Renormalization process and its application to «bad» functions

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Theory of Functions and its Applications
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Content

1) Introduction

2) Problem statement and definitions

3) Mathematical approach

4) Physical approach

7) Conclusion and remarks

Historical background

Renormalization (in context QFT)

Not perturbative methods

Perturbative methods

A functional integration

Formal series

"Bad behavior"

?

Divergences

Renormalization:
Faddeev, Slavnov (1978)
Collins (1984)
Zavialov (1990)
Kazakov (2009)

Standard QFT:
Bogoliubov, Shirkov (1959)
Faddeev, Slavnov (1978)
Itzykson, Zuber (1980)
Peskin, Schroeder (1995)

Functional integral:
Daniell (1919)
Feynman (1948)
Cartier, DeWitt-Morette (1997)
Zinn-Justin (2005)
Takhtajan (2008)
Shavgulidze, Smolyanov (2015)
Ivanov, Russkikh (2023)
Bhattacharya, Cotler,
Dersy, Schwartz (2024)

FQFT:
Atiyah (1988)
Segal (2004)
Reshetikhin (2010)
Stolz, Teichner (2011)
Cattaneo, Mnev, Reshetikhin (2018)
Kandel, Mnev, Wernli (2021)

Can we present a simple example to illustrate?

First functional

Let V **denote a set of sequences** $a = \{a_j \in \mathbb{R}\}_{j=1}^{+\infty}$. **The symbol** p_n **denotes a projector from** V **to** V :

$$\big(p_n(a)\big)_j = a_j \ \textbf{ for } \ j \in \{1,\ldots,n\}, \ \ \big(p_n(a)\big)_j = 0 \ \textbf{ for } \ j > n\,.$$

Let $k \in \mathbb{N}$. **The symbol** $\mathscr{B}_k$ **denotes a subset of** V, **all elements** $\beta \in \mathscr{B}_k$ **of which satisfy** :

$$\beta_j > 0 \textbf{ for all } j \in \mathbb{N}, \textbf{ and } b_k(\beta) = \sum_{j=1}^{+\infty} \beta_j^{-k} < +\infty\,.$$

Let $s \in \mathbb{R}$, $n \in \mathbb{N}$, **we define**

$$\Phi_n(s,\beta) = \prod_{j=1}^{n} \int_{\mathbb{R}} \frac{\mathrm{d}a_j}{\sqrt{\pi/\beta_j}}\, e^{-\beta_j a_j^2 + isa_j^2}\,.$$

What do we get in $n \to +\infty$?

Example

Let $\mathbb{N} \ni n > 1$ **and** $\mathbf{B} = \{x \in \mathbb{R}^n : |x| \leqslant 1\}$. **We consider**

$$-\sum_{i=1}^{n} \partial_{x_i}^2 u(x) = \lambda u(x), \;\; u(x)\Big|_{|x|=1} = 0.$$

$$\Big\downarrow$$

$$\beta = \{\beta_j\}_{j=1}^{+\infty}, \;\; \{\phi_j(x) \in L^2(\mathbf{B})\}_{j=1}^{+\infty}$$

The Green's function is

$$G(x,y) = \sum_{j=1}^{+\infty} \phi_j(x)\beta_j^{-1}\phi_j(y)\,.$$

For $x \sim y$ **the main part is proportional to** $\ln(|x-y|)$ **for** $n = 2$ **and** $|x-y|^{2-n}$ **for** $n > 2$. **Hence** $b_1(\beta) = +\infty$.

For other numbers we get $b_k(\beta) = \int_{\mathbf{B}} \mathrm{d}^n x_1 \ldots \int_{\mathbf{B}} \mathrm{d}^n x_k \, G(x_1, x_2) \cdot \ldots \cdot G(x_{k-1}, x_k)\,.$ **Hence** $\beta \in \mathscr{B}_2$ **for** $n = 2{,}3$.

Representations

1)

$$\Phi_n(s,\beta) = \left(\prod_{j=1}^{+\infty}\int_{\mathbb{R}} \frac{\mathrm{d}a_j}{\sqrt{\pi/\beta_j}}\right) e^{-S_0[a,\beta]+isS_1[p_n(a)]},$$

$$S_0[a,\beta] = \sum_{j=1}^{+\infty} \beta_j\, a_j^2, \quad S_1[a] = \sum_{j=1}^{+\infty} a_j^2\,.$$

2)

$$\Phi_n(s,\beta) = \prod_{j=1}^{n}\left(1 - is/\beta_j\right)^{-\frac{1}{2}} = f_n(s)e^{ig_n(s)/2},$$

$$f_n(s) = \prod_{j=1}^{n}\left(1 + s^2/\beta_j^2\right)^{-\frac{1}{4}}, \quad g_n(s) = \sum_{j=1}^{n} \arctan\left(s/\beta_j\right)\,.$$

Some properties

Let $\Phi(s,\beta)$, $f(s)$, **and** $g(s)$ **denote the limit values of** $\Phi_n(s,\beta)$, $f_n(s)$, **and** $g_n(s)$, **if such exist**.

Let $s \in \mathbb{R}\backslash\{0\}$ **and** $\beta \in \mathscr{B}_2$. **Then we have**

1) $0 < f(s) < f_{n+1}(s) < f_n(s) < 1$ **for all** $n \in \mathbb{N}$;

2) $f(\cdot) \in L^p(\mathbb{R})$ **for all** $p > 0$;

3) $g(s) < +\infty$ **if and only if** $b_1(\beta) < +\infty$;

4) $\Phi_n(s,\beta)$ **has a limit when** $n \to +\infty$ **if and only if** $b_1(\beta) < +\infty$.

If $s = 0$, **then we have** $\Phi(0,\beta) = 1$, $f(0) = 1$, **and** $g(0) = 0$.

What do we need to do if $\beta \in \mathscr{B}_2\backslash\mathscr{B}_1$?

We need to renormalize!

Regularization

Let $\beta \in \mathscr{B}_2 \backslash \mathscr{B}_1$.

A regularizing function is called a function $\rho(\,\cdot\,) \in C\big(\mathbb{R}_+, [0,1]\big)$, **such that for all** $\Lambda > N$:

1) $\rho(0) = 1$;

2) $\displaystyle\sum_{j=1}^{+\infty} \frac{1}{\beta_j(\Lambda)} = r(\Lambda) + \kappa + o(1) < +\infty$, **where** $\beta_j(\Lambda) = \beta_j \big/ \rho\big(\sqrt{\beta/\Lambda}\big)$.

singular behavior

Example: characteristic function of the interval $[0,a]$ **for some fixed** $a > 0$.

A regularization of the function $\Phi(s,\beta)$ **is the transition to** $\Phi(s,\beta(\Lambda))$ **using the deformation**

$$\beta = \{\beta_j\}_{j=1}^{+\infty} \to \beta(\Lambda) = \{\beta_j(\Lambda)\}_{j=1}^{+\infty} .$$

Renormalization

The object $\Phi(s, \beta(\Lambda))$ **is renormalizable if we can find such functions that**

$$e^{h_1+ih_2}\Phi\left(s + h_3, \{\beta_j(\Lambda) + h_{3+j}\}_{j=1}^{+\infty}\right)$$

has a finite limit at $\Lambda \to +\infty$.

Here $h_i = h_i(\Lambda, s, \beta)$ **satisfy:** $1)$ **real**;

$2)$ **they diverge at** $\Lambda \to +\infty$ **for all fixed** s;

$3)$ **and they tend to zero at** $s \to 0$ **for each fixed** Λ .

Not unique! $\longrightarrow$ **We obtain a family of functions!**

The functional is renormalizable!

$$\Phi(s,\beta(\Lambda)) \longrightarrow \Phi(s,\beta(\Lambda))e^{-is(r(\Lambda)+\theta)/2}$$

Additional phase factor!

Not exist!

$$\Phi(s,\beta) \xrightarrow{reg.} \Phi(s,\beta(\Lambda)) \xrightarrow{ren.} \Phi(s,\beta(\Lambda))e^{-is(r(\Lambda)+\theta)/2} \xrightarrow{\Lambda\to+\infty} \Phi(s,\beta,\theta).$$

$$\Phi(s,\beta,\theta) = \Phi(s,\beta(\Lambda))e^{-is(r(\Lambda)+\theta)/2}\Big|_{\Lambda\to+\infty} = f(s)\exp\left(-\frac{i}{2}\big(s\theta + g_{\mathrm{r}}(s)\big)\right),$$

$$g_{\mathrm{r}}(s) = -s\kappa + s^3\int_0^1 \mathrm{d}t \sum_{j=1}^{+\infty} \frac{t^2\beta_j^{-3}}{1+s^2t^2/\beta_j^2}.$$

Second functional

Let $\lambda \geqslant 0$ **and** $\beta \in \mathscr{B}_2 \backslash \mathscr{B}_1$, **then**

$$Z_n(\lambda, \beta) = \left(\prod_{j=1}^{n} \int_{\mathbb{R}} \frac{\mathrm{d}a_j}{\sqrt{\pi/\beta_j}} \right) \exp\left(-S_0[p_n(a)] - \lambda \Big(S_1[p_n(a)] \Big)^2 \right).$$

Nonlocal quartic model

$Z(\lambda, \beta) = \lim_{n \to +\infty} Z_n(\lambda, \beta) = 0$ **for all** $\lambda > 0$ **and** $Z(0, \beta) = 1.$

Second type of "divergence"!

Renormalization

Note that

$$Z_n(\lambda,\beta) = \mathrm{T}\big(\Phi_n(\,\cdot\,,\beta)\big)(\lambda) = \frac{1}{\sqrt{4\pi\lambda}}\int_{\mathbb{R}} \mathrm{d}s\, e^{-s^2/(4\lambda)}\Phi_n(s,\beta)\,.$$

Then we get

$$\begin{tikzcd}
\Phi(s,\beta) \arrow[r, "reg."] & \Phi(s,\beta(\Lambda)) \arrow[r, "ren."] \arrow[d, "\mathrm{T}"] & \Phi(s,\beta(\Lambda))e^{-is(r(\Lambda)+\theta)/2} \arrow[r, "\Lambda\to+\infty"] \arrow[d, "\mathrm{T}"] & \Phi(s,\beta,\theta) \arrow[d, "\mathrm{T}"] \\
Z(\lambda,\beta) \arrow[r, "reg."] & Z(\lambda,\beta(\Lambda)) \arrow[r, "ren."] & Z(\lambda,\beta_{\mathrm{r}}(\Lambda))N(\Lambda) \arrow[r, "\Lambda\to+\infty"] & Z(\lambda,\beta,\theta)
\end{tikzcd}$$

$$N(\Lambda) = e^{-(r(\Lambda)+\theta)^2/4}\prod_{j=1}^{+\infty}\sqrt{\beta(\Lambda)/\beta_{\mathrm{r}}(\Lambda)}, \qquad \beta_{\mathrm{r}}(\Lambda) = \{\beta_j(\Lambda) - r(\Lambda) - \theta\}_{j=1}^{+\infty}\,.$$

Spectrum! What do you do!?

$$Z(\lambda,\beta,\theta) = \frac{1}{\sqrt{\pi\lambda}}\int_{\mathbb{R}} \mathrm{d}s\, e^{-s^2/(4\lambda)} f(s)\cos\big(s\theta/2 + g_{\mathrm{r}}(s)/2\big)\,.$$

Not zero! For almost all values θ.

Physical approach

Let us see

$$\left(\prod_{j=1}^{n}\int_{\mathbb{R}}\frac{\mathrm{d}a_j}{\sqrt{\pi/\beta_j}}\,e^{-\beta_j a_j^2}\right)\Big(S_1[p_n(a)]\Big)^k = \left(\sum_{i=1}^{n}\partial_{c_i}^2\right)^k\left(\prod_{j=1}^{n}\int_{\mathbb{R}}\frac{\mathrm{d}a_j}{\sqrt{\pi/\beta_j}}\,e^{-\beta_j a_j^2+c_j a_j}\right)\Bigg|_{c=0}$$

$$= \left(\sum_{i=1}^{n}\partial_{c_i}^2\right)^k\exp\left(\frac{1}{4}\sum_{j=1}^{n}c_j^2/b_j\right)\Bigg|_{c=0}.$$

Integration ⟶ Formal series

Let $n = 1$ and $\beta_1 = 1$, then

$$\partial_t^{2k}e^{t^2/4}\Big|_{t=0} = \frac{(2k)!}{k!2^{2k}},$$

$$\sum_{k=0}^{+\infty}\frac{(is)^k}{k!}\int_{\mathbb{R}}\frac{\mathrm{d}t}{\sqrt{\pi}}\,e^{-t^2}t^{2k} = \sum_{k=0}^{+\infty}\frac{(is)^k}{k!}\frac{(2k)!}{k!2^{2k}}.$$

It is $(1-is)^{-1/2}$ for $|s|<1$.

Diagrammatic technique

Let us define three elements :

1) **The dot denotes the summation operator for all index values;**

2) **The line with one index denotes the component** a_j **for** $j \in \mathbb{N}$;

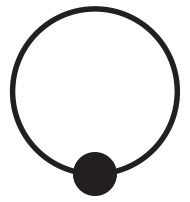
3) **The line with two indices denotes the value** $\delta_{ij}(2\beta_j(\Lambda))^{-1}$, **where** $i, j \in \mathbb{N}$.

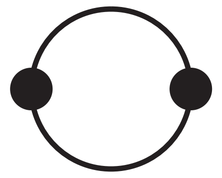
Example 1 :

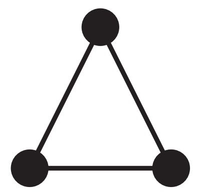
$$S_1[a] = \sum_{j=1}^{+\infty} a_j^2 =$$ 

$$\sum_{i,j=1}^{+\infty} a_i \delta_{ij}(2\beta_j(\Lambda))^{-1} a_j =$$ 

Example 2 :

$b_1(\beta(\Lambda)) = 2$ 

$b_2(\beta(\Lambda)) = 4$ 

$b_3(\beta(\Lambda)) = 8$ 

Auxiliary operators

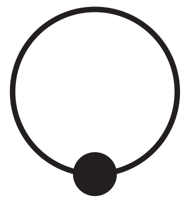
The symbol $\mathbb{H}$ **denotes a linear operator from the set of polynomials** $\mathbb{C}[S_1]$ **to** $\mathbb{C}$,

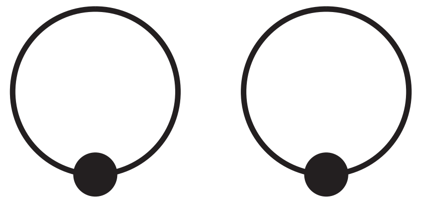
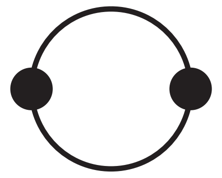
which is defined on monomials S_1^k **by two rules** :

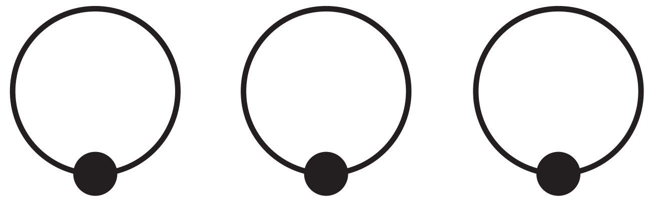
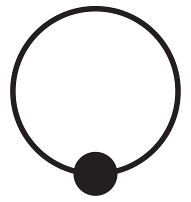
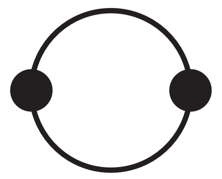
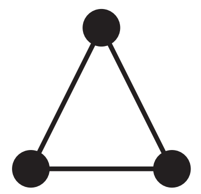
1) **if** $k = 0$, **then** $\mathbb{H}(S_1^0) = \mathbb{H}(1) = 1$;

2) **if** $k \in \mathbb{N}$, **then** $\mathbb{H}(S_1^k)$ **is equal to the sum of all possible pairwise connections of** $2k$ **external lines for** k **vertices**.

Example :

$\mathbb{H}(S_1) =$  $= \dfrac{1}{2} b_1(\beta(\Lambda))$

$\mathbb{H}(S_1^2) =$  $+2$  $= \dfrac{1}{4} b_1^2(\beta(\Lambda)) + \dfrac{1}{2} b_2(\beta(\Lambda))$

$\mathbb{H}(S_1^3) =$  $+6$   $+8$  $= \dfrac{1}{8} b_1^3(\beta(\Lambda)) + \dfrac{3}{4} b_1(\beta(\Lambda)) b_2(\beta(\Lambda)) + b_3(\beta(\Lambda))$

Physical «integral»

Let $f(\cdot)$ be a function that can be represented using a Taylor series

$$f(z) = \sum_{j=0}^{+\infty} f_j z^j \, .$$

A functional integral of $f(S_1[\cdot])$ with a weight $\exp(-S_0[\cdot\;,\beta(\Lambda)])$ is a formal series

$$\sum_{j=0}^{+\infty} f_j \, \mathbb{H}(S_1^j),$$

which is indicated by a symbolic entry of the form

$$\int_V \mathscr{D}a \, e^{-S_0[a,\beta(\Lambda)]} f(S_1[a]) \, .$$

Our functionals

$$\int_V \mathscr{D}a\, e^{-S_0[a,\beta(\Lambda)]+isS_1[a]} = \sum_{j=0}^{+\infty} \frac{(is)^j}{j!} \mathbb{H}(S_1^j)$$

$$\int_V \mathscr{D}a\, e^{-S_0[a,\beta(\Lambda)]-\lambda S_1^2[a]} = \sum_{j=0}^{+\infty} \frac{(-\lambda)^j}{j!} \mathbb{H}(S_1^{2j})$$

$$\Big\downarrow$$

Renormalization!

$$\int_V \mathscr{D}a\, e^{-S_0[a,\beta(\Lambda)]+is(S_1[a]-r(\Lambda)/2-\theta/2)} = \sum_{j=0}^{+\infty} \frac{(is)^j}{j!} \mathbb{H}\big((S_1 - r(\Lambda)/2 - \theta/2)^j\big)$$

$$\int_V \mathscr{D}a\, e^{-S_0[a,\beta(\Lambda)]-\lambda(S_1[a]-r(\Lambda)/2-\theta/2)^2} = \sum_{j=0}^{+\infty} \frac{(-\lambda)^j}{j!} \mathbb{H}((S_1 - r(\Lambda)/2 - \theta/2)^{2j})$$

Many thanks!