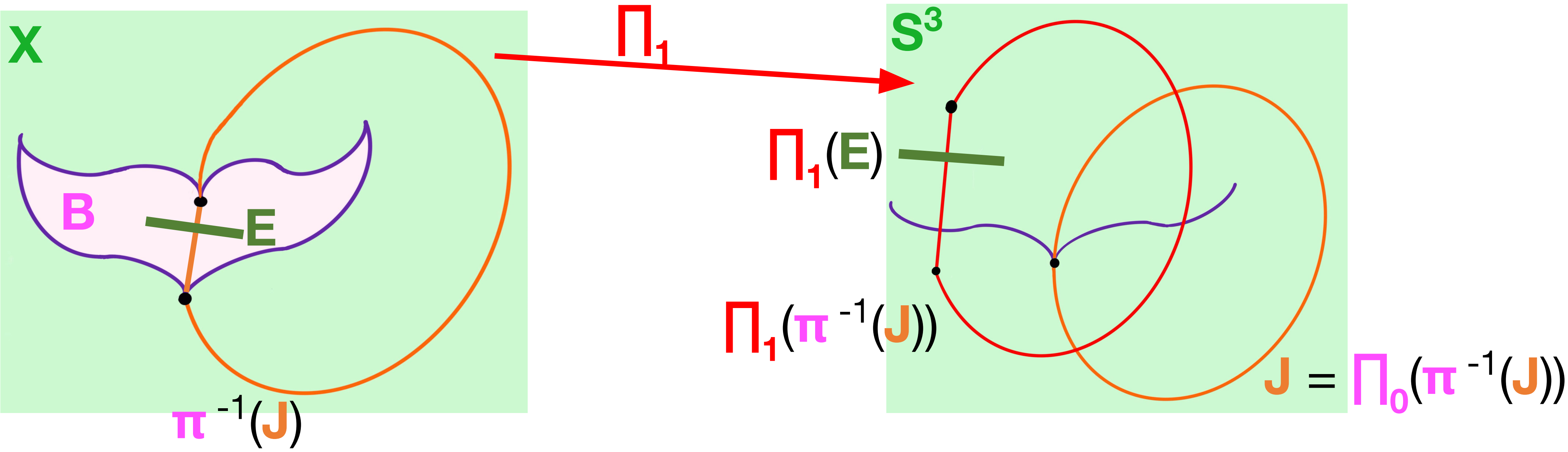
Apply Π_t to $\pi^{-1}(J)$ to obtain a non-ambient isotopy from
$J = \Pi_0(\pi^{-1}(J))$ to $\Pi_1(\pi^{-1}(J))$.
and ***reparametrize***.





∴ J is non-ambiently isotopic

$\pi^{-1}(J)$ pierces a tame disk E.



∴ $\Pi_1(\pi^{-1}(J))$ pierces a tame disk $\Pi_1(E)$.

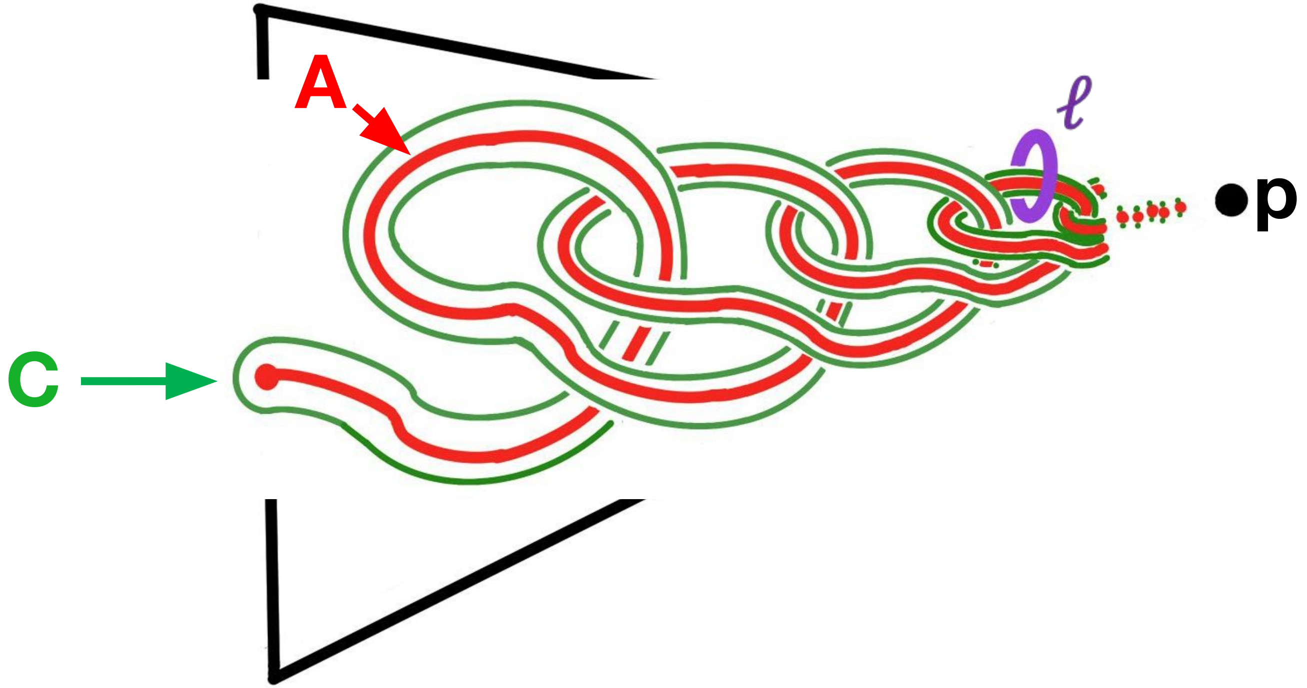
∴ $\Pi_1(\pi^{-1}(J))$ is non-ambiently isotopic to an unknot.

∴ J is non-ambiently isotopic to an unknot. □

Example: ∃ a wild knot in S^3 that pierces a wild disk but pierces no tame disk.

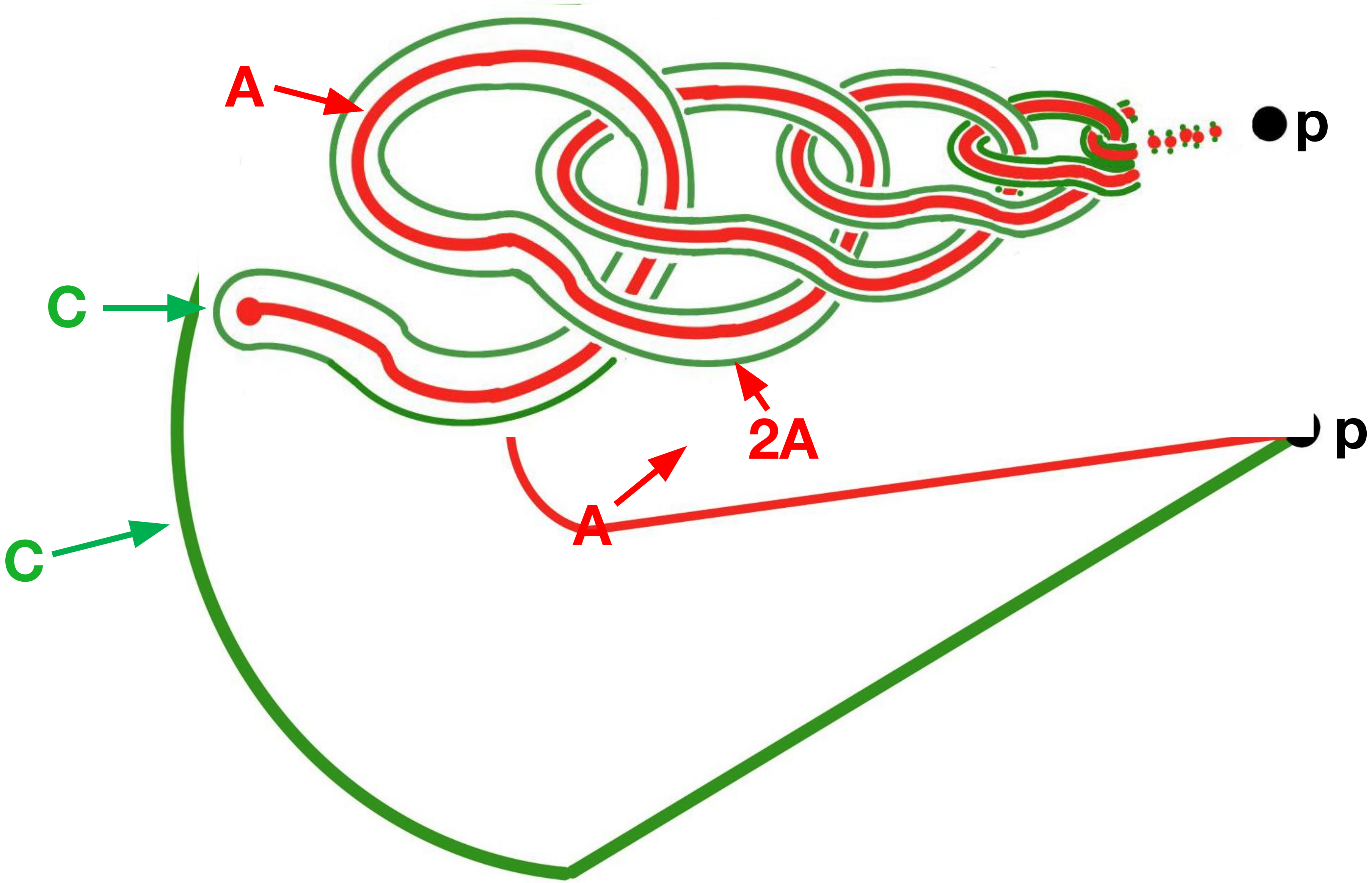
A – arc – **wild** at endpoint **p,** tame elsewhere (Fox-Artin, 1948).

A is **wild** at **p** because
∃ ***loop*** ℓ near **p** in $S^3 - A$
that is **not** null-homotopic
near **p** in $S^3 - A$.



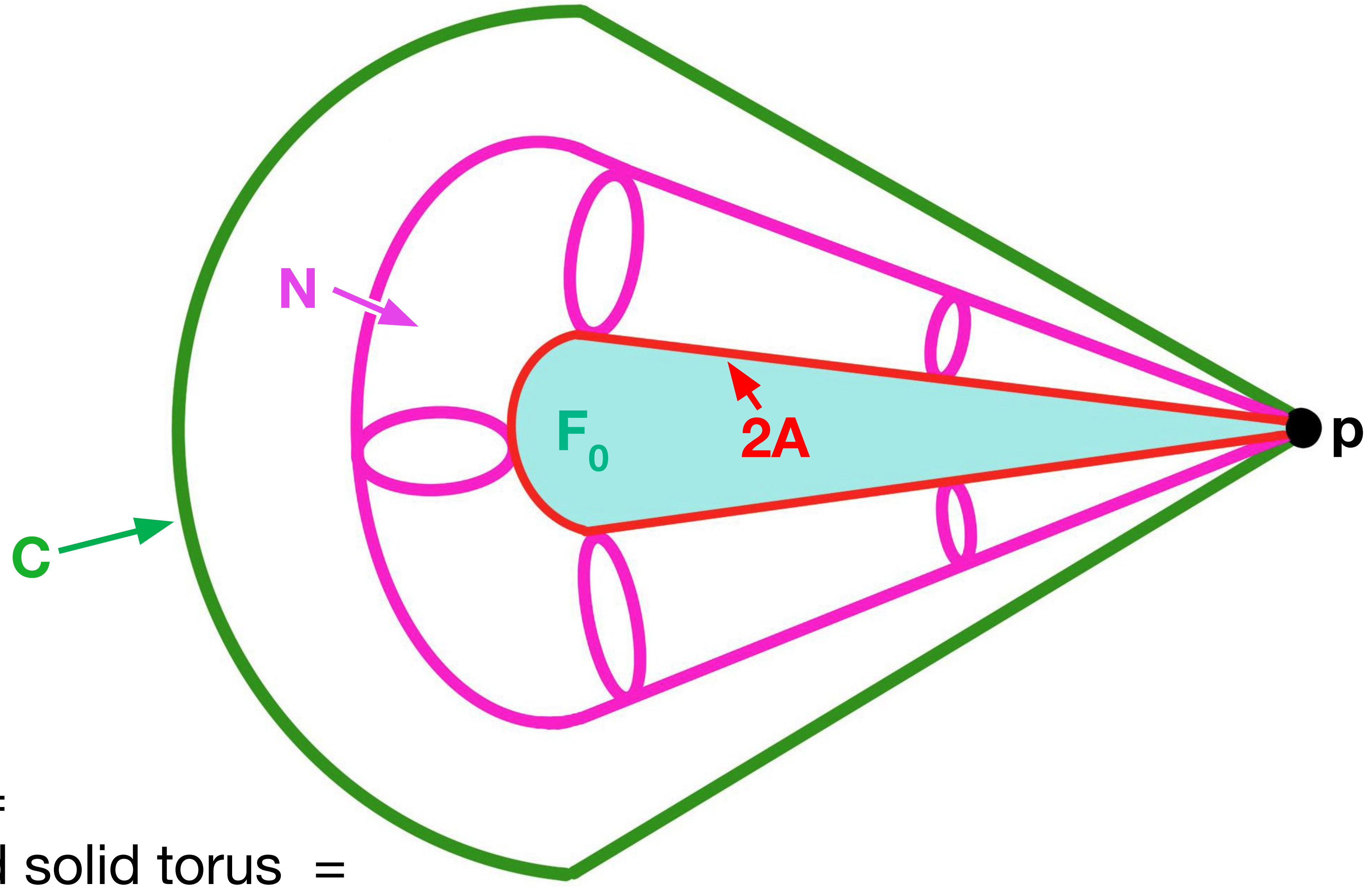
Thicken **A** to "tapered" 3-ball **C** with **p** $\in \partial$**C**, **A** – {**p**} ⊂ int(**C**).
C is **wild** at **p** and **tame** elsewhere:
ℓ is **not** null-homotopic near **p** in S^3 – **C**.

Temporarily work inside C.



Replace A by its ***double*** – 2A.

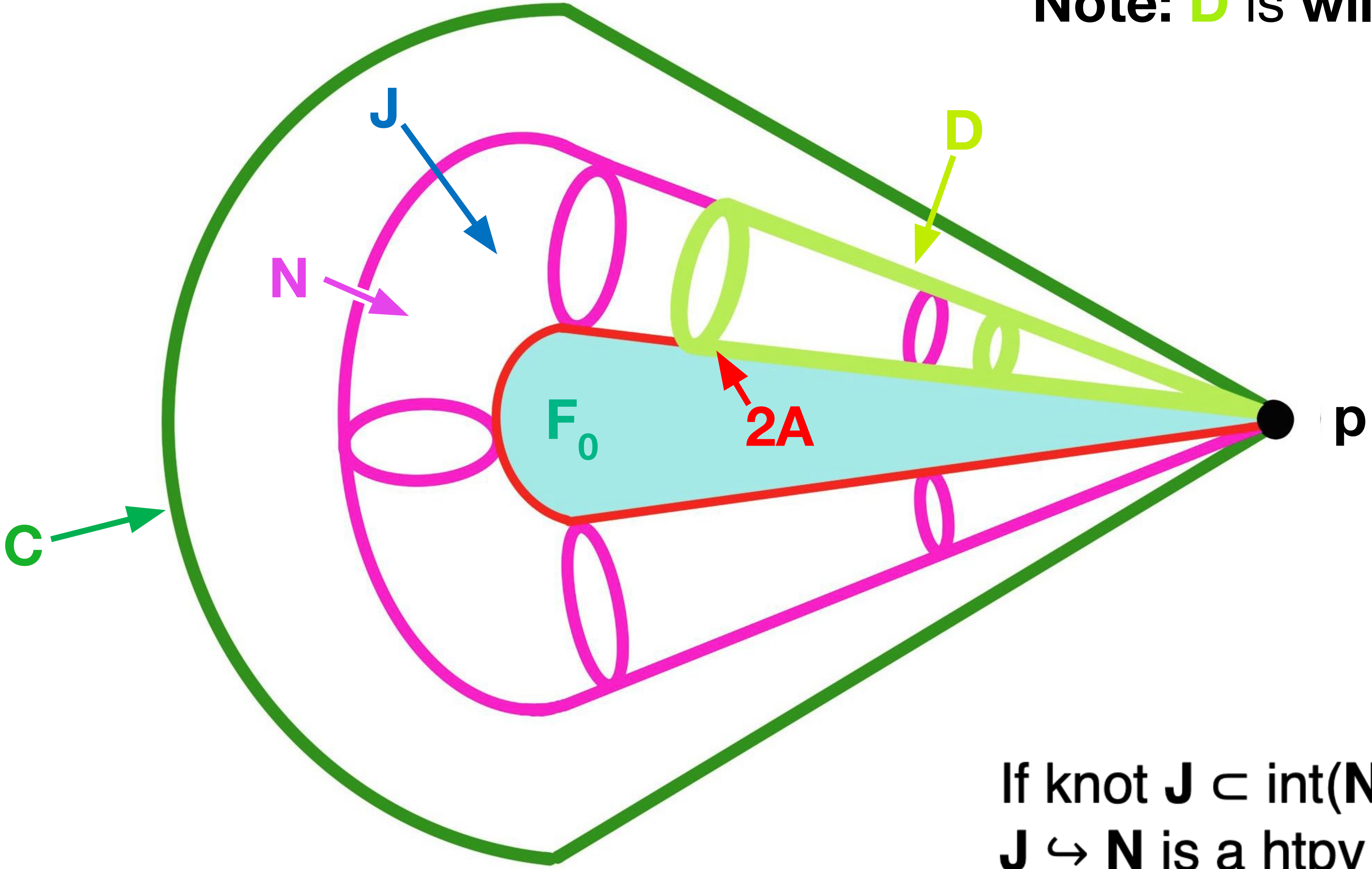
2A bounds a disk F_0.



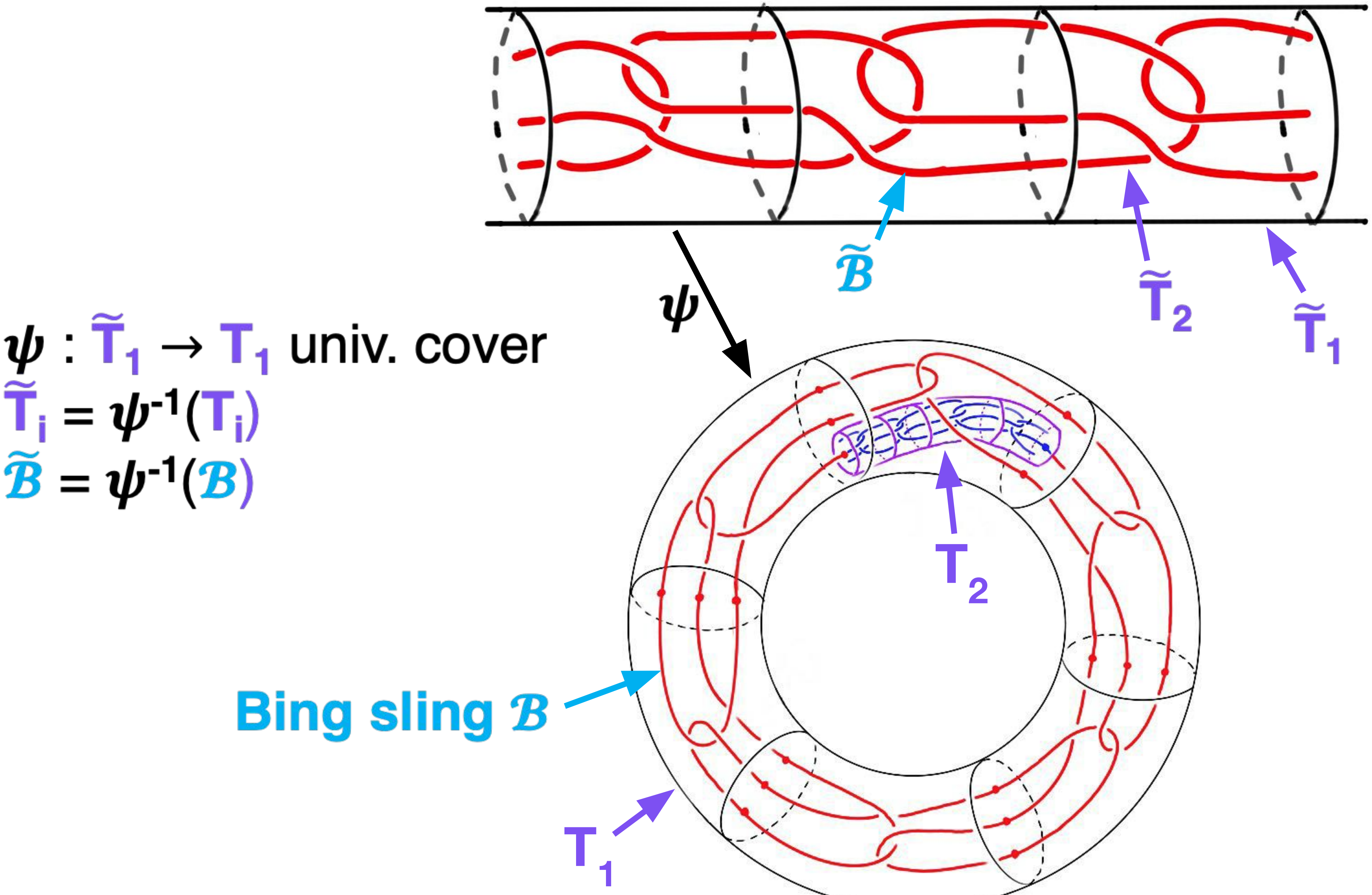
Let N =
pinched solid torus =
solid torus / meridinal disk such that $2A \subset \partial N$ and $N \cap F_0 = 2A$.

D is a disk

Note: D is **wild** at **p** in S^3.



If knot **J** $\subset$ int(**N**) $\cup$ {**p**} and **J** $\hookrightarrow$ **N** is a htpy equiv, then **J** pierces **D** at **p**.



$\psi : \widetilde{T}_1 \to T_1$ univ. cover
$\widetilde{T}_i = \psi^{-1}(T_i)$
$\widetilde{\mathcal{B}} = \psi^{-1}(\mathcal{B})$
ψ
$\widetilde{\mathcal{B}}$
$\widetilde{T}_2$
$\widetilde{T}_1$
T_2
Bing sling $\mathcal{B}$
T_1

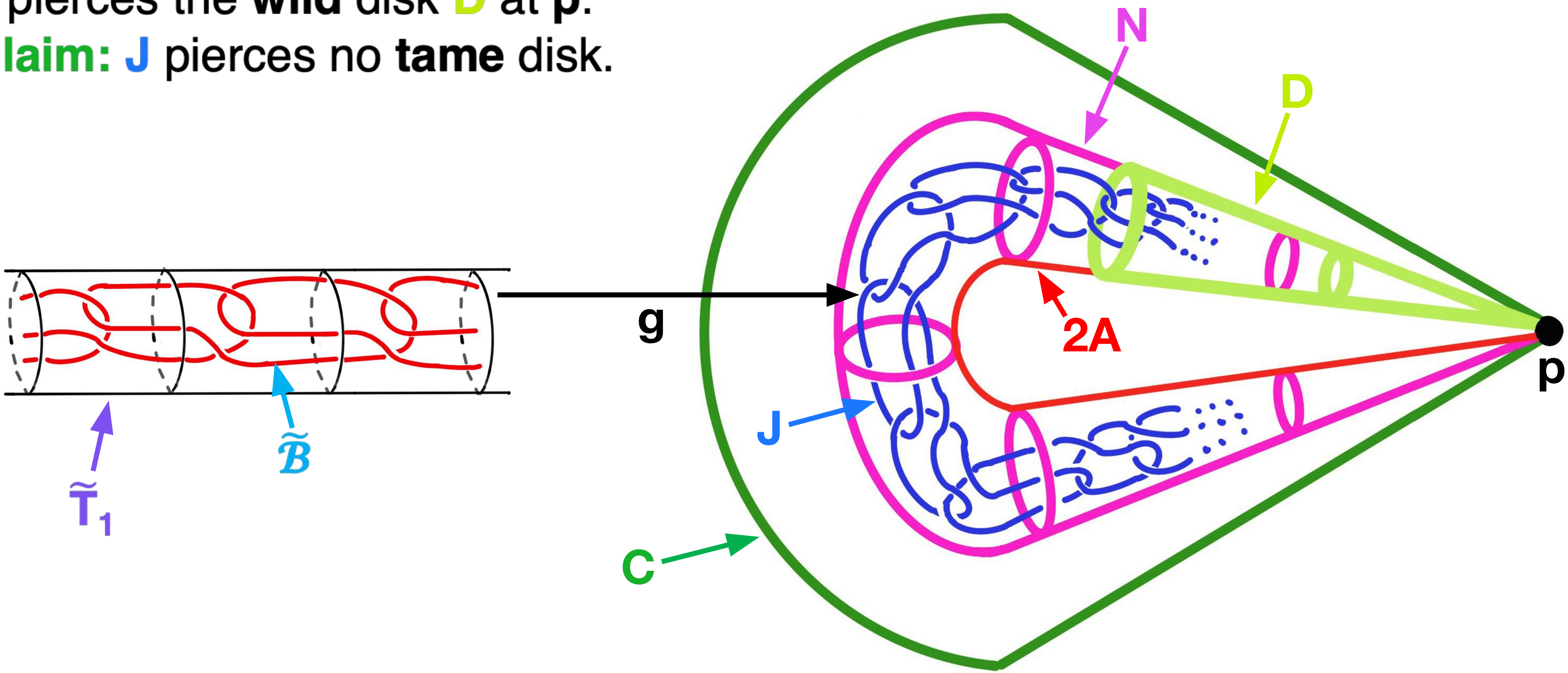
Let **g** : $\widetilde{T}_1 \to$ **N** – {**p**} be a homeomorphism.

Let **J** = **g**($\widetilde{\mathcal{B}}$) ∪ {**p**}.

∴ **J** is a knot in S^3.

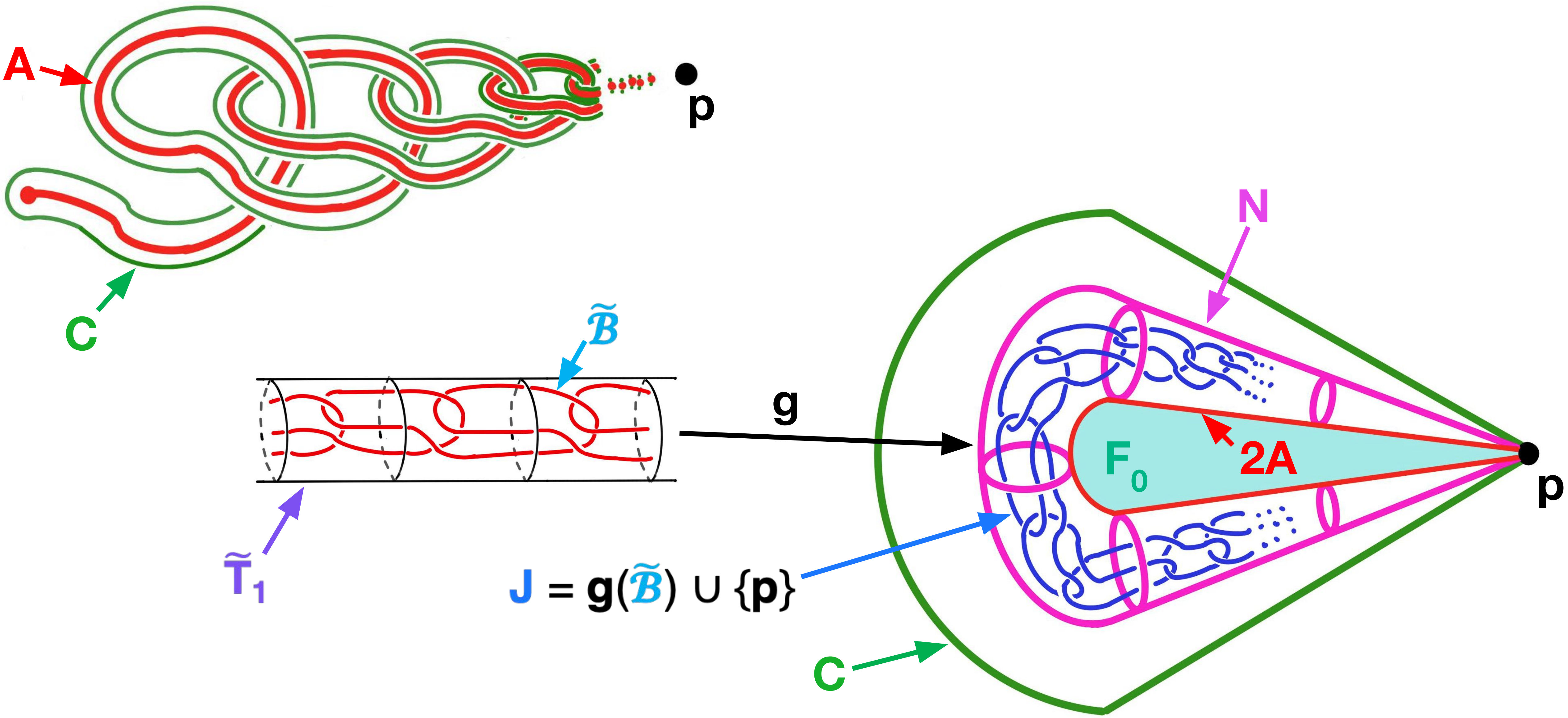
J pierces the **wild** disk **D** at **p**.

Claim: J pierces no **tame** disk.

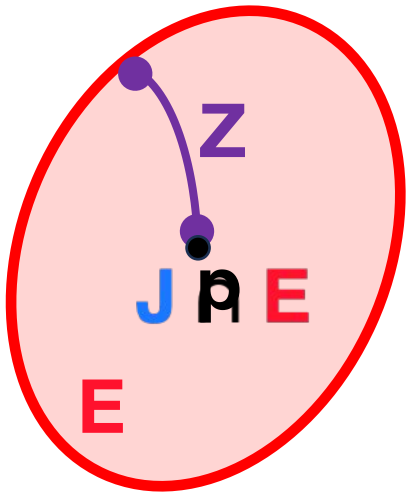


Claim: J pierces no **tame** disk.

Remember these pictures:



Claim: J pierces no **tame** disk.



Idea of proof: Assume J pierces a **tame** disk E.

$\therefore$ J can't pierce E at a point of $J - \{p\}$

because $J - \{p\} = g(\widetilde{\mathcal{B}})$ is locally homeo to the **Bing sling**.

$\therefore$ J pierces E at p.

Step 1: $\exists$ arc Z in $E \cap C$ such that $p \in \partial Z$ and $Z - \{p\} \subset \text{int}(C)$.

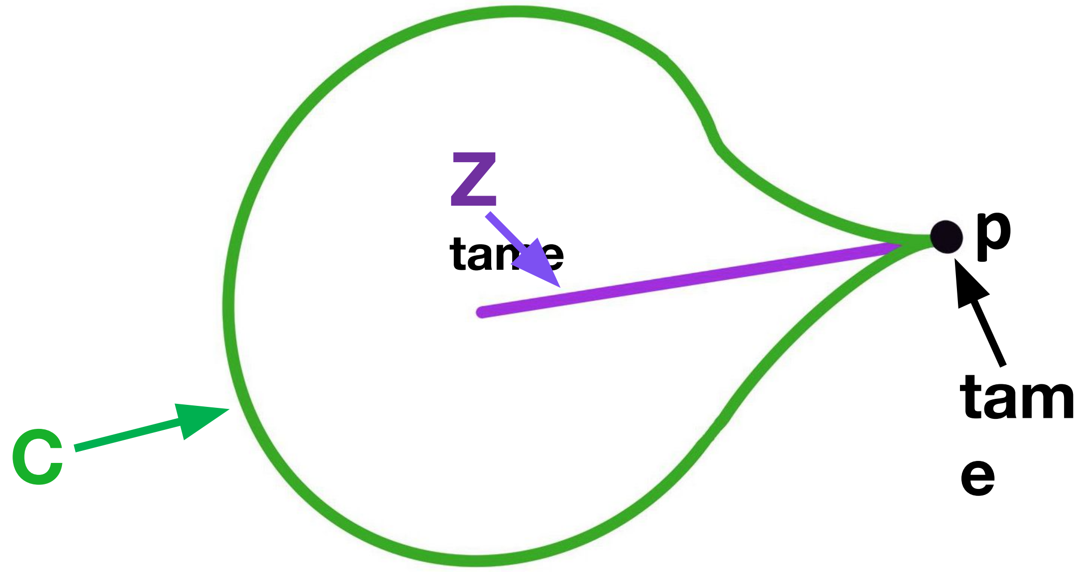
(**Hint:** Z = component of $E \cap F$ where F = surface "bounded by" J.)

Z is **tame** because $Z \subset$ tame disk E.

Recall: C is **wild** at p and **tame** elsewhere.

Step 2: C wild at **p** and **Z tame** contradicts:

Proposition: If **C** is a 3-ball in S^3, $\mathbf{p} \in \partial\mathbf{C}$, **C** is **tame** at every point of $\mathbf{C} - \{\mathbf{p}\}$, and **Z** is a **tame** arc **Z** in S^3 such that $\mathbf{p} \in \partial\mathbf{Z}$ and $\mathbf{Z} - \{\mathbf{p}\} \subset \mathrm{int}(\mathbf{C})$, then **C** is **tame** at **p**.

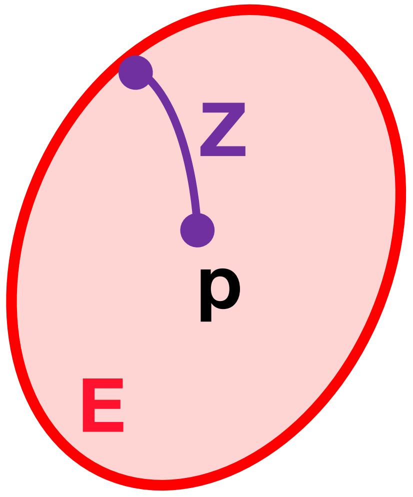


Game over!

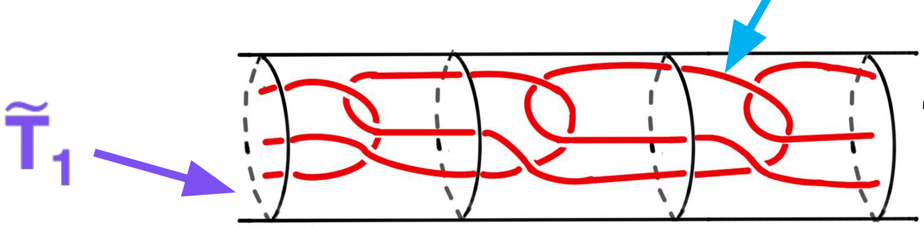
Proposition proved below.

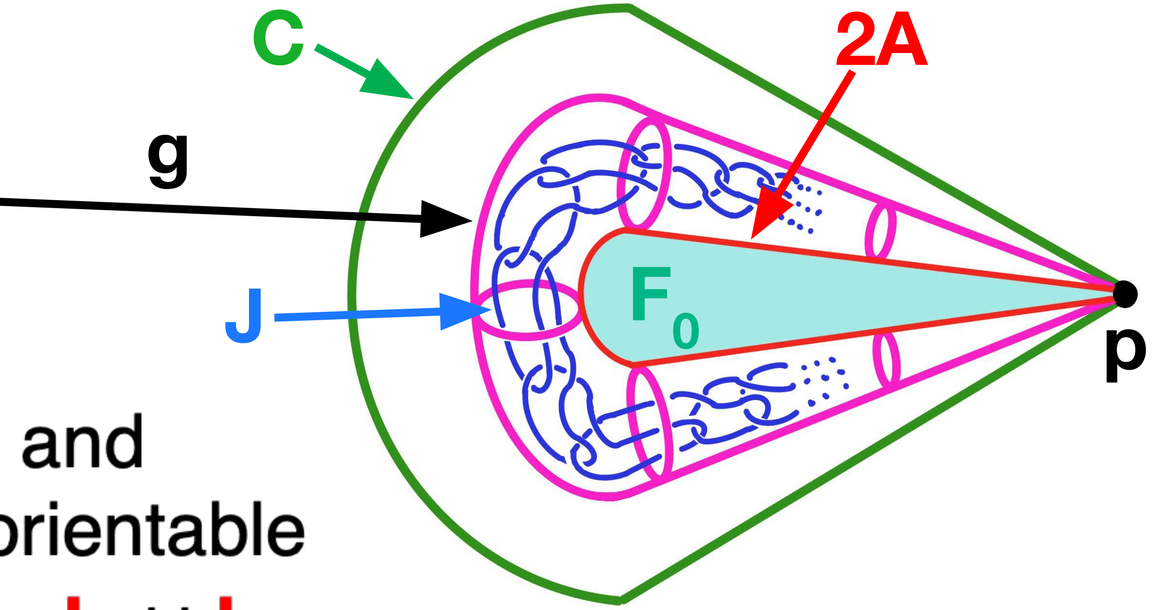
Completion of **Step 1:**

Proof that: $\exists$ arc $\mathbf{Z}$ in $\mathbf{E} \cap \mathbf{C}$ such that $\mathbf{p} \in \partial \mathbf{Z}$ and $\mathbf{Z} - \{\mathbf{p}\} \subset \text{int}(\mathbf{C})$.



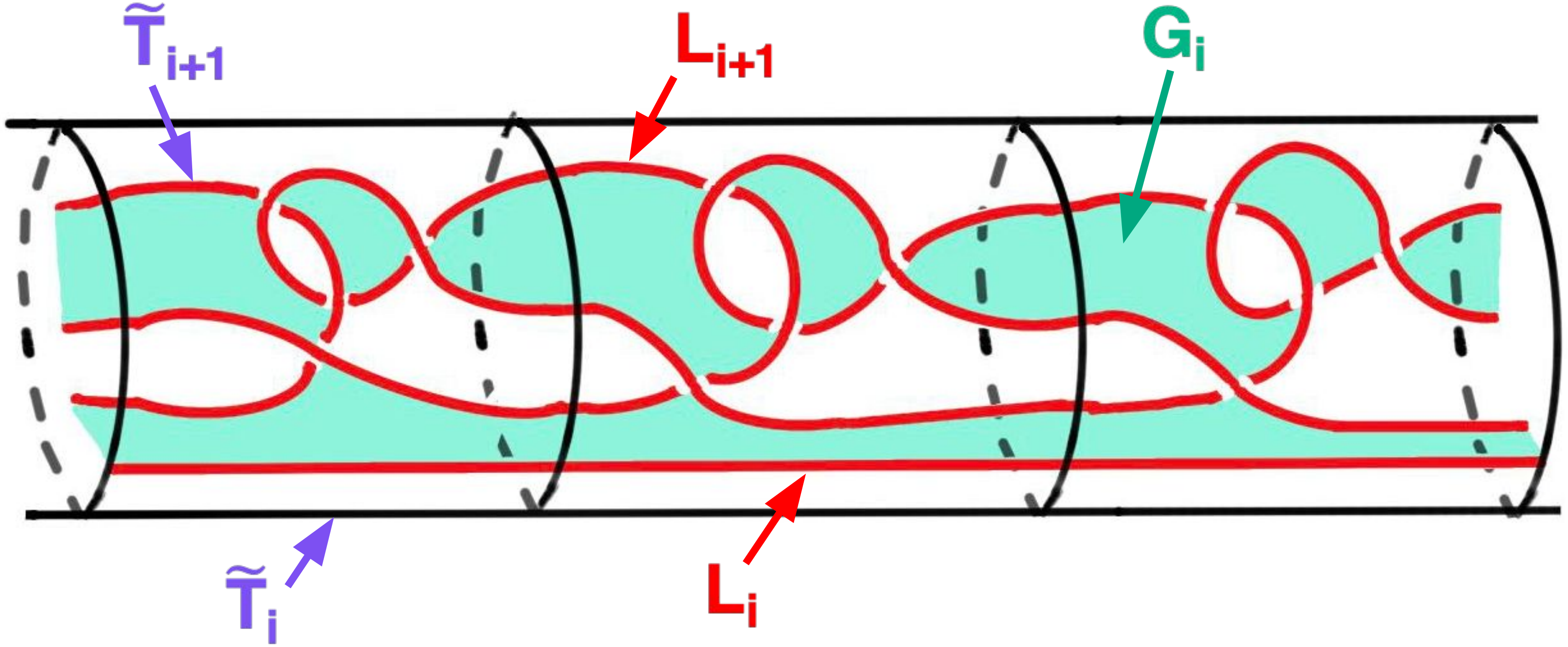
Recall **2A** bounds disk F_0 in **C**.





In $\tilde{T}_1$, ∃ properly embedded lines $L_i \subset \partial \tilde{T}_i$ and and ∃ properly embedded infinite genus orientable surfaces $G_i \subset \tilde{T}_i - \text{int}(\tilde{T}_{i+1})$ such that $\partial G_i = L_i \cup L_{i+1}$.

Also $\psi(L_1) =$ **2A** .

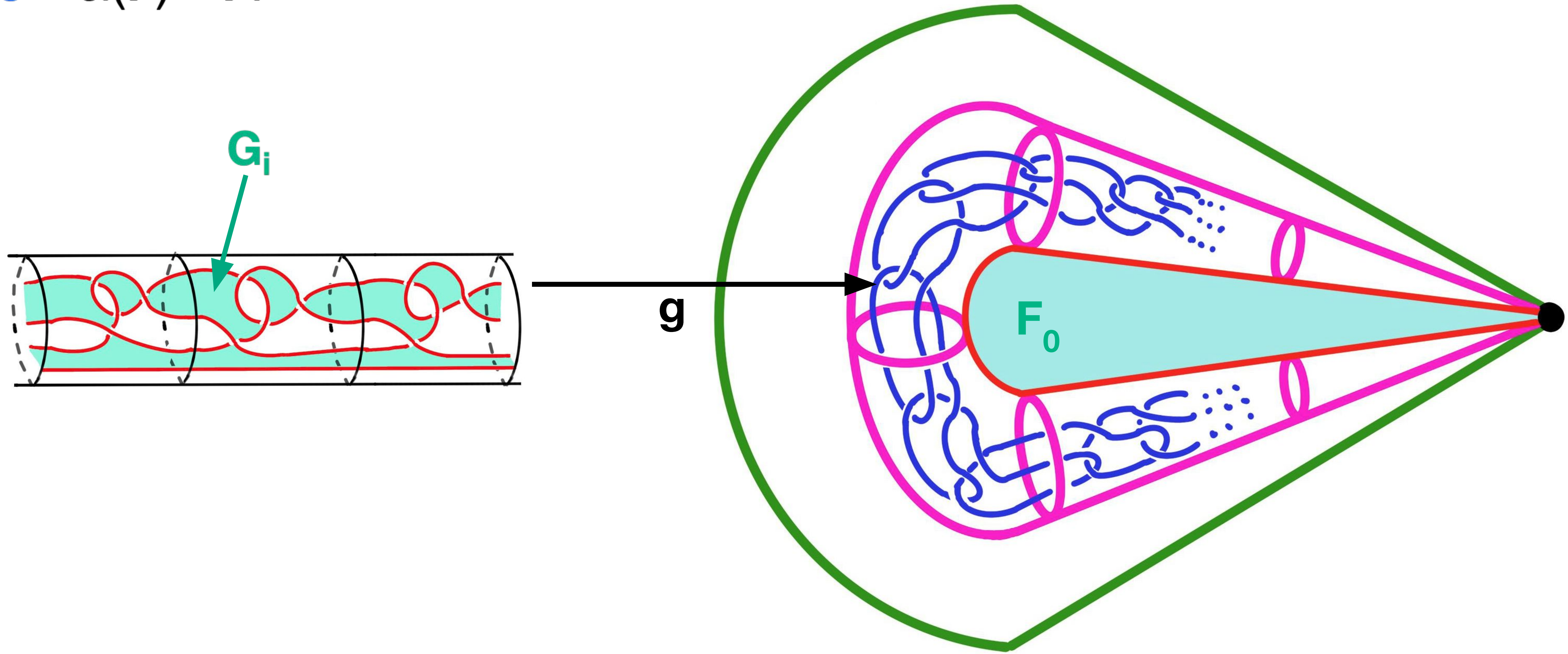


Let $F_i = \mathbf{g}(G_i)$ for $i \geq 1$.

Let $F = \bigcup_{i \geq 0} F_i$.

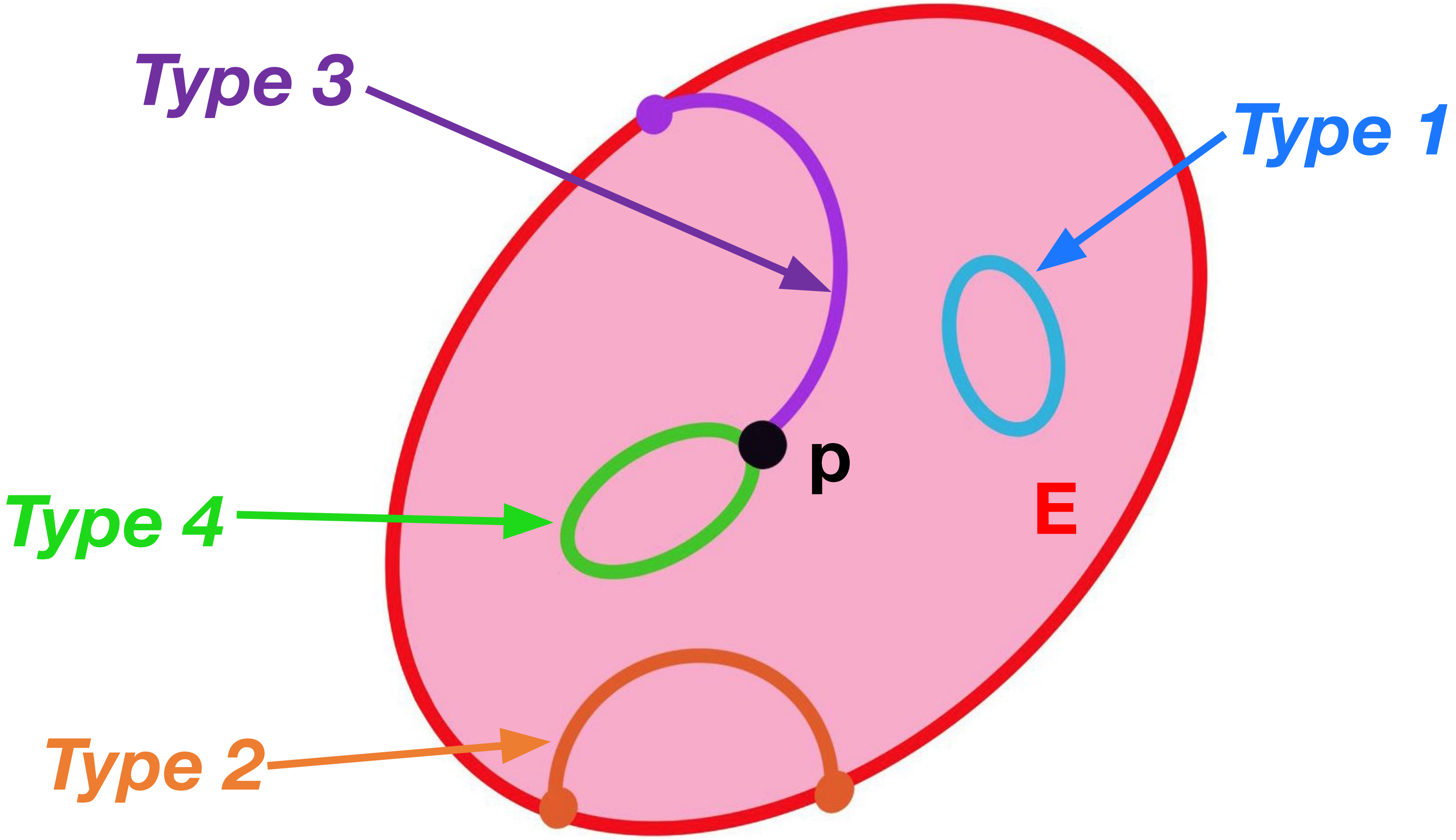
F is an infinite genus orientable surface in int(**C**) “bounded by” **J**:

J = cl(**F**) – **F**.



Perturb **F** to make it ***transverse*** to **E** – {**p**}.

∃ **4** ***types*** of components of **F** ∩ (**E** – {**p**}):



Claim: ***Type 3*** must exist.

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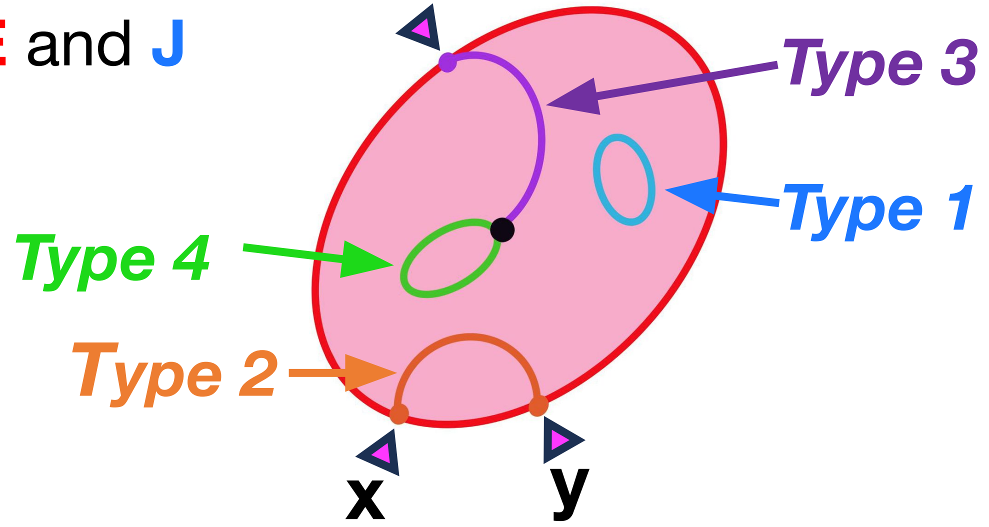
linking number of ∂E and J



Proof:

$$\ell k(\partial E, J) = \Sigma \{ \cap(\partial E, F)_x : x \in \partial E \cap F \}$$

oriented intersection number of ∂E and F at x

Let $\mathcal{E}_i$ = { endpoints of ***Type i*** components of $F \cap (E - \{p\})$.

$$\therefore \ell k(\partial E, J) = \Sigma \{ \cap(\partial E, F)_x : x \in \mathcal{E}_2 \} + \Sigma \{ \cap(\partial E, F)_x : x \in \mathcal{E}_3 \}.$$

x and y endpts of same ***Type 2*** component $\Rightarrow \cap(\partial E, F)_x = -\cap(\partial E, F)_y$.

$$\therefore 0 \neq \ell k(\partial E, J) = \Sigma \{ \cap(\partial E, F)_x : \mathcal{E}_3 \}.$$

$\therefore$ A ***Type 3*** component θ of $F \cap (E - \{p\})$ must exist. □

$Z = cl(\theta)$ is an arc in $E \cap C$ such that $p \in \partial Z$ and $Z - \{p\} \subset int(C)$.

Completion of **Step 2:**

Proposition: If **C** is a 3-ball in S^3, **p** $\in \partial$**C**, **C** is **tame** at every point of **C** $-$ {**p**}, and **Z** is a **tame** arc **Z** in S^3 such that **p** $\in \partial$**Z** and **Z** $-$ {**p**} $\subset$ int(**C**), then **C** is **tame** at **p**.

We need:

Lemma: If **A** is a spanning arc in a 3-ball **C**, then $\exists$ a retraction **r** : **C** $-$ **A** $\rightarrow \partial$**C** $-$ ∂**A**.

Proof sketch: Triangulate (**C** $-$ **A**,∂**C** $-$ ∂**A**).

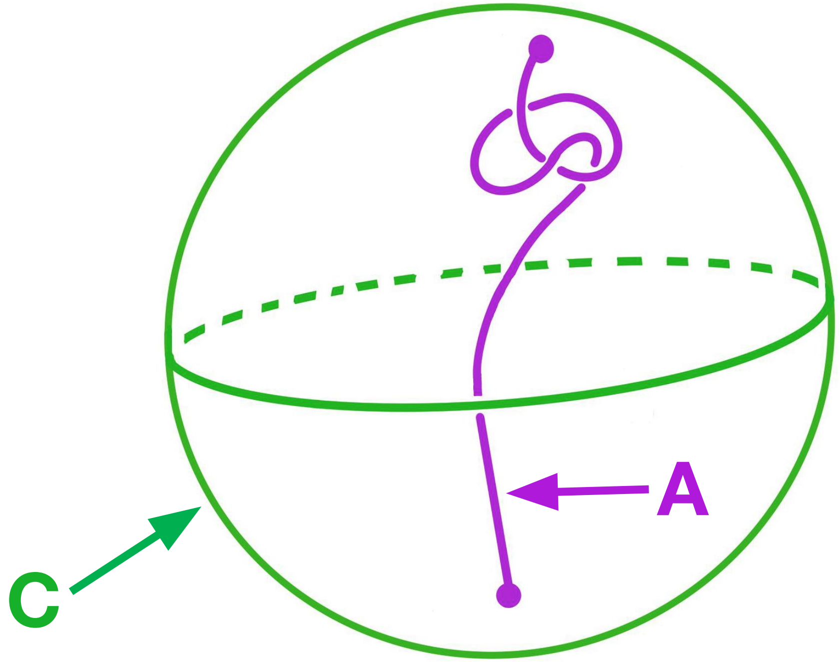
Let T_0 = max. tree in ∂**C** $-$ ∂**A**.

Extend T_0 to a max. tree T in **C** $-$ **A**.

Let $\mathbf{r}|_T : T \rightarrow T_0$ be a retraction.

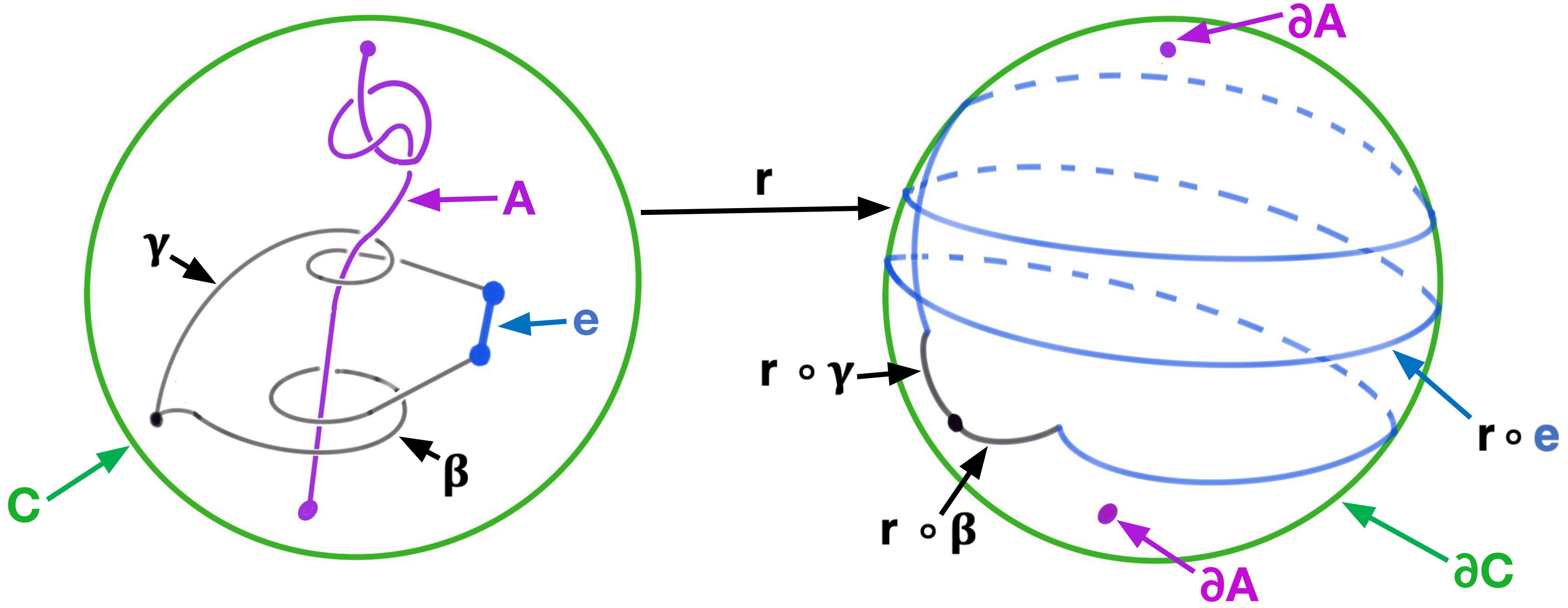
For an oriented edge **e** in **C** $-$ **A**, **e** $\not\subset$ T,

define $\mathbf{r}|_{\mathbf{e}} : \mathbf{e} \rightarrow \partial$**C** $-$ ∂**A** as follows:



Let β, γ be arcs in T from a basepoint in $\partial C - \partial A$ to the endpoints of **e**.

Choose $r \mid e$ so that $\ell k((r \circ \beta) * (r \circ e) * (r \circ \gamma^{-1})), \partial A) = \ell k(\beta * e * \gamma^{-1}, A)$.



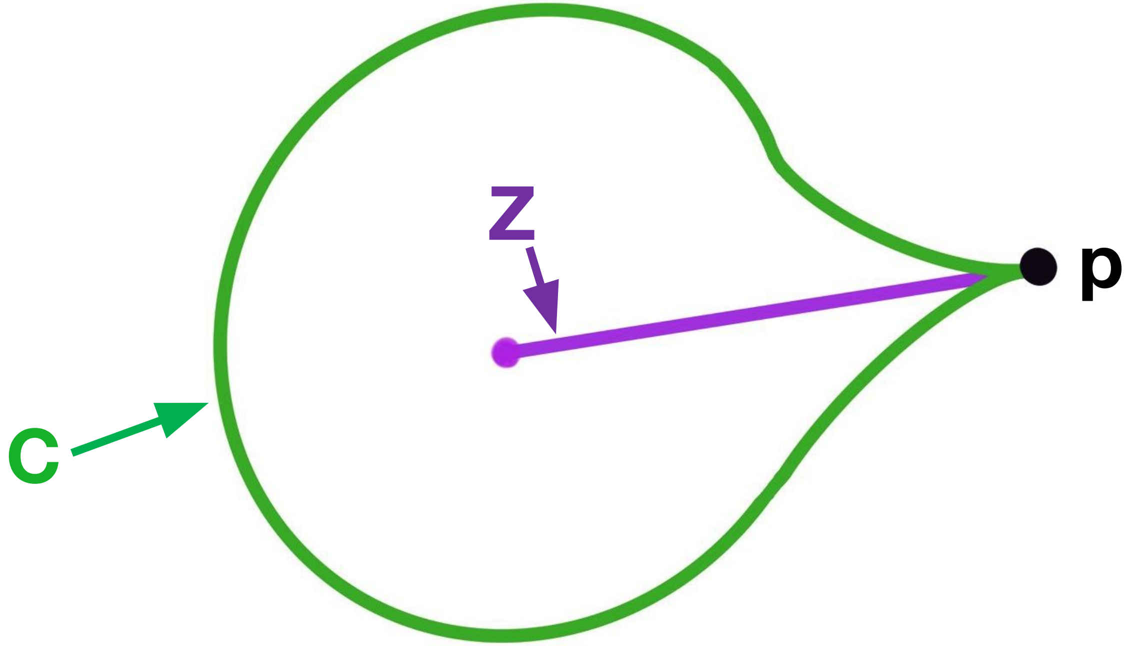
Then **r** extends over the 2-skeleton of **C** – **A**.

r extends over the 3-skeleton of **C** – **A** because $\pi_2(\partial C - \partial A) = 0$.

Proof of the Proposition:

C is a 3-ball in S^3, **p** ∈ ∂**C**, **C** is **tame** at every point of **C** – {**p**}, and **Z** is a **tame** arc in **C** such that **p** ∈ ∂**Z** and **Z** – {**p**} ⊂ int(**C**).

Fact: **C** is **tame** at **p** if: every open neighborhood U of **p** in S^3 contains an contains an open neighborhood V of **p** in S^3 such that every loop in V – **C** is null-homotopic in U – **C**. (Bing (1961)).

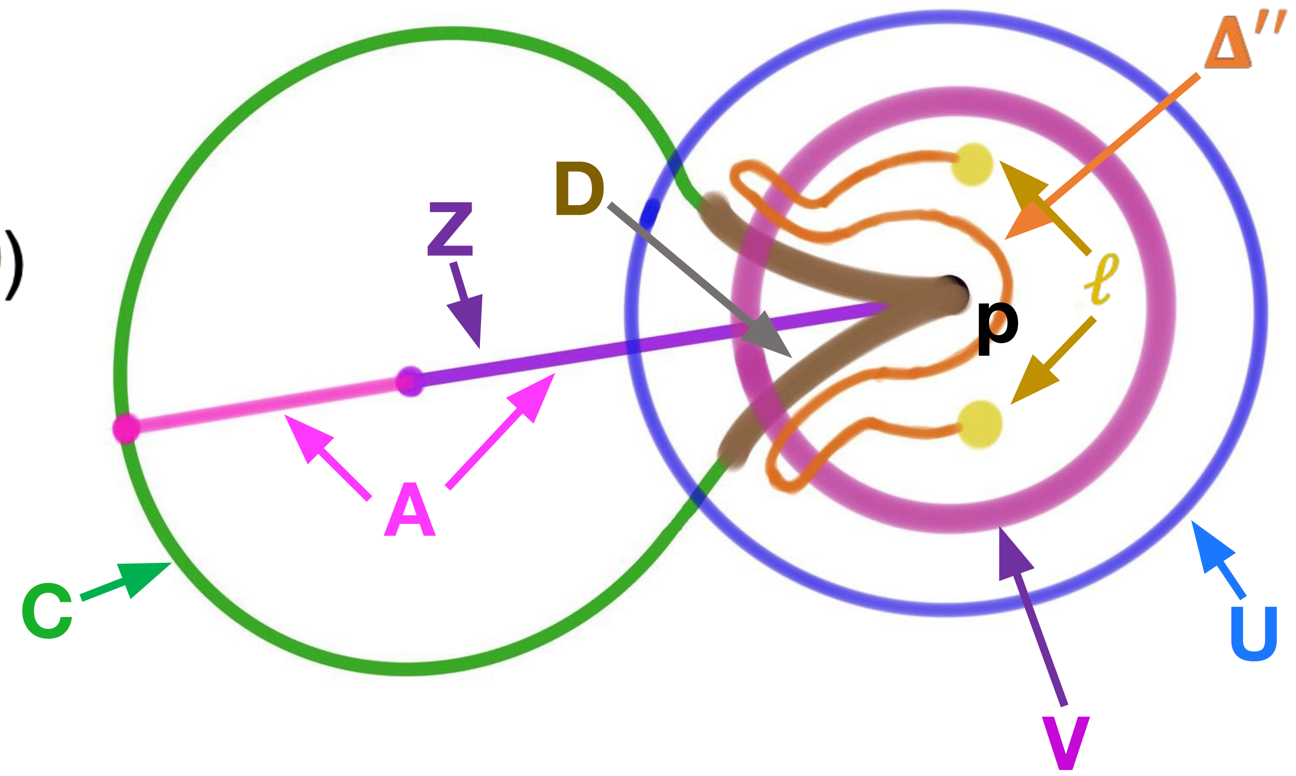


Extend Z spanning arc A of C.

Let U = open nbhd of **p** in S^3.

Let D = disk in ∂C such that **p** ∈ int(D) and D ⊂ U.

Let V = open nbhd of **p** in U such that V ∩ ∂C ⊂ int(D) and V – A is simply connected. (Possible because Z is **tame**.)



Compose retraction **r** : C – A → ∂C – ∂A with retraction ∂C – ∂A → D – {**p**}, to get retraction **r′** : C – A → D – {**p**}.

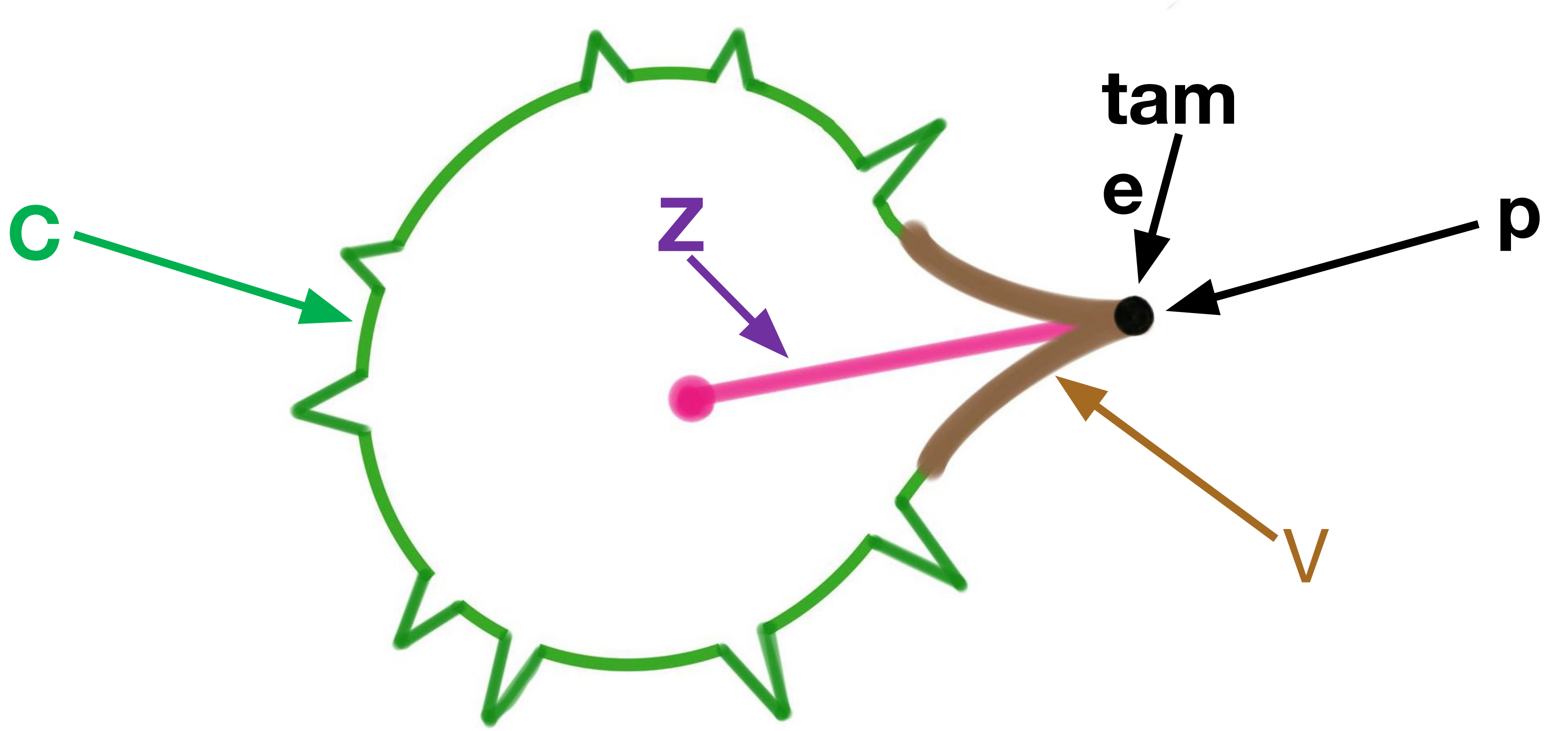
Must prove ℓ contracts to point in U – C.

ℓ bounds r′(Δ) = Δ′ in (V – int(C)) ∪ D.

Use external collar on C – {**p**} to move Δ′ to Δ′′ ⊂ U – C fixing ℓ. □

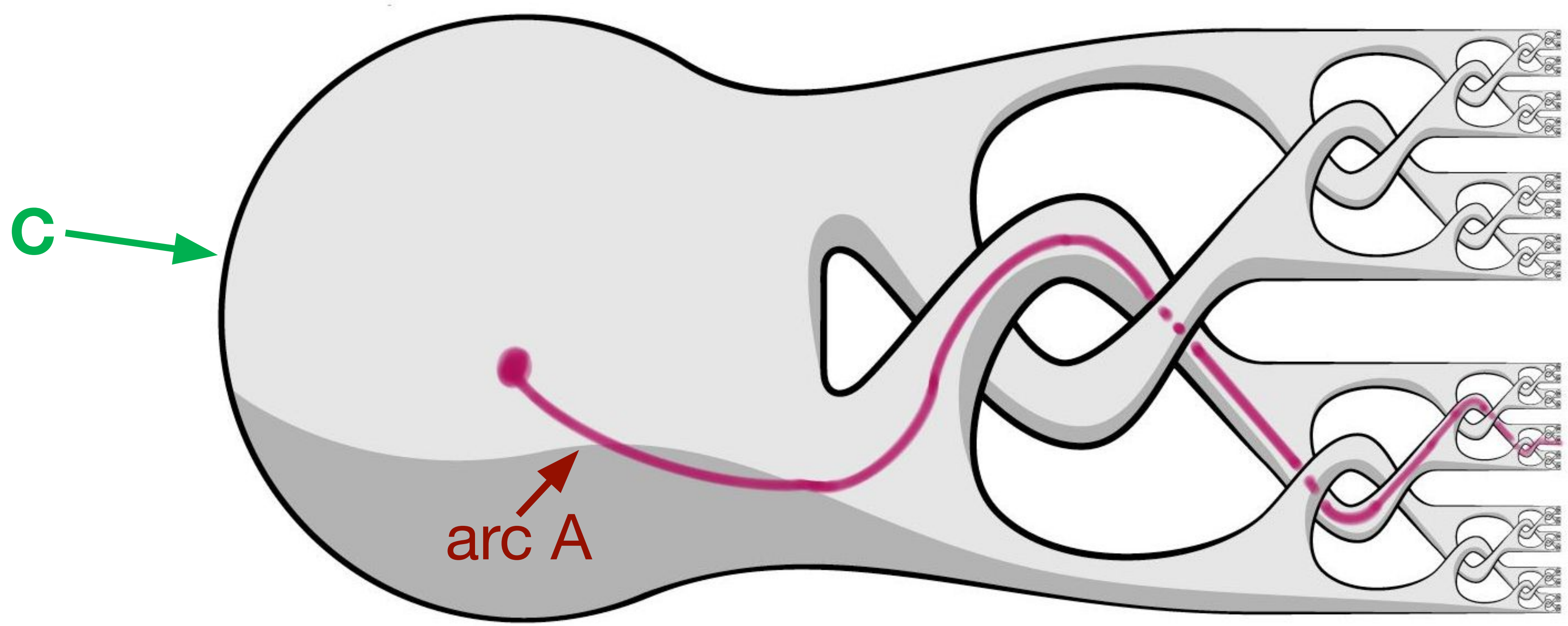
Addendum: The preceding proof actually proves a stronger result:

Proposition +: If **C** is a 3-ball in S^3, **p** ∈ ∂**C**,
V is an open nbhd of **p** in ∂**C** such that **C** is **tame** at every point of V – {**p**},
and **Z** is a **tame** arc in S^3 such that **p** ∈ ∂**Z** and **Z** – {**p**} ⊂ int(**C**),
then **C** is **tame** at **p**.



The hypothesis of **Proposition +** –
"∃ an open nbhd V of **p** in ∂**C** such that **C** is **tame** at every point of V – {**p**}"
– can't be omitted:

Example. If **C** is the ∂**C** is an Alexander horned sphere, then every point of ∂**C** can be approached from int(**C**) by an arc that is **tame** in S^3.



(no isolated wild points)

A is ambiently isotopic to a horizontal straight line segment.

Proposition + is a **strictly 3-dimensional** phenomenon because:

Fact 1: For $n \geq 4$, arcs in S^n have no isolated **wild** points.

Fact 2: For $n \geq 4$, n-balls in S^n have no isolated **wild** points (Kirby, 1968).

$\therefore$ For $n \geq 4$, if **C** is an n-ball in S^n, then every point of ∂**C** can be approached from int(**C**) by an arc that is **tame** in S^n.

Proof:

Approach a point $\mathbf{p} \in \partial$**C** from int(**C**) by an arc A that is possibly **wild** in S^n.
Approximate A mod **p** by an arc A′ such that $A' - \{\mathbf{p}\}$ is **tame** in S^n.
$\therefore$ A′ is **tame** in S^n. □