Spectral and functional inequalities on antisymmetric functions

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We investigate so-called (d, N)-antisymmetric functions that appear in the area of interest as wavefunctions of fermions' systems. Here N and d are natural numbers and we consider $x = (x_1, \ldots, x_N) \in \mathbb{R}^{dN}$, where $x_i = (x_{i1}, \ldots, x_{id}) \in \mathbb{R}^d$ for all $1 \leqslant i \leqslant N$. Every function u defined on $\mathbb{R}^{dN}$ we call *antisymmetric*, if for all $1 \leqslant i, j \leqslant N$ and $x_1, \ldots, x_N \in \mathbb{R}^d$

$$u(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_N) = -u(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_N).$$

As we expected, the constants in classical inequalities are much better for antisymmetric functions. Indeed, one of our main results in [HLS23] is Hardy inequality for antisymmetric functions. It claims that for any $u \in \mathcal{H}^1_A(\mathbb{R}^{dN})$ we have

$$\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 \, dx \geqslant H_A(dN) \int_{\mathbb{R}^{dN}} \frac{|u(x)|^2}{|x|^2} \, dx \tag{1}$$

The constant in (1) is sharp and $H_A(dN) \sim N^{2+2/d}$ unlike classical N^2.

Similarly, we improve Sobolev inequality restricted to antisymmetric functions. For every $u \in \mathcal{H}^1_A(\mathbb{R}^{dN})$, $dN \geqslant 3$ the following inequality holds

$$\int_{\mathbb{R}^{dN}} |\nabla u(x)|^2 dx \geqslant S_A(dN) \left(\int_{\mathbb{R}^{dN}} |u(x)|^{\frac{2dN}{dN-2}} dx \right)^{\frac{dN-2}{dN}}.$$

The constant $S_A(dN)$ is sharp and is $(N!)^{\frac{2}{dN}}$ times bigger than the classical constant $S(dN)$.

The spectrum of Laplace-Beltrami operator restricted to antisymmetric function is fully investigated in [Shc24]. The main theorem claims that for $d = 1$ the eigenvalues of the Laplace-Beltrami operator $-\Delta_\theta$ on antisymmetric functions on sphere equals

$$\lambda_l = \left(l + \frac{N(N-1)}{2}\right)\left(l + \frac{N(N+1)}{2} - 2\right),$$

whose multiplicity κ_l satisfies the inequality

$$\frac{2}{N!(N-2)!}\ell^{N-2} \leqslant \kappa_\ell \leqslant \frac{2}{N!(N-2)!}\left(\ell + \frac{N(N+1)}{2}\right)^{N-2}.$$

Spectral inequalities for Schrödinger operator are presented as Cwikel-Lieb-Rozenblum and Lieb-Thirring inequalities in [LS25]. Let $d = 1$ and $V \geqslant 0$ be a symmetric potential such that $V \in L^{\gamma+N/2}(\mathbb{R}^N)$. Then for any γ satisfying the conditions

$$\begin{cases} \gamma \geqslant 1/2 & \text{if } N = 1, \\ \gamma > 0 & \text{if } N = 2, \\ \gamma \geqslant 0 & \text{if } N \geqslant 3. \end{cases}$$

we have

$$\operatorname{Tr}(-\Delta - V)_-^\gamma \leqslant \frac{L_{\gamma,N}}{N!} \int\limits_{\mathbb{R}^N} V^{\gamma+N/2}\, dx,$$

where $L_{\gamma,N}$ is classical LT constant.

[HLS23] T. Hoffmann-Ostenhof, A. Laptev, and I. Shcherbakov, *Hardy and Sobolev inequalities on antisymmetric functions*, Bull. Math. Sci. (2023), p. 2350010.

[LS25] A. Laptev and I. Shcherbakov, *Spectral and functional inequalities on antisymmetric functions*, Ufa Math. Journal **17**:1 (2025), pp. 142–153.

[Shc24] I. Shcherbakov, *Spectrum of the Laplace–Beltrami operator on antisymmetric functions*, Springer, Journal of Mathematical Sciences **279**:1-3 (2024), pp. 563–572.