Extension operators on Sobolev spaces and Neumann eigenvalue problem

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1. Neumann spectral problem.

Main problem :

Estimates of the first non-trivial Neumann eigenvalues of the Laplace operator in bounded domains $\Omega \subset \mathbb{R}^n$, $n \geq 2$, in terms of the norms of the extension operators.

The classical Neumann spectral problem for the Laplacian:

$$\begin{cases} -\Delta u = \mu u & \text{in} \quad \Omega \\ \frac{\partial u}{\partial n} = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

The weak statement of this spectral problem: find a function $u \in W_2^1(\Omega)$ that satisfies the equality

$$\int\limits_{\Omega} \nabla u(x) \cdot \nabla v(x)\ dx = \mu \int\limits_{\Omega} u(x)v(x)\ dx,$$

for any function $v \in W_2^1(\Omega)$.

By the Min–Max Principle the first non-trivial Neumann eigenvalue $\mu_1(\Omega)$ for the Laplacian can be characterized as

$$\mu_1(\Omega) = \min\left\{ \frac{\int\limits_{\Omega} |\nabla u(x)|^2\,dx}{\int\limits_{\Omega} |u(x)|^2\,dx} : u \in W^1_2(\Omega)\setminus\{0\},\ \int\limits_{\Omega} u\,dx = 0 \right\}.$$

Moreover, $\mu_1(\Omega)^{-\frac{1}{2}}$ is the best constant $B_{2,2}(\Omega)$ in the following Poincaré inequality

$$\inf_{c\in\mathbb{R}} \|u - c \mid L_2(\Omega)\| \leq B_{2,2}(\Omega)\|\nabla u \mid L_2(\Omega)\|, \quad u \in W^1_2(\Omega).$$

The Sobolev space $W^1_p(\Omega)$, $1 \leq p < \infty$, is defined as a Banach space of locally integrable weakly differentiable functions $u : \Omega \to \mathbb{R}$ equipped with the following norm:

$$\|u \mid W^1_p(\Omega)\| = \left(\int\limits_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} + \left(\int\limits_{\Omega} |\nabla u(x)|^p \, dx \right)^{\frac{1}{p}}.$$

Short historical review.

Explicit values of $\mu_1(\Omega)$ are known only for several particular domains. For example:

- Rectangle $a \times b$: $\mu_1 = \left(\frac{\pi}{\max\{a,b\}}\right)^2$;
- 45° right triangle: $\mu_1 = \left(\frac{\pi}{a}\right)^2$, a is the leg length;
- 30° right triangle: $\mu_1 = \frac{16}{3}\left(\frac{\pi}{c}\right)^2$, c is the hypotenuse length;
- Equilateral triangle: $\mu_1 = \left(\frac{4\pi}{3a}\right)^2$, a is the side length;
- n-ball of the radius R: $\mu_1(B_R) = \left(\frac{p_{n/2}}{R}\right)^2$,

$p_{n/2}$ denotes the first positive zero of the function $\left(t^{1-n/2}J_{n/2}(t)\right)'$.

The classical upper estimate of the first non-trivial Neumann eigenvalue of the Laplace operator is related to the Szegö–Weinberger inequality (1954/56):

$$\mu_1(\Omega) \leq \mu_1(\Omega^*) = \frac{(p_{n/2})^2}{R_*^2}, \qquad (1)$$

where $p_{n/2}$ denotes the first positive zero of the derivative of the function $(t^{1-n/2}J_{n/2}(t))'$, and Ω^* is a ball of the same volume as Ω with R_* as its radius.

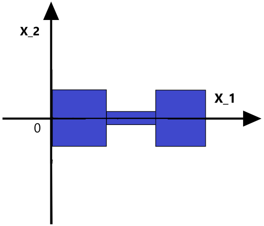
The classical lower estimate of the first non-trivial Neumann eigenvalue of the Laplace operator in convex domains $\Omega \subset \mathbb{R}^n$ is related to the Payne–Weinberger inequality (1960):

$$\mu_1(\Omega) \geq \frac{\pi^2}{d(\Omega)^2}, \tag{2}$$

where $d(\Omega)$ is a diameter of a convex domain Ω.

In non-convex domains the first non-trivial Neumann eigenvalues can not be estimated in the terms of Euclidean diameters of domains.

It can be seen by considering a domain consisting of two identical squares connected by a thin corridor:



When the width of the corridor tends to zero, the first non-trivial Neumann eigenvalue tends to zero also.

2. Extension operators on Sobolev spaces.

Recall that a linear bounded operator

$$E : L^1_p(\Omega) \to L^1_p(\mathbb{R}^n)$$

satisfying the conditions

$$Eu|_\Omega = u \quad \text{and} \quad \|E\| := \sup_{u \in L^1_p(\Omega)} \frac{\|Eu \mid L^1_p(\mathbb{R}^n)\|}{\|u \mid L^1_p(\Omega)\|} < \infty$$

is called a linear bounded extension operator.

We say that Ω is *a Sobolev L^1_p-extension domain* if there exists a linear bounded extension operator.

It is well known (A. P. Calderón 1961, E. M. Stein 1970) that

$$\text{if } \Omega \in \mathbb{R}^n,\ n \geq 2,\ \text{ be a Lipschitz domain}$$

then there exists a bounded extension operator

$$E : L^1_p(\Omega) \to L^1_p(\mathbb{R}^n),\ 1 \leq p \leq \infty.$$

P. W. Jones (1981) introduced the notion of (ε, δ)-domains and proved that in every (ε, δ)-domain there exists the bounded extension operator

$$E : L^k_p(\Omega) \to L^k_p(\mathbb{R}^n),$$

for all $k \geq 1$ and $p \geq 1$.

The complete description of extension operators of the Sobolev space $L^1_2(\Omega)$, $\Omega \subset \mathbb{R}^2$,

$$E : L^1_2(\Omega) \to L^1_2(\mathbb{R}^2)$$

was obtained by (**S. K. Vodop'yanov, V. M. Gol'dshtein, T. G. Latfullin, 1979**) in terms of quasiconformal geometry of domains.

Namely, it was proved that a simply connected domain $\Omega \subset \mathbb{R}^2$ is the L^1_2-extension domain iff Ω is a quasidisc.

A domain Ω is called a K-quasidisc, $K \geq 1$, if it is the image of the unit disc $\mathbb{D} \subset \mathbb{R}^2$ under K-quasiconformal mappings of the plane onto itself.

• Let $\Omega \subset \mathbb{R}^n$ be a Sobolev L^1_2-extension domain. Then the embedding operator

$$i_\Omega : W^1_2(\Omega) \hookrightarrow L^2(\Omega)$$

is compact.

• The spectrum of the Neumann Laplace operator in Ω is discrete, and can be written in the form of a non-decreasing sequence:

$$0 = \mu_0(\Omega) < \mu_1(\Omega) \leq \mu_2(\Omega) \leq ... \leq \mu_n(\Omega) \leq ...,$$

where each eigenvalue is repeated as many time as its multiplicity.

3. Main results.

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a Sobolev L^1_2-extension domain. Then for any bounded Lipschitz domain $\widetilde{\Omega} \supset \Omega$*

$$\mu_1(\widetilde{\Omega}) \leq \|E_\Omega\|^2 \cdot \mu_1(\Omega), \tag{3}$$

where $\|E_\Omega\|$ denotes the norm of a linear bounded extension operator

$$E_\Omega : L^1_2(\Omega) \to L^1_2(\widetilde{\Omega}).$$

Theorem 2. *Let $\Omega \subset \mathbb{R}^n$ be a Sobolev L^1_2-extension domain. Then*

$$\mu_1(\Omega) \geq \left(\frac{1}{\|E_\Omega\|} \frac{p_{n/2}}{R_\Omega}\right)^2, \tag{4}$$

where R_Ω is a radius of a minimum enclosing ball B_Ω of Ω, $p_{n/2}$ denotes the first positive zero of the function $\left(t^{1-n/2} J_{n/2}(t)\right)'$ and $\|E_\Omega\|$ denotes the norm of a linear bounded extension operator

$$E_\Omega : L^1_2(\Omega) \to L^1_2(B_\Omega).$$

Remark 1. *If Ω is a convex domain and a norm of an extension operator*

$$E_\Omega : L^1_2(\Omega) \to L^1_2(B_\Omega)$$

satisfies to the following condition

$$\|E_\Omega\| \leq \frac{2p_{n/2}}{\pi}$$

then estimate (4) is better than the classical estimate (2).

Example 1. Consider n-dimensional half-ball $B^- = \{x \in \mathbb{R}^n : |x| < 1 \ \& \ x_n < 0\}$. Define the extension operator

$$E_{B^-}(u) = \begin{cases} u(x_1, ..., x_{n-1}, x_n), & \text{if } x \in B^-, \\ u(x_1, ..., x_{n-1}, -x_n), & \text{if } x \in B^+, \end{cases}$$

where $B^+ = \{x \in \mathbb{R}^n : |x| < 1 \ \& \ x_n > 0\}$. Then

$E_{B^-} : L^1_2(B^-) \to L^1_2(B)$, $B = \{x \in \mathbb{R}^n : |x| < 1\}$, and $\|E_{B^-}\| = \sqrt{2}$.

Hence by Theorem 2

$$\mu_1(B^-) \geq \frac{p^2_{n/2}}{2} > \frac{\pi^2}{4}, \text{ if } n \geq 4.$$

3.1. Spectral estimates in planar domains.

• Any conformal mapping $\varphi : \Omega \to \Omega'$ generates an isometry

$$\varphi^* : L^1_2(\Omega') \to L^1_2(\Omega).$$

• **S. K. Vodop'yanov and V. M. Gol'dshtein (1975)**. A homeomorphism $\psi : \Omega \to \widetilde{\Omega}$ is a K-quasiconformal mapping if and only if ψ generates by the composition rule $\psi^*(\widetilde{u}) = \widetilde{u} \circ \psi$ a bounded composition operator on Sobolev spaces $L^1_2(\Omega)$ and $L^1_2(\widetilde{\Omega})$:

$$\|\psi^*(\widetilde{u}) \mid L^1_2(\Omega)\| \leq K^{\frac{1}{2}} \|\widetilde{u} \mid L^1_2(\widetilde{\Omega})\|$$

for any $\widetilde{u} \in L^1_2(\widetilde{\Omega})$.

Consider the following diagram

$$\begin{array}{ccc}
L_2^1(\Omega) & \xrightarrow{\varphi^*} & L_2^1(H^+) \\
{\scriptstyle (\widetilde{\varphi}^{-\mathbf{1}})^* \circ \omega \circ \varphi^*}\Big\downarrow & & \Big\downarrow{\scriptstyle \omega} \\
L_2^1(\mathbb{R}^2) & \xleftarrow{(\widetilde{\varphi}^{-\mathbf{1}})^*} & L_2^1(\mathbb{R}^2)
\end{array}$$

where ω is a symmetry with respect to the real axis, that extend any function $u \in L_2^1(H^+)$ to a function $\widetilde{u}$ from $L_2^1(\mathbb{R}^2)$.

In accordance to this diagram we define the extension operator on Sobolev spaces

$$E : L^1_2(\Omega) \to L^1_2(\mathbb{R}^2)$$

by the formula

$$(Eu)(x) = \begin{cases} u(x) & \text{if } x \in \Omega, \\ \widetilde{u}(x) & \text{if } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \end{cases}$$

where $\widetilde{u} : \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ is defined as $\widetilde{u} = (\widetilde{\varphi}^{-1})^* \circ \omega \circ \varphi^* u$.

In this case the norm of this extension operator $\|E\| \leq 1 + K$.

Theorem 3. *Let Ω be a K-quasidisc. Then*

$$\mu_1(\Omega) \geq \left(\frac{j'_{1,1}}{R_\Omega}\right)^2 \cdot \left(\frac{1}{1+K}\right)^2,$$

where R_Ω is a radius of a minimum enclosing ball B_Ω of Ω and $j'_{1,1} \approx 1.84118$ denotes the first positive zero of the derivative of the Bessel function J_1.

Star-shaped domains. We say that a domain Ω^* is β-star-shaped (with respect to $z_0 = 0$) if the function $\varphi(z)$, $\varphi(0) = 0$, conformally maps the unit disc $\mathbb{D}$ onto Ω^* and the condition satisfies:

$$\left|\arg \frac{z\varphi'(z)}{\varphi(z)}\right| \leq \beta\pi/2, \quad 0 \leq \beta < 1, \quad |z| < 1.$$

In the paper (**M. Fait, Y. Krzyz, Y. Zygmunt, 1976**) proved the following: the boundary of the β-star-shaped domain Ω^* is a K-quasicircle with $K = \cot^2(1-\beta)\pi/4$.

Then by Theorem 3 we have

$$\mu_1(\Omega^*) \geq \sin^4\left((1-\beta)\pi/4\right) \cdot \left(\frac{j'_{1,1}}{R_{\Omega^*}}\right)^2 .$$

Spiral-shaped domains. We say that a domain Ω_s is β-spiral-shaped (with respect to $z_0 = 0$) if the function $\varphi(z)$, $\varphi(0) = 0$, conformally maps the unit disc $\mathbb{D}$ onto Ω_s and the condition satisfies:

$$\left| \arg e^{i\gamma} \frac{z\varphi'(z)}{\varphi(z)} \right| \leq \beta\pi/2, \quad 0 \leq \beta < 1, \quad |\gamma| < \beta\pi/2, \quad |z| < 1.$$

In the paper (**M. A. Sevodin, 1987**) proved the following: the boundary of the β-spiral-shaped domain Ω_s is a K-quasicircle with $K = \cot^2(1-\beta)\pi/4$.

Then by Theorem 3 we have

$$\mu_1(\Omega_s) \geq \sin^4\left((1-\beta)\pi/4\right) \cdot \left(\frac{j'_{1,1}}{R_{\Omega_s}}\right)^2.$$

3.2. Spectral estimates in space domains.

Let $u(x)$ be a nonnegative continuous function of class W^1_∞ in some two-sided neighborhood of the unit sphere $S := \{x : |x| = 1\}$. We set $\rho = |x|$, $\theta = x/\rho$ and denote

$$M_1 = \min_{x\in S} u(x) > 0, \quad M_2 = \max_{x\in S} u(x), \quad M_3 = \sup_{x\in S} |\nabla u(x)|.$$

Let Ω_1 and Ω_R, $R > 1$, be a star-shaped domains having the form

$$\Omega_\beta := \{x : \rho < \beta u(\theta)\}, \quad \beta > 0.$$

In the paper (**S. G. Mihlin, 1979**) obtained the estimate of the norm of extension operator

$$E^* : W_2^1(\Omega_1) \to W_2^1(\Omega_R)$$

which have the form

$$\|E^*\|^2 \leq 1 + \left(\frac{M_2}{M_1}\right)^2 (N_1 N_2)^2 \left(\|E_R\|^2 - 1\right),$$

where

$$N_1^2 = \max\left\{\frac{M_1^2 + (n-1)M_3^2}{M_1^4}, \frac{2}{M_1^2}, 1\right\}$$

and

$$N_2^2 = \max\left\{M_2^2 + 2(n-1)M_3^2, 2M_2^2, 1\right\}.$$

Then by Theorem 1 we have

Theorem 4. *Let Ω_1 and Ω_R, $R > 1$, be star-shaped domains in Euclidean space $\mathbb{R}^n$. Then*

$$\mu_1(\Omega_R) \leq \mu_1(\Omega_1)\|E^*\|^2 \leq \mu_1(\Omega_1)\left(1 + \left(\frac{M_2}{M_1}\right)^2 (N_1 N_2)^2 \left(\|E_R\|^2 - 1\right)\right).$$

Gol'dshtein V. Pchelintsev V., Ukhlov A. Sobolev extension operators and Neumann eigenvalues. J. Spectr. Theory **10** (2020), 337–353.

THANKS