Operators in Kahler Spaces and Quantum Mechanics

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Outlook

- i) Real & Complex analysis

Outlook

- ii) Quantum Mechanics in complex Hilbert Space & real Kähler Space

Outlook

- iii) Quantum Mechanics as Symplectic Dynamical System
  - Ergodicity of Quantum Mechanics

Outlook

- iv) Quantum Mechanics & Elasticity Theory

Real & Complex analysis

Complex	Real
$z = x + iy$	(x, y)
$f = f(z, \bar{z})$	$f = u(x, y) + iv(x, y)$
Analyticity	Cauchy-Riemann
$\partial_{\bar{z}} f = 0$	
Schrodinger	Hamilton
$(\partial_t + iH)\psi = 0$	

Quantum systems and integrability

I.Volovich., Complete Integrability of Quantum and Classical Dynamical Systems, p-Adic Numbers, Ultr.An. and Appl. (2019)

- **Quantum dynamical system $\mathcal{H}$**
- **System of harm. oscillators (X, μ, V)**

$$V_t(x) = e^{it\omega(x)}$$

$$\phi_x(t) = e^{it\omega(x)}\phi_x(0)$$

$$\phi_x = q_x + ip_x$$

$$\dot{q}_x = \omega_x p_x \qquad \dot{p}_x = -\omega_x p_x$$

Measures of system of oscillators

- Sakbaev V.Zh., I.V.,Analogues of Jacobi and Weyl Theorems for Infinite-dimensional Tori, JOSA 2024
- Sakbaev V.Zh., I.V.,Measures of Systems of Oscillators and Properties of Trajectories, Arxiv:2506.18093
- Discrete, absolute continuous, singular measures
- Properties of trajectories

Hamiltonian

- V.V. Kozlov, O.G. Smolyanov...

ii)Quantum Mechanics in Real Kähler Space

- In this talk: formulation of real QM in real Kähler space
- Proof of the equivalence of Real Kähler QM to QM in Hilbert space
- Based on: I.V. “Real Quantum Mechanics in a Kähler Space,” arXiv:2504.16838

Hilbert Space & Kähler Space

A Kähler space is a real Hilbert space equipped with a symplectic form and automorphism called complex structure.
A Kähler space is a quadruplet

$$(\mathcal{K}, g, \omega, \mathcal{J})$$

ii) QM over real numbers

- QM and real numbers: Birkhoff, von Neumann, Stueckelberg,...

Recent: **M. O. Renou et al, "Quantum theory based on real numbers can be experimentally falsified," Nature (2021)**

Trushechkin et al, "Quantum mechanics based on real numbers: A consistent description," 2503.17307

T. Hoffreumon and M. P. Woods, "Quantum theory does not need complex numbers," 2504.02808 .

T. Feng, C. Ren and V. Vedral, "Locality Implies Complex Numbers in Quantum Mechanics," 2504.07808.

$(\mathcal{K}, g, \omega, \mathcal{J})$

- $\mathcal{K}$ is a real vector space,
- $g : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is a positive defined the bilinear form,
- $\omega : \mathcal{K} \times \mathcal{K} \to \mathbb{R}$ is a a symplectic,
- $J : \mathcal{K} \to \mathcal{K} \quad J^2 = -\mathrm{id}$ is a complex structure,
- $g(\cdot, \cdot) = \omega(\,\cdot\,, J\,\cdot\,)$ and $\omega(J\,\cdot\,, J\,\cdot\,) = \omega(\,\cdot\,, \cdot\,)$,

One can assume that $(\mathcal{K}, g)$ is a real Hilbert space and

$$\mathcal{K} = \mathrm{K} \oplus \mathrm{K}, \quad \psi \Leftrightarrow \begin{pmatrix} q \\ p \end{pmatrix}$$

$$\text{then } \mathcal{J} \begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}, \quad p, q \in \mathrm{K},$$

K is a real vector space.

From Hilbert to Kähler

- **If $\langle .,. \rangle$ is inner product in the complex Hilbert space then**

$$\langle .,. \rangle = \mathrm{g}(.,.) + \mathrm{i}\omega(.,.)$$

- **$g(.,.)$ – positive defined**

  $\omega(.,.)$ – skew-symmetric

$\gamma \,\&\, \gamma^{-1}$

- $\gamma : \mathbb{R}^{2N} \to \mathbb{C}^{N}$
- $\gamma \begin{pmatrix} q \\ p \end{pmatrix} = q + ip = \psi$

From Kähler to Hilbert

Let $\mathcal{K} = K \oplus K$

Define $\psi \in \mathcal{H}$ as $\psi = q + ip, \quad q, p \in K.$

The inner product on $\mathcal{H}$

$$\begin{aligned} \langle \psi_1, \psi_2 \rangle &= g\big((q_1, p_1), (q_2, p_2)\big) \\ &+ i\,\omega\big((q_1, p_1), (q_2, p_2)\big) \end{aligned}$$

g and ω are the metric and symplectic form on $\mathcal{K}$

$(g, \omega, \mathcal{J})$ on $\mathbb{R}^{2N} = \mathbb{R}^N \oplus \mathbb{R}^N$

- the scalar product on $\mathbb{R}^{2N}$

$$g\left(\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}\right) = \sum_{a=1}^{N} (q_{1,a}q_{2,a} + p_{1,a}p_{2,a})$$

- the symplectic form on $\mathbb{R}^{2N}$

$$\omega\left(\begin{pmatrix} q_1 \\ p_1 \end{pmatrix}, \begin{pmatrix} q_2 \\ p_2 \end{pmatrix}\right) = \sum_{a=1}^{N} (q_{1,a}p_{2,a} - q_{2,a}p_{1,a})$$

- the complex structure

$$\mathcal{J}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} q \\ p \end{pmatrix} = \begin{pmatrix} -p \\ q \end{pmatrix}$$

Consistency condition on $(\boldsymbol{g}, \boldsymbol{\omega}, \boldsymbol{J})$

- One can check explicitly

$$\langle \psi_1, \psi_2 \rangle = g(\gamma^{-1}\psi_1, \gamma^{-1}\psi_2) + i\omega(\gamma^{-1}\psi_1, \gamma^{-1}\psi_2)$$

Tensor product & Paradox

- $$\mathbb{C} = \mathbb{R}^2,$$
- $$\mathbb{C} \otimes \mathbb{C} = \mathbb{C}$$
- $$\mathbb{R}^2 \otimes \mathbb{R}^2 = \mathbb{R}^4$$
- $$\mathbb{R}^2 =? \mathbb{R}^4$$

Tensor product over real number fields

- $\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{R}^{2n} = \mathbb{R}^{4mn}$

Tensor product over complex number fields

- $\mathbb{R}^{2m} \otimes_{\mathbb{R}} \mathbb{R}^{2n} = \mathbb{R}^{4mn}$
- $\xi_A = (\mathbb{R}^{2m}_A, g_A, \omega_A, J_A)$
- $\xi_B = (\mathbb{R}^{2n}_B, g_B, \omega_B, J_B)$,
- Transform them into complex Hilbert spaces $(\mathbb{C}^m_A, \langle\cdot,\cdot\rangle_A)$ and $(\mathbb{C}^n_B, \langle\cdot,\cdot\rangle_B)$.

Tensor product over complex number fields

- We then form the tensor product of these complex Hilbert spaces: $(\mathbb{C}_A^m, \langle\cdot,\cdot\rangle_A) \otimes (\mathbb{C}_B^n, \langle\cdot,\cdot\rangle_B) = (\mathbb{C}_{AB}^{mn}, \langle\cdot,\cdot\rangle_{AB})$,

Tensor product over complex number fields

- Transform the resulting complex Hilbert space back into a real Kähler space: $\xi_{AB} = (\mathbb{R}^{2mn}_{AB}, g_{AB}, \omega_{AB}, J_{AB})$.

Operators

Let $L : \mathbb{C}^N \to \mathbb{C}^N$ be a linear operator.

An associated operator $\mathcal{L} : \mathbb{R}^{2N} \to \mathbb{R}^{2N}$ which is defined by using

$$\langle \gamma(\cdot), L\gamma(\cdot) \rangle = g(\cdot, \mathcal{L}\cdot) + i\omega(\cdot, \mathcal{L}\cdot).$$

$\mathcal{K}$-Hermitian operators

Hermitian $\boldsymbol{L}$ in $\mathbb{C}^{\boldsymbol{N}}$ $\Leftrightarrow$ $\mathcal{K}$-Hermitian $\boldsymbol{\mathcal{L}}$

Hermiticity $\langle \boldsymbol{\psi}, \boldsymbol{L\phi} \rangle = \langle \boldsymbol{L\psi}, \boldsymbol{\phi} \rangle$

$$\Downarrow$$

$$\begin{aligned} \boldsymbol{g}\left(\boldsymbol{\mathcal{L}}(\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{r},\boldsymbol{s})\right) &= \boldsymbol{g}\left((\boldsymbol{q},\boldsymbol{p}),\boldsymbol{\mathcal{L}}(\boldsymbol{r},\boldsymbol{s})\right) \\ \boldsymbol{\omega}\left(\boldsymbol{\mathcal{L}}(\boldsymbol{q},\boldsymbol{p}),(\boldsymbol{r},\boldsymbol{s})\right) &= \boldsymbol{\omega}\left((\boldsymbol{q},\boldsymbol{p}),\boldsymbol{\mathcal{L}}(\boldsymbol{r},\boldsymbol{s})\right) \end{aligned}$$

$$\Downarrow$$

$$\boldsymbol{\mathcal{L}}^{\mathrm{T}} = \boldsymbol{\mathcal{L}}, \quad \boldsymbol{\mathcal{L}\mathcal{J}} = \boldsymbol{\mathcal{J}\mathcal{L}}$$

Quantum Mechanics in Real Kähler Space

- i) To a physical system one assigns a real Kähler space $\mathcal{K}$;

  its state is represented by vectors $\eta \in \mathcal{K}$, $g(\eta, \eta) = 1$,

  $g(\cdot, \cdot)$ - an inner product in $\mathcal{K}$,

  (or a density matrix ρ)

Quantum Mechanics in Real Kähler Space

- ii) To the observable L corresponds the

  $\mathcal{K}$- Hermitian operator $\mathcal{L}$, which spectrum

  is observable.

  The spectral decomposition for $\mathcal{L}$:

  $\sum_{i=1}^{n} \lambda_i \mathcal{E}_i = \mathcal{L}, \quad \sum_{i=1}^{n} \mathcal{E}_i = \mathcal{I}, \quad \text{rank}(\mathcal{E}_i) \geq 2.$

Quantum Mechanics in Real Kähler Space

- **iii) Born rule:**
  if we measure L in the normalized state

  η, the probability of obtaining result λ_i

  is given by $g(\eta, \mathcal{E}_i \eta)/\mathrm{rank}(\mathcal{E}_i)$.

Quantum Mechanics in Real Kähler Space

- **iv) the Kähler space $\mathcal{K}$ corresponding to the composition of two systems $\mathfrak{N}$ and $\mathfrak{M}$ is $\mathcal{K}_{\mathfrak{N}} \underset{\mathcal{K}}{\otimes} \mathcal{K}_{\mathfrak{M}}$.**

Quantum Mechanics in Real Kähler Space

- **v*) A compact Lie group $\mathfrak{G}$ of internal symmetries is realized in the Kähler space $\mathcal{K}$ by symplectic orthogonal representation . $U(g), g \in \mathfrak{G}$.**

Quantum Mechanics in Real Kähler Space

- **vi*) time and space - symplectic orthogonal representation of the Galilean group**

iii) Ergodicity of Finite-Dimensional Quantum Systems

- Boltzmann. Ergodicity hypothesis
- G.Weyl, Birkhoff
- von Neumann, Kolmogorov,...
- We will prove that almost any finite dimensional quantum system is ergodic

Quantum =Classical

- N-dimensional Hilbert space $\mathbb{C}^N$
- $H = (H_{ab})$ Hermitian $N \times N$ matrix, $a, b = 1, ..., N$
- Schr. eq. $i\,\dot{\psi}_a = H_{ab}\,\psi_b$, $\psi_a : \mathbb{R} \to \mathbb{C}$
- $H_{ab} = K_{ab} + iL_{ab} \quad K_{ab} = K_{ba}$, $L_{ab} = -L_{ba}$ are real
- Real form of the Schr. eq.: $\psi_a = q_a + ip_a$

$$\dot{q}_a = K_{ab}p_b + L_{ab}q_b, \quad \dot{p}_a = -K_{ab}q_b + L_{ab}p_b.$$

Unitary-symplectic

- $\langle .,. \rangle = g(.,.) + i\omega(.,.)$
- $U(N) = O(2N, \mathbb{R}) \cap Sp(2N, \mathbb{R})$

Quantum =Classical

- $\dot{q}_a = K_{ab}p_b + L_{ab}q_b, \quad \dot{p}_a = -K_{ab}q_b + L_{ab}p_b.$ are classical Hamiltonian equations with the Hamiltonian

$$H_{sym} = \frac{1}{2}K_{ab}(p_a p_b + q_a q_b) + L_{ab}p_a q_b.$$

- Diagonalization of H by unitary transformation is equivalent to the diagonalization of H_{sym} by symplectic orthogonal transformation. We get

$$H_{sym} = \frac{1}{2}\sum_{a=1}^{N}\lambda_a(\xi_a^2 + \eta_a^2)$$

Ergodicity

- **Definition. A finite-dimensional quantum dynamical system is called ergodic if the associated classical dynamical system $\{M, \varphi_t, \mu\}$ on the surface of the integrals of motion is ergodic.**

- **The time average for an integrable function f coincides with the spatial average almost everywhere**

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\varphi_t(x))dt = \int_M f d\mu.$$

Ergodicity

- Constants of motion $F_a = \xi_a^2 + \eta_a^2, a = 1, ...N$.
- In the action-angle variables a linear flow on a torus:
  $\dot{J}_a = 0, \quad \dot{\theta}_a = \lambda_a \ (\mathrm{mod}\ 1), \quad a = 1, ...N$
- Theorem. A finite-dimensional quantum dynamical system with rationally independent eigenvalues of the Hamiltonian is ergodic.
- Thus, in a finite-dim. Hilbert space, almost any quantum dynamical system is ergodic.

Conclusion for points iii)

- **In a finite-dimensional Hilbert space, almost all quantum dynamical systems are ergodic.**

iv) Quantum Mechanics & Elasticity Theory

- On the equivalence between Schrodinger & Euler-Bernoulli eqs.

I.V. "On the Equivalence Between the Schrodinger Equation in Quantum Mechanics and the Euler-Bernoulli Equation in Elasticity Theory", 2411.03261

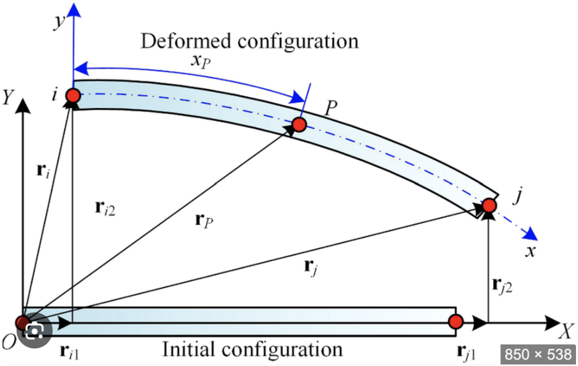
Euler–Bernoulli

In 1750, Euler and Bernoulli proposed an equation to describe the vibrations of beams and plates in elasticity theory.

Euler–Bernoulli eq.: $\ddot{u} + \Delta^2 u = 0,$

Δ - Laplace operator.

Euler–Bernoulli eq. for beams and plates



Schrodinger equation

In 1926, Schrodinger discovered the wave equation that governs quantum mechanical particles. For a free particle, this equation is

$$\mathbf{i}\dot{\psi} = -\boldsymbol{\Delta}\psi. \tag{1}$$

Relation of Euler–Bernoulli & Schrodinger equations

From Schrodinger eq. one has

$$\ddot{\psi} + \mathbf{\Delta}^2 \psi = \mathbf{0}. \tag{2}$$

Indeed,

$$\dot{\psi} = \mathbf{i}\mathbf{\Delta}\psi, \qquad \ddot{\psi} = \mathbf{i}\mathbf{\Delta}\dot{\psi}$$

Therefore,

$$\ddot{\psi} = -\mathbf{\Delta}^2 \psi \tag{3}$$

The Cauchy Problem for the Schrodinger Equation

$$i\dot{\psi} = -\Delta\psi, \quad \Delta = \partial_{x_1}^2 + ...\partial_{x_d}^2$$
$$\psi(0,x) = \psi_0(x), \quad x \in \mathbb{R}^d, \quad t \in \mathbb{R}$$

$\psi = \psi(t,x)$ - complex function

Solutions $\psi \in C^\infty(\mathbb{R}^{d+1})$
with initial data $\psi_0 = \psi_0(x)$
belonging to the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

Solution to Cauchy problem

is given by the integral

$$\psi(t,x) = \int_{\mathbb{R}^d} \tilde{\psi}_0(k)\, e^{-ik^2 t - ik\cdot x}\, dk,$$

$\tilde{\psi}_0(k)$ - Fourier transform of $\psi_0(x)$,

$$k^2 = \sum_{i=1}^{d} k_i^2 \quad k\cdot x = \sum_{i=1}^{d} k_i x_i$$

Additionally, we have $\dot{\psi}(0,x) = i\Delta\psi_0(x)$.

Symplectic form

Schrodinger eq. as a system of 2 eqs:

$$\psi(t,x) = u(t,x) + iv(t,x),$$

which leads to the system

$$\dot{u} = -\Delta v, \quad \dot{v} = \Delta u \quad (*)$$

(*) represents Hamiltonian dynamics with the Hamiltonian

$$H_{\text{sym}} = \frac{1}{2}\int_{\mathbb{R}^d} [u(-\Delta)u + v(-\Delta)v]\, dx.$$

Theorem

$$i\,\dot{\psi} = -\Delta\psi, \quad \psi(0,x) = \psi_0(x)$$

$$\psi(t,x) = u(t,x) + iv(t,x),$$

$$\psi_0(x) = u_0(x) + iv_0(x)$$

$$\begin{aligned} \ddot{u} + \Delta^2 u = 0, \quad u(0,x) &= u_0(x) \\ \dot{u}(0,x) &= -\Delta v_0(x) \\ \ddot{v} + \Delta^2 v = 0, \quad v(0,x) &= v_0(x), \\ \dot{v}(0,x) &= \Delta u_0(x). \end{aligned}$$

Generalized Euler–Bernoulli Equation with Potential

We start from the Schrodinger eq with a potential:

$$\mathbf{i}\dot{\psi} = \mathbf{H}\psi \, (*) \qquad \mathbf{H} = -\boldsymbol{\Delta} + \mathbf{V} \qquad \psi(\mathbf{0}, \mathbf{x}) = \psi_0$$

Differentiating (*) with respect to time, we obtain a generalization of Euler–Bernoulli eq.

$$\ddot{\psi} + H^2\psi = 0.$$

Conclusion for points iv)

- **Euler-Bernoulli is equivalent to Schroedinger equation.**
- **In a finite-dimensional Hilbert space, almost all quantum dynamical systems are ergodic.**

Conclusion

- **Euler-Bernoulli is equivalent to Schroedinger equation.**
- **In a finite-dim. Hilbert space, almost all quantum dynamical systems are ergodic.**
- **Formulation of real QM in real Kähler space and proof of the equivalence of real Kähler QM to QM in Hilbert space are given**