# Invertibility conditions for some operators in weight spaces

Международная конференция «Теория функций и ее приложения», посвященная 120-летию со дня рождения академика РАН Сергея Михайловича Никольского

**Кабанко Михаил Владимирович**

Department: КГУ, кафедра математическогоа нализа и прикладной математики

Abstract: By the representation of the operator in tandem with sparse weights, a criterion for the invertibility of the operator in a general Hilbert couple is obtained. This criterion is based on the invertibility of the main diagonal of the matrix corresponding to the operator as well as on the weight conditions which are superimposed on diagonals parallel to the main one.

# Representation

## Definition

A representation of the operator algebra $\mathfrak{A}$ is the pair $(H_\theta, \theta)$, where $H_\theta$ is some Hilbert space, which is called a representation and $\theta$ is a homomorphism of the form

$$\theta : \mathfrak{A} \to B(H_\theta).$$

A representation will be called faithful if the homomorphism $\theta$ is injective.

Murphy, G.J.: $C^\star$-algebras and Operator Theory. Academic Press, San Diego (1990)

Helemskii, A.Y.: Lectures and Exercises on Functional Analysis. Translations of mathematical monographs, vol. 233. AMS Bookstore, Providence, Rhode Island (2004).

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# Irreducible representation

## Definition

A representation $(H_\theta, \theta)$ of the operator algebra $\mathfrak{A}$ is called topological irreducible (or acting irreducibly on $H_\theta$) if the only closed vector subspaces of $H_\theta$ that are invariant for $\mathfrak{A}$ are trivial subspaces, i.e. 0 and $H_\theta$.

Murphy, G.J.: $C^\star$-algebras and Operator Theory. Academic Press, San Diego (1990)

Helemskii, A.Y.: Lectures and Exercises on Functional Analysis. Translations of mathematical monographs, vol. 233. AMS Bookstore, Providence, Rhode Island (2004).

ФМИ

Some auxiliary results

Main results

# Interpolation

**Definition**

Two Banach spaces $X_0$ and $X_1$ form a Banach couple if both spaces are contained in the Hausdorff topological space $\mathfrak{X}$ and the embeddings $X_i$, $(i = 0, 1)$ into $\mathfrak{X}$ are continuous. A Banach couple $\{X_0, X_1\}$ is called regular if the space $X_0 \cap X_1$ is dense in $X_0$ and $X_1$.

1. Bergh, J., Loöfström, J.: Interpolation Spaces. An Introduction. Springer, Berlin, Heidelberg, New York (1976)
2. Dmitriev, V.I., Krein, S.G., Ovchinnikov, V.I.: Fundamentals of the theory of interpolation. Geom. Linear Spaces Oper. Theory (Russian), pp. 31–74. Jaroslav. Gos. Univ., Yaroslavl (1977)

# Interpolation

Let $\{H_0, H_1\}$ be a Hilbert couple and $\mathfrak{A}$ is the algebra of operators acting in the couple, i.e. $\mathfrak{A} = B(H_0) \cap B(H_1)$. We shell construct the map $\pi$ from the algebra $\mathfrak{A}$ into the algebra of operators $B(H_\pi)$ where $H_\pi$ is an interpolation Hilbert space in the regular Hilbert couple $\{H_0, H_1\}$ and $\pi(T) = T\,|_{H_\pi}$ for $T \in \mathfrak{A}$.

# Representation in interpolation space

Some auxiliary results

Main results

## Proposition

The $(H_\pi, \pi)$ is a faithful representation of the algebra of operators $\mathfrak{A}$ acting in a Hilbert couple.

## Theorem

The $(H_\pi, \pi)$ is topological irreducible, where $H_\pi$ is the interpolation Hilbert space.

Proof Th1

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# General case

**Theorem 2**

Let $T$ be a linear bounded operator acting in the Hilbert couple $\overline{H}$. If $T$ is a topological isomorphism in the sum $H_0 + H_1$ and it is the same in the intersection $H_0 \cap H_1$ then $T$ is a topological isomorphism in the algebra $B(\overline{H})$.

Proof Th2

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# Hilbert couple

$$\{H_0, H_1\} = \left\{l_2\left(G_n\right), l_2\left(2^{-n} G_n\right)\right\}$$

$$\|x\|_{H_0} = \sum_{n=-\infty}^{\infty} \|\xi_n\|_{G_n}^2 < \infty.$$

$$\|x\|_{H_1} = \sum_{n=-\infty}^{\infty} 2^{-2n} \|\xi_n\|_{G_n}^2 < \infty.$$

Donoghue, W.F. The interpolation of quadratic norms. Acta Math. 118, 251–270 (1967). https://doi.org/10.1007/BF02392483

Ovchinnikov, V.I.: Operator ideals and interpolation in Hilbert couples. Vestn. VSU. Phys. Math. 2 142–161 (2014). http://www.vestnik.vsu.ru/pdf/physmath/2014/02/2014-02-13.pdf

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# Operators in Hilbert couple



Let $A \in B(\overline{H})$ and $(a_{ij})_{i,j=-\infty}^{\infty}$, where $a_{ij} : G_j \to G_i$ are operators acting from $G_j$ to $G_i$.
$A = A' + A''$ when

$$A' = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k,\ A'' = \frac{A_0}{2} + \sum_{k=-\infty}^{-1} A_k$$

and operators $A_k$ have matrices $(a_{i\ i-k})_{i,k=-\infty}^{\infty}$ (it is $k$-th diagonal).

Gohberg, I.C., Krein, M.G.: Theory and Applications of Volterra operators on Hilbert spaces, Translations of Mathematical Monographs, vol. 24. AMS Bookstore, Providence, Rhode Island (1970).
https://doi.org/10.1090/mmono/024

# Operators in Hilbert couple

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Some auxiliary results

Main results

Operator $A$ acts in the space $l_2(G_n)$, the norms of elements of block-matrices satisfy the inequality

$$\|a_{ij}\|_{G_j \to G_i} \leq \|A\|_{B(\overline{H})}.$$

If the operator $A$ is acting in $l_2(2^{-j}G_j)$ then the inequality

$$\|a_{ij}\|_{G_j \to G_i} \leq 2^{i-j}\|A\|_{B(\overline{H})} \qquad (1)$$

holds. Combining the two conditions, we obtain

$$\|a_{ij}\|_{G_j \to G_i} \leq \min(1, 2^{i-j})\|A\|_{B(\overline{H})}.$$

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Some auxiliary results

Main results

# Operators in Hilbert couple

$$\|A_k\|_{l_2(G_j)\to l_2(G_j)} \leq \min(1, 2^k)\|A\|_{B(\overline{H})}.$$


Kabanko, M.V.: Algebra of operators acting in Hilbert couple. Trans. Math. Fac. VSU 6, 54–60 (2001) https://doi.org/10.13140/RG.2.2.32163.98086

Kabanko, M.V., Ovchinnikov, V.I.: On irreducible representations of the algebra of operators of Hilbert couple. Trans. Math. Fac. VSU 5 (New series), pp. 52–60 (2001) https://www.researchgate.net/publication/325631534_On_irreducible_representations_of_the_algebra_of_operators_of_Hilbert_couple/citations


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Some auxiliary results

Main results

# Sparse Hilbert couple

Let $\overline{K} = \{l_2(2^{2^j}), l_2(2^{-2^j})\}$ be a Hilbert couple.

Ovchinnikov, V.I.: Optimal interpolation theorem for quasi-Banach spaces lp with weights and for operators from von Neumann-Schatten classes in Hilbert pairs. Math. Notes 53(2), 94–99 (1993). https://doi.org/10.1007/BF01208323

Ovchinnikov, V.I.: Interpolation of cross-normed ideals of operators defined on different spaces. Funct. Anal. Appl. 28(3), 213–215 (1994). https://doi.org/10.1007/BF01078458

## Schatten-von Neumann class

**Definition**

Let $H$ be a Hilbert space. If $0 < \alpha < \infty$ and the sequence of s–numbers of operator $T$ is p-summable (i.e. $\{s_j(T)\}_{j=1}^{\infty} \in l_p$), then operator $T$ is said to belong to the Schatten-von Neumann class $S_\alpha(H)$.

It is known that $S_\alpha(H)$ is a Banach space, and

$$\|T\|_{S_\alpha(H)} = \left(\sum_{j=1}^{\infty} (s_j(T))^\alpha\right)^{\frac{1}{\alpha}}$$

is a norm on $S_\alpha(H)$ if $1 < \alpha < \infty$. If $0 < \alpha < 1$, then $\|T\|_{S_\alpha(H)}$ is a quasi-norm and $S_\alpha(H)$ is a quasi-Banach space.

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# Operators in sparse Hilbert couple

**Theorem**

*Each operator in $B(\overline{K})$ can be represented by a sum of the diagonal operator $D$ and the operator $N \in S_\alpha(l_2)$ when $S_\alpha(l_2)$ is the Neumann-Schatten class in the space $l_2$ for any index $\alpha > 0$.*

Proof Th3

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# Represantation in middle space $l_2$



Now we consider a representation $(l_2, \varphi)$ of the algebra $B(\overline{K})$ in the interpolation space $l_2$:

$$\varphi : B(\overline{K}) \to B(l_2), \varphi(A) = A\mid_{l_2}.$$

Recall that the representation is faithful and irreducible and the operator $A\mid_{l_2}$ is equal to the sum of diagonal operator $D$ and the compact operator $N$, i.e. $A\mid_{l_2} = D + N$. Further, consider the image of the representation in the Calkin algebra $B(l_2)/C(l_2)$

$$\pi : B(l_2) \to B(l_2)/C(l_2), \pi(A\mid_{l_2}) = \widetilde{A},$$

where $C(l_2)$ is an ideal of compact operators in $l_2$.

# Calkin algebra

The operator $A\mid_{l_2} = D + N$ is invertible when it is acting in $l_2$ if and only if the image of the operator is not in the null-coset of Calkin algebra, i.e. the diagonal $D = \{a_{nn}\}_{n=1}^{\infty}$ has $\inf\limits_n \mid a_{nn} \mid > 0$.

## Main case

In this part of the paper, we passed to the question of the invertibility of an operator, which acts in the couple $\overline{H} = \left\{ l_2(2^{\frac{n}{2}} G_n), l_2(2^{-\frac{n}{2}} G_n) \right\}$. Each operator, which acts in the couple, can be represented as

$$A = D + \sum_{k \in \mathbb{Z}, k \neq 0} A_k,$$

when $D$ is an operator corresponding to the main diagonal in the matrix representation of $A$. The operators $A_k$ are the diagonals which "parallel" to the main diagonal.

Gil, M.I.: Spectrum of infinite block matrices and pi-triangular operators. Electron. J. Linear Algebra 16, 216–231 (2007). https://doi.org/10.13001/1081-3810.1197

Some auxiliary results

Main results

# Theorem 4

## Theorem

*Let operator $A$ be in $B(\overline{H})$. Suppose that the diagonal operator $D$ in the space $l_2(G_n)$ is invertible and the inequality*

$$\|D^{-1}\|_{l_2(G_n)} < \frac{\sqrt{2}-1}{2\|A\|_{B(\overline{H})}}$$

*holds. Then $A$ will be invertible in space $l_2(G_n)$.*

Proof Th4

Kabanko, M.V. On invertibility of some classes of operators in weighted Hilbert spaces. Adv. Oper. Theory 7, 21 (2022).
https://doi.org/10.1007/s43036-022-00187-0

Some auxiliary results

Main results

THANK YOU!

# Proof of Theorem 1

Assume on the contrary, that some non-trivial subspace in $H_\pi$ is invariant for $\pi(\mathfrak{A})$. It is true if and only if each vector in $H_\pi$ is not cyclic vector for the image $\pi$. Let $e$ be such not cyclic vector i.e. the closure $\overline{\mathfrak{A}(e)}$ does not coincide with $H_\pi$.
Let

$$f(x) : H_0 + H_1 \to \mathbb{C}$$

be an arbitrary linear functional acting in a sum of spaces. We define a linear operator acting in a sum of spaces in the following way

$$T : H_0 + H_1 \to H_0 \cap H_1, \qquad Tx = f(x)a,$$

where $a$ is fixed vector from $H_0 \cap H_1$. Obviously, $T$ images sum of spaces $H_0 + H_1$ into the intersection $H_0 \cap H_1$.

# Proof of Theorem 1

Then

$$T : H_\pi \to H_\pi$$

and hence $T \in \mathfrak{A}$.

We will choose a functional $f$ in such a way that $f(e) = 1$. Then $Te = f(e)a = a$ and, therefore, $\overline{\mathfrak{A}(e)}$ contains the $H_0 \cap H_1$. The property stating that the intersection is dense in an interpolation Hilbert space is well known. Then the closure of $H_0 \cap H_1$ (and $\overline{\mathfrak{A}(e)}$) coincides with $H_\pi$. This contradicts the fact that the vector $e$ is not cyclic. Thus the representation is irreducible.

Back to Th1

# Proof of Theorem 2

First, we observe that if $T$ is injective in $H_0 + H_1$ then it has trivial kernels in $H_0$ and $H_1$. To see that it is bijective we must prove that $T$ is surjective in spaces of couple i.e. the image $T(H_0)$ coincides with $H_0$ and image $T(H_1)$ coincides with $H_1$.
Let $z$ be an element in $H_1$ such that it is not in $T(H_1)$. Recall that if an operator $T$ acting in the sum $H_0 + H_1$ then $z = T(x_0 + x_1)$ with $x_0 \in H_0$, $x_1 \in H_1$. Thus on the one hand $z - T(x_1) = T(x_0) \in H_0$ and, on the other hand, $z - T(x_1) \in H_1$. Consequently, $T(x_0) \in H_0 \cap H_1$. As the operator $T$ is invertible in $H_0 \cap H_1$ hence $x_0 \in H_0 \cap H_1$ and $x_0 + x_1 \in H_1$. Therefore, $z = T(x_0 + x_1) \in T(H_1)$. We obtain contradiction and the theorem is proved.

Back to Th2

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Some auxiliary results

Main results

# Proof of Theorem 3

Suppose $A$ is some operator acting in the couple $\overline{K}$ and $(a_{ij})_{ij}^{\infty}$ is a matrix corresponding to the operator $A$ in the space $l_2$ (with $a_{ij} \in \mathbb{C}$).
Thus

$$| a_{ij} | \leq \min(2^{2^i - 2^j}, 2^{2^j - 2^i}) \|A\|_{\overline{K}}.$$

Let us consider an operator in the form

$$N = \sum_{m,n=1, m \neq n}^{\infty} A_{mn},$$

where the operators $A_{mn}$ map the vector $\{\xi_j\}_{j=1}^{\infty}$ into the vector $\{\zeta_i\}_{i=1}^{\infty}$ by the rules $\zeta_m = a_{mn}\xi_n$ and $\zeta_i = 0$ if $i \neq m$.

Back to Th3

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Some auxiliary results

Main results

# Proof of Theorem 3

The series

$$\sum_{m,n=1,m\neq n}^{\infty} \|A_{mn}\|_{l_2}^{\alpha}$$

is convergent.
Thus the operator

$$N = \sum_{m,n=1,m\neq n} A_{mn},$$

is an absolutely convergent series and $N$ belongs to the Neumann-Schatten class (for each $\alpha > 0$).

Back to Th3

Some auxiliary results

Main results

# Proof of Theorem 4

Operator $A$ can be represented in the form

$$A = D\left(I + D^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}} A_k\right).$$

$$\|A_k\|_{l_2(G_n)} \leq 2^{-\left|\frac{k}{2}\right|}\|A\|_{B(\overline{H})}.$$

by the condition of the theorem we have

$$\left\|D^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}} A_k\right\|_{l_2(G_n)} \leq \frac{2\|D^{-1}\|_{l_2(G_n)}\|A\|_{B(\overline{H})}}{\sqrt{2}-1} < 1.$$

Thus, the operator

$$I + D^{-1}\sum_{k\in\mathbb{Z}\setminus\{0\}} A_k$$

is invertible in the space $l_2(G_n)$. Back to Th4