# Inverse iteration method for higher eigenvalues of the $p$-Laplacian

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Let $1 < p < \infty$ and let $\Omega \subset \mathbb{R}^D$ be a domain of finite measure, $D \geqslant 1$. Consider the problem of finding an *eigenvalue* $\lambda \in \mathbb{R}$ and an *eigenfunction* $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$\int_\Omega |\nabla u|^{p-2} \langle \nabla u, \nabla v \rangle \, dx = \lambda \int_\Omega |u|^{p-2} u v \, dx \quad \text{for any } v \in W_0^{1,p}(\Omega).$$

This is a weak form of the Dirichlet eigenvalue problem for the $p$-Laplace operator

$$\Delta_p(\cdot) := \operatorname{div}(|\nabla(\cdot)|^{p-2} \nabla(\cdot)),$$

turning into the classical Dirichlet Laplace eigenvalue problem for $p = 2$. Eigenfunctions are precisely critical points of the Rayleigh quotient functional $u \mapsto R[u]$ defined on $W_0^{1,p}(\Omega) \setminus \{0\}$ as

$$R[u] := \frac{\int_\Omega |\nabla u|^p \, dx}{\int_\Omega |u|^p \, dx},$$

and eigenvalues are critical levels of $R$.

It is known (see, for example, [Lin]) that there exists the smallest eigenvalue $\lambda_1(\Omega, p)$, called the *first* eigenvalue, and that $\lambda_1(\Omega, p)$ is also the only eigenvalue admitting sign-constant eigenfunctions (all of which being constant multipliers of one-another). The properties of $\lambda_1(\Omega, p)$ allow to develope different numerical methods for its numerical approximation, among which algorithms based on inverse iteration schemes play an important role. All the other eigenvalues are called *higher*, and they are significantly less studied from both the theoretic and the numeric viewpoints.

We propose a novel algorithm using inverse iterations to approximate some higher eigenvalue alongside the corresponding eigenfunctions. Given an arbitrary initial guess $u_0 \in L^\infty(\Omega)$ such that

$$\min\left\{\|u_0^+\|_p, \|u_0^-\|_p\right\} > 0 \quad \text{and} \quad \Big|\{x \in \Omega : \, u_0(x) = 0\}\Big| = 0,$$

this algorithm generates a sequence $\{u_{k+1}\} \subset W_0^{1,p}(\Omega) \setminus \{0\}$ by solving consecutive $p$-Poisson equations with the property that their solutions $u_{k+1}$ satisfy

$$R[u_{k+1}] = R[u_{k+1}^+] = R[u_{k+1}^-],$$

where for any $u\colon\ \Omega \to \mathbb{R}$ we let $u^\pm := \max\{\pm u, 0\}$. We prove the following convergence results for $\{u_{k+1}\}$.

**Theorem 1.** *Let $\mathcal{U}$ be the set of all strong limit points of the sequence $\{u_k\}$ in $W_0^{1,p}(\Omega)$. Then the following assertions hold:*

1. *The sequence $\{R[u_k]\}$ monotonically decreases towards a higher eigenvalue $R^*$:*

$$R^* := \inf_{k \geqslant 1} R[u_k] = \lim_{k\to\infty} R[u_k].$$

2. *$\mathcal{U} \neq \varnothing$ and it is a subset of the (sign-changing) eigenfunctions corresponding to the eigenvalue $R^*$.*
3. *Any subsequence $\{u_{k_n}\}$ contains a sub-subsequence having a strong limit in $W_0^{1,p}(\Omega)$. If $\{u_{k_n}\}$ converges strongly to some $u \in \mathcal{U}$, then, for any $i \in \mathbb{N}$, the shifted subsequence $\{u_{k_n+i}\}$ also converges strongly to $u$.*
4. *$\lim_{k\to\infty} \rho(u_k, \mathcal{U}) = 0$, where $\rho$ stands for the distance function in $W_0^{1,p}(\Omega)$.*
5. *$\mathcal{U}$ either consists of a single eigenfunction or has no isolated elements. In particular, if $R^*$ is a simple eigenvalue, then the whole sequence $\{u_k\}$ converges to the corresponding eigenfunction.*

The talk is based on the paper [BG25].


[BG25] V. Bobkov and T. Galimov, *Inverse iteration method for higher eigenvalues of the $p$-Laplacian*, 2025, arXiv: 2504.12836.

[Lin] P. Lindqvist, *A nonlinear eigenvalue problem*, Topics in mathematical analysis, pp. 175–203.