Payne nodal set conjecture for the Riesz fractional p-Laplacian in Steiner symmetric domains

24.06
11:30-12:00

Sergey Kolonitskii

Saint Petersburg Electrotechnical University "LETI"

sbkolonitskii@etu.ru

Let $p \in (1, +\infty)$, $s \in (0, 1)$ and let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 1$. Consider the eigenvalue problem with Dirichlet boundary condition, i.e. the boundary value problem

$$\begin{cases} (-\Delta)_p^s u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{$\mathcal{D}$}$$

where $(-\Delta)_p^s$, $p > 1$, is the Riesz, or semi-restricted, fractional p-Laplacian defined for sufficiently regular functions as

$$(-\Delta)_p^s u(x) = \text{const} \cdot \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^N \setminus B(0,\varepsilon)} \frac{|u(y) - u(x)|^{p-2}(u(y) - u(x))}{|y - x|^{N+ps}} \, dy.$$

The second eigenfunction, defined per standard Lyusternik-Shnirelmann argument, is a sign-changing function. The Payne nodal set conjecture for Steiner symmetric domains asserts that the nodal set of any second eigenfunction intersects the boundary $\partial\Omega$. In the local settings with $s = 1$, the conjecture was proven in [Pay73] and [Dam00] in the linear case $p = 2$ and in [BK19] in the nonlinear case $p \in (1, +\infty)$. We extend and generalize these results to the nonlocal nonlinear case $s \in (0, 1)$ and $p \in (1, +\infty)$.

Theorem 1. *Assume that Ω is Steiner symmetric with respect to the hyperplane $H_0 := \{(x_1, \ldots, x_N) \in \mathbb{R}^N : x_1 = 0\}$. Let u be a second eigenfunction of ($\mathcal{D}$). Then*

$$\operatorname{dist}(\operatorname{supp} u^-, \partial\Omega) = 0 \quad \textit{and} \quad \operatorname{dist}(\operatorname{supp} u^+, \partial\Omega) = 0.$$

A similar result is obtained for least energy nodal solutions of the equation $(-\Delta)_p^s u = f(u)$ under nonlocal Dirichlet boundary conditions, where the model case of f being a subcritical and superlinear nonlinearity, i.e. $f(u) = |u|^{q-2} u$ with $q \in (p, p_s^*)$.

The proof is based on properties of polarization specific to the nonlocal case $s \in (0, 1)$. Most importantly, in a strong contrast to the local case, the polarization strictly decreases certain strong functionals associated with the Slobodetski seminorm of a given function unless the polarized function coincides with either the original function or its reflection. Curiously, we do not require any assumptions on smoothness of $\partial\Omega$, and even connectedness and boundedness of Ω can be weakened.

[BK19] V. Bobkov and S. Kolonitskii, *On a property of the nodal set of least energy sign-changing solutions for quasilinear elliptic equations*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **149**:5 (2019), pp. 1163–1173.

[Dam00] L. Damascelli, *On the nodal set of the second eigenfunction of the laplacian in symmetric domains in* $\mathbb{R}^N$, Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Matematica e Applicazioni **11**:3 (2000), pp. 175–181.

[Pay73] L.E. Payne, *On two conjectures in the fixed membrane eigenvalue problem*, Zeitschrift für angewandte Mathematik und Physik ZAMP **24** (1973), pp. 721–729.