# Local renormalized solution of anisotropic elliptic equation with variable exponents in nonlinearities in $\mathbb{R}^n$

24.06
10:00-10:30

Larisa M. Kozhevnkova

*Ufa University of Science and Technology*

kosul@mail.ru

M. F. Bidaut-Veron [Bid03] introduced the concept of a local renormalized solution for the following equation with the $p$-Laplacian, absorption, and a Radon measure $\mu$:

$$-\Delta_p u + |u|^{p_0-2}u = \mu, \quad p \in (1,n), \quad 0 < p-1 < p_0. \tag{1}$$

In particular, M. F. Bidaut-Veron proved the existence of a local renormalized solution in $\mathbb{R}^n$ of the equation (1) with $\mu \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$.

In the present work, the concept of local renormalized solution is adapted to the anisotropic elliptic equation of the second order with variable growth exponents and locally integrable function $f$:

$$-\mathrm{div}\, \mathrm{a}(\mathrm{x}, \nabla u) + b(\mathrm{x}, u, \nabla u) = f, \quad \mathbb{R}^n. \tag{2}$$

Denote $C^+(\mathbb{R}^n) = \{p \in C(\mathbb{R}^n) \mid 1 < \overline{p} \leq \widehat{p} < +\infty\}$, where $\overline{p} = \inf\limits_{\mathrm{x}\in\mathbb{R}^n} p(\mathrm{x})$ and $\widehat{p} = \sup\limits_{\mathrm{x}\in\mathbb{R}^n} p(\mathrm{x})$. Denote $\overrightarrow{\mathrm{p}}(\cdot) = (p_1(\cdot), p_2(\cdot), \dots, p_n(\cdot)) \in (C^+(\mathbb{R}^n))^n$, $\overrightarrow{\mathbf{p}}(\cdot) = (p_0(\cdot), \overrightarrow{\mathrm{p}}(\cdot)) \in (C^+(\mathbb{R}^n))^{n+1}$ and

$$p_+(\mathrm{x}) = \max_{i=\overline{1,n}} p_i(\mathrm{x}), \quad p_-(\mathrm{x}) = \min_{i=\overline{1,n}} p_i(\mathrm{x}), \quad \mathrm{x} \in \mathbb{R}^n.$$

**Assumption** $P$. We assume that functions

$$\mathrm{a}(\mathrm{x},\mathrm{s}) = (a_1(\mathrm{x},\mathrm{s}), \dots, a_n(\mathrm{x},\mathrm{s})) : \mathbb{R}^{2n} \to \mathbb{R}^n, \quad b(\mathrm{x}, s_0, \mathrm{s}) : \mathbb{R}^{2n+1} \to \mathbb{R},$$

included in the equation (2) are Carathéodory functions. Assume that there exist nonnegative functions $\Phi_i \in L_{p_i'(\cdot),\mathrm{loc}}(\mathbb{R}^n)$ and positive numbers $\widehat{a}, \overline{a}$ such that for a.e. $\mathrm{x} \in \mathbb{R}^n$ and all $\mathrm{s}, \mathrm{t} \in \mathbb{R}^n$, the following inequalities hold:

$$|a_i(\mathrm{x},\mathrm{s})| \leq \widehat{a}\left(\mathrm{P}(\mathrm{x},\mathrm{s})^{1/p_i'(\mathrm{x})} + \Phi_i(\mathrm{x})\right), \quad i = 1, \dots, n;$$

$$(\mathrm{a}(\mathrm{x},\mathrm{s}) - \mathrm{a}(\mathrm{x},\mathrm{t})) \cdot (\mathrm{s} - \mathrm{t}) > 0, \quad \mathrm{s} \neq \mathrm{t};$$

$$\mathrm{a}(\mathrm{x},\mathrm{s}) \cdot \mathrm{s} \geq \overline{a}\mathrm{P}(\mathrm{x},\mathrm{s}).$$

Hereinafter, we use the notation $p_i'(\cdot) = p_i(\cdot)/(p_i(\cdot) - 1)$, $\mathrm{P}(\mathrm{x},\mathrm{s}) = \sum\limits_{i=1}^{n} |s_i|^{p_i(\mathrm{x})}$, $\mathrm{s}\cdot\mathrm{t} = \sum\limits_{i=1}^{n} s_i t_i$, $\mathrm{s} = (s_1,\dots,s_n)$, $\mathrm{t} = (t_1,\dots,t_n)$.

In addition, let there exist a nonnegative function $\Phi_0 \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$, a continuous nonnegative function $\widehat{b} : \mathbb{R}^+ \to \mathbb{R}^+$, and a positive number $\overline{b}$ such that for a.e. $\mathrm{x} \in \mathbb{R}^n$ and all $s_0 \in \mathbb{R}$, $\mathrm{s} \in \mathbb{R}^n$, the following inequalities hold:

$$|b(\mathrm{x}, s_0, \mathrm{s})| \le \widehat{b}(|s_0|)\,(\Phi_0(\mathrm{x}) + \mathrm{P}(\mathrm{x},\mathrm{s}))\,;$$

$$b(\mathrm{x}, s_0, \mathrm{s}) s_0 \ge \overline{b} |s_0|^{p_0(\mathrm{x})+1}, \quad p_+(\cdot) - 1 < p_0(\cdot).$$

Here we assume that

$$\underline{p}(\mathrm{x}) = n\left(\sum_{i=1}^{n} 1/p_i(\mathrm{x})\right)^{-1} < n, \quad \underline{p}^*(\mathrm{x}) = \frac{n\underline{p}(\mathrm{x})}{n - \underline{p}(\mathrm{x})}.$$

Denote $q_0(\cdot) = \underline{p}^*(\cdot)/\overline{p}_-'$, $\overline{p}_-' = \overline{p}_-/(\overline{p}_- - 1)$, let the following additional assumption be satisfied:

$$p_+(\cdot) - 1 < q_0(\cdot),$$

which is possible provided that $p_+(\cdot) < \underline{p}^*(\cdot)$.

In an anisotropic Sobolev space with variable exponents, the existence and regularity of a local renormalized solution of equation is established, and it is proved that the solution is sign-constant.

**Theorem 1.** *Let $f \in L_{1,\mathrm{loc}}(\mathbb{R}^n)$ and Assumption P be satisfied. Then there exists a local renormalized solution $u$ of the equation* (2). *If $f \ge 0$ for a.e. $\mathrm{x} \in \mathbb{R}^n$, then $u \ge 0$ for a.e. $\mathrm{x} \in \mathbb{R}^n$.*

In [BD07], conditions on the exponents $p_i(\cdot)$, $i = 0,\dots,n$, sufficient for the uniqueness of a local weak solution of the anisotropic equation (1) were found. For a local renormalized solution without additional restrictions on the growth of the solution at infinity, the uniqueness is not known.

[BD07] M. Bokalo and O. Domanska, *On well-posedness of boundary problems for elliptic equations in general anisotropic Lebesgue-Sobolev spaces*, Mat. Stud **28**:1 (2007), pp. 77–91.

[Bid03] M.F. Bidaut-Véron, *Removable singularities and existence for a quasilinear equation with absorption or source term and measure data*, Advanced Nonlinear Studies **3**:1 (2003), pp. 25–63.