On the Hénon problem with different fractional Laplacians

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Let B be a unit ball $\mathbb{R}^n$ $(n \geq 2)$, $q > 2$ and $\alpha > 0$. The Dirichlet problem

$$\begin{cases} -\Delta u = |x|^\alpha |u|^{q-2} u \\ u > 0 \text{ in } B \\ u\Big|_{\partial\Omega} = 0 \end{cases} \tag{1}$$

was introduced by M. Hénon in [Hén73] . This problem was proposed as a model for spherically symmetric stellar clusters and was investigated numerically for some definite values of q and α. In the last decades it became evident that the Hénon equation and its modifications exhibits very interesting features concerning existence, multiplicity and symmetry properties of solutions.

First, for $s \in (0, 1)$ we consider the problem

$$(-\Delta)^s_{\mathcal{D}} u = |x|^\alpha |u|^{q-2} u \quad \text{in } B, \qquad u \in \widetilde{H}^s(B), \tag{2}$$

where $(-\Delta)^s_{\mathcal{D}}$ stands for the restricted or spectral Dirichlet fractional Laplacian. We use the notation $\widetilde{H}^s(B) = \Big\{u \in H^s(\mathbb{R}^n) \;:\; \operatorname{supp}(u) \subset \overline{B}\Big\}$, where $H^s(\mathbb{R}^n)$ is the standard Sobolev–Slobodetskii space.

It was shown in [Shc20] that for sufficiently large α, problem (2) has an arbitrary number of different positive solutions (not obtained from each other by orthogonal transformations). For original problem (1) similar results were obtained in [SSW02]. A similar problem driven by p-Laplacian was considered in [Naz01, KN07].

Further, we consider the problem

$$(-\Delta)^s_{\mathcal{N}} u + u = |x|^\alpha |u|^{q-2} u \quad \text{in } B, \qquad u \in H^s(B), \tag{3}$$

where $(-\Delta)^s_{\mathcal{N}}$ is the spectral Neumann fractional Laplacian.

A problem (3) driven by standard Laplacian $(s = 1)$ was investigated in [GS08]. A similar problem with p-Laplacian was considered in [Shc18].

We show that for $q \in \left(2; \frac{2(n+\alpha)}{n-2s}\right)$ the problem (3) has a positive radial solution. However, for $s > 1/2$, $q \in \left(\frac{2(n-1)}{n-2s}, \frac{2n}{n-2s}\right)$ and α sufficiently large there exists also a nonradial positive solution of (3). We also show that for q sufficiently close to 2, the radial function is a local minimizer of the energy functional corresponding to equation (3) in the space $H^s(B)$.

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