Approximate sampling recovery

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Logic of development

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3. Smoothness classes and linear recovery.
4. Classes with structural condition and nonlinear recovery.

Trigonometric polynomials. Dirichlet kernel.

Functions of the form

$$t(x) = \sum_{|k|\le n} c_k e^{ikx} = a_0/2 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx)$$

are called trigonometric polynomials of order n. The set of such polynomials we denote by $\mathcal{T}(n)$.

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The Dirichlet kernel of order n

$$\mathcal{D}_n(x) := \sum_{|k|\le n} e^{ikx} = e^{-inx}(e^{i(2n+1)x} - 1)(e^{ix} - 1)^{-1}$$

$$= \left(\sin(n+1/2)x\right) / \sin(x/2).$$

Interpolation

Denote

$$x^j := 2\pi j/(2n+1), \qquad j = 0, 1, ..., 2n.$$

Clearly, the points x^j, $j = 1, \dots, 2n$, are zeros of the Dirichlet kernel $\mathcal{D}_n$ on $[0, 2\pi]$.

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Clearly, the points x^j, $j = 1, \dots, 2n$, are zeros of the Dirichlet kernel $\mathcal{D}_n$ on $[0, 2\pi]$. Consequently, for any continuous f

$$I_n(f)(x) := (2n+1)^{-1} \sum_{j=0}^{2n} f(x^j)\mathcal{D}_n(x - x^j)$$

interpolates f at points x^j: $I_n(f)(x^j) = f(x^j)$, $j = 0, 1, ..., 2n$.

Error of interpolation

It is easy to check that for any $t \in \mathcal{T}(n)$ we have $I_n(t) = t$. Using this and the inequality

$$\big|\mathcal{D}_n(x)\big| \le \min\big(2n+1, \pi/|x|\big), \qquad |x| \le \pi,$$

we obtain

$$\|f - I_n(f)\|_\infty \le C \ln(n+1) E_n(f)_\infty,$$

where $E_n(f)_p$ is the best approximation of f in the L_p norm by polynomials from $\mathcal{T}(n)$.

The de la Vallée Poussin kernels

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Consider the following recovery operator

$$R_n(f) := (4n)^{-1} \sum_{j=1}^{4n} f\big(x(j)\big)\mathcal{V}_n\big(x - x(j)\big), \qquad x(j) := \pi j/(2n).$$

Properties of $R_n(f)$

It is easy to check that for any $t \in \mathcal{T}(n)$ we have $R_n(t) = t$. Using this and the above majorant we obtain

$$\|f - R_n(f)\|_\infty \leq CE_n(f)_\infty.$$

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What about error in the L_p, $p \in [1, \infty)$? Let $\varepsilon := \{\epsilon_k\}_{k=0}^\infty$ be a non-increasing sequence of non-negative numbers. Define

$$E(\varepsilon, p) := \{f \in \mathcal{C} : \; E_k(f)_p \leq \epsilon_k, \; k = 0, 1, \dots\}.$$

Error of recovery

Theorem (VT, 1985)

Assume that a sequence ε satisfies the conditions: for all $s = 0, 1, \dots$ we have

$$\sum_{\nu=s+1}^{\infty} \epsilon_{2^\nu} \le B\epsilon_{2^s}, \qquad \epsilon_s \le D\epsilon_{2s}.$$

Then for $p \in [1, \infty)$

$$\sup_{f \in E(\varepsilon,p)} \|f - R_n(f)\|_p \asymp \sum_{\nu=0}^{\infty} 2^{\nu/p} \epsilon_{n2^\nu}.$$

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This theorem for $1 \le p \le 2$ was proved in VT, 1985. A similar proof works for other p.

Linear optimal recovery

For a fixed m and a set of points $\xi := \{\xi^j\}_{j=1}^m \subset \Omega$, let Φ_ξ be a linear operator from $\mathbb{C}^m$ into $L_p(\Omega,\mu)$. Denote for a class $\mathbf{F}$ (usually, centrally symmetric and compact subset of $L_p(\Omega,\mu)$)

$$\varrho_m(\mathbf{F}, L_p) := \inf_{\text{linear}\, \Phi_\xi;\, \xi} \sup_{f\in\mathbf{F}} \|f - \Phi_\xi(f(\xi^1),\dots,f(\xi^m))\|_p.$$

The above described recovery procedure is a linear procedure. The characteristic $\varrho_m(\mathbf{F}, L_p)$ was introduced in VT (1993).

Define

$$W^r_q := \{f \,:\, f = J_r(\varphi),\ \|\varphi\|_q \le 1\},$$

where

$$J_r(f)(x) := (2\pi)^{-1} \int_{\mathbb{T}} f(x-y) F_r(y) dy,$$

$$F_r(y) := 1 + \sum_{k=1}^{\infty} k^{-r} \cos(ky - r\pi/2).$$

Univariate smoothness classes

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Theorem (VT, 1993)

Let $1 \le q, p \le \infty$ *and* $r > 1/q$. *Then*

$$\varrho_{4m}(W^r_q, L_p) \asymp \sup_{f\in W^r_q} \|f - R_m(f)\|_p \asymp m^{-r+(1/q-1/p)_+}.$$

In the case $1 < p < \infty$ *the above estimates are valid for the operator* I_m *instead of the operator* R_m.

Multivariate case. Classes

For $\mathbf{r} = (r_1, \dots, r_d) \in \mathbb{R}^d_+)$ define

$$J_{\mathbf{r}}(f)(\mathbf{x}) := (2\pi)^{-d} \int_{\mathbb{T}^d} f(\mathbf{x} - \mathbf{y}) F_{\mathbf{r}}(\mathbf{y}) d\mathbf{y},$$

$$F_{\mathbf{r}}(y) := \prod_{j=1}^{d} F_{r_j}(y_j)$$

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and

$$\mathbf{W}^{\mathbf{r}}_q := \{f \,:\, f = J_{\mathbf{r}}(\varphi),\ \|\varphi\|_q \le 1\}.$$

Recovery operators

Let for $i = 1, \dots, d$ operator R_n^i be the operator R_n acting with respect to the variable x_i. Denote

$$\Delta_s^i := R_{2^s}^i - R_{2^{s-1}}^i, \quad R_{1/2} = 0,$$

and for $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}_0^d$

$$\Delta_{\mathbf{s}} := \prod_{i=1}^{d} \Delta_{s_i}^i.$$

Consider the recovery operator (Smolyak operator)

$$T_n := \sum_{\mathbf{s}:\|\mathbf{s}\|_1 \le n} \Delta_{\mathbf{s}}.$$

Operator T_n uses m function values with $m \ll \sum_{k=1}^{n} 2^k k^{d-1} \ll 2^n n^{d-1}$.

The following bound was obtained by S. Smolyak in 1960. Let $\mathbf{r} = (r, \dots, r)$. In this case write $\mathbf{W}^{\mathbf{r}}_q = \mathbf{W}^r_q$. Then

$$\sup_{f \in \mathbf{W}^r_\infty} \|f - T_n\|_\infty \ll 2^{-rn} n^{d-1}, \quad r > 0.$$

It was extended to the case $p < \infty$ in VT, (1985):

$$\sup_{f \in \mathbf{W}^r_p} \|f - T_n\|_p \ll 2^{-rn} n^{d-1}, \quad r > 1/p.$$

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Open problem. Find the right order of the optimal sampling recovery $\varrho_m(\mathbf{W}^r_p, L_p)$ in case $1 \le p \le \infty, p \neq 2$, and $r > 1/p$.

Further results

We have (VT, (1993))

$$\varrho_m(\mathbf{W}^r_2)_\infty \asymp m^{-r+1/2}(\log m)^{r(d-1)}, \quad r > 1/2.$$

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$$\varrho_m(\mathbf{W}^r_2)_\infty \asymp m^{-r+1/2}(\log m)^{r(d-1)}, \quad r > 1/2.$$

The order of optimal recovery is provided by the Smolyak operator T_n. Also we know (VT, (1993))

$$\sup_{f\in\mathbf{W}^r_q} \|f - T_n(f)\|_\infty \asymp 2^{-(r-1/q)n} n^{(d-1)(1-1/q)}.$$

Kolmogorov width

It is natural to compare quantities $\varrho_m(\mathbf{F}, L_p)$ and $\varrho_m^*(\mathbf{F}, L_p)$ with the Kolmogorov widths. Let $\mathbf{F} \subset L_p$ be a centrally symmetric compact. The quantities

$$d_n(\mathbf{F}, L_p) := \inf_{\{u_i\}_{i=1}^n \subset L_p} \sup_{f \in \mathbf{F}} \inf_{c_i} \left\| f - \sum_{i=1}^{n} c_i u_i \right\|_p, \quad n = 1, 2, \dots,$$

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$$d_m(\mathbf{F},L_p)\le \varrho_m(\mathbf{F},L_p).$$

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We have the following obvious inequalities

$$d_m(\mathbf{F}, L_p) \le \varrho_m(\mathbf{F}, L_p).$$

We consider the case $p = 2$, i.e. recovery takes place in the Hilbert space L_2.

Inequality

Theorem (VT, 2020)

There exist two positive absolute constants b and B such that for any compact subset Ω of $\mathbb{R}^d$, any probability measure μ on it, and any compact subset $\mathbf{F}$ of $\mathcal{C}(\Omega)$ we have

$$\varrho_{bn}(\mathbf{F}, L_2(\Omega,\mu)) \le B d_n(\mathbf{F}, L_\infty).$$

Further results I

The following generalization of the above theorem to the case $2 < p \le \infty$ was obtained by D. Krieg, K. Pozharska, M. Ullrich, and T. Ullrich (2024).

Theorem (KPUU, 2024)

Let $2 \le p \le \infty$. There exists a positive absolute constant C such that for any compact subset Ω of $\mathbb{R}^d$, any probability measure μ on it, and any compact subset $\mathbf{F}$ of $\mathcal{C}(\Omega)$ we have

$$\varrho_{4n}(\mathbf{F}, L_p(\Omega,\mu)) \le Cn^{1/2-1/p} d_n(\mathbf{F}, L_\infty). \quad (1)$$

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$$\varrho_{4n}(\mathbf{F}, L_p(\Omega,\mu)) \le C n^{1/2-1/p} d_n(\mathbf{F}, L_\infty). \quad (1)$$

Note that we have $d_n(\mathbf{F}, L_\infty)$ in the right side of the two above inequalities, which is larger than $d_n(\mathbf{F}, L_2)$. However, it is known that for many function classes we have $d_n(\mathbf{F}, L_\infty) \asymp d_n(\mathbf{F}, L_2)$.

Further results II

For special sets $\mathbf{F}$ (in the reproducing kernel Hilbert space setting) the following inequality was proved (see M. Dolbeault , D. Krieg, and M. Ullrich (2023); N. Nagel, M. Schäfer, T. Ullrich (2020); D. Krieg and M. Ullrich (2020)):

$$\varrho_{cn}(\mathbf{F}, L_2) \le \left(\frac{1}{n} \sum_{k \ge n} d_k(\mathbf{F}, L_2)^2 \right)^{1/2} \tag{2}$$

with an absolute constant $c > 0$. Here, $d_k(\mathbf{F}, L_2)$ is the Kolmogorov width of $\mathbf{F}$ in the space L_2.

Further results III

Known results on the $d_k(\mathbf{W}^r_p, L_2)$: For $1 < q \le 2$, $r > 1/q - 1/2$ and $2 < q < \infty$, $r > 0$

$$d_k(\mathbf{W}^r_q, L_2) \asymp \left(\frac{(\log k)^{d-1}}{k}\right)^{r-(1/q-1/2)_+}, \quad (a)_+ := \max(a,0), \tag{3}$$

combined with the (2) give for $1 < q \le \infty$, $r > \max(1/q, 1/2)$ the following bounds

$$\varrho_m(\mathbf{W}^r_q, L_2) \ll \left(\frac{(\log m)^{d-1}}{m}\right)^{r-(1/q-1/2)_+},$$

which gives the right orders of decay of the sequences $\varrho_m(\mathbf{W}^r_q, L_2)$ in the case $1 < q < \infty$ and $r > \max(1/q, 1/2)$ because, obviously, $\varrho_m(\mathbf{W}^r_q, L_2) \ge d_m(\mathbf{W}^r_q, L_2)$.

Define the optimal sampling recovery on specific classes of multivariate functions on a domain Ω. Precisely, for a function class $\mathbf{F} \subset \mathcal{C}(\Omega)$, we consider the asymptotics of the following characteristics,

$$\varrho_m^o(\mathbf{F}, L_p) := \inf_{\xi} \inf_{\mathcal{M}} \sup_{f \in \mathbf{F}} \|f - \mathcal{M}(f(\xi^1), \dots, f(\xi^m))\|_p ,$$

where $\mathcal{M}$ ranges over all mappings $\mathcal{M} : \mathbb{C}^m \to L_p(\Omega, \mu)$ and ξ ranges over all subsets $\{\xi^1, \cdots, \xi^m\}$ of m points in Ω. The value $\varrho_m^o(\mathbf{F}, L_p)$ shows the optimal value of recovery of $f \in \mathbf{F}$ from a sampling set ξ, and we use the upper index o to mean optimality. The characteristic $\varrho_m^o(\mathbf{F}, L_p)$ was introduced in Traub, Wasilkowski, and Woźniakowski (1988).

Function classes

Our specific choice for $\mathbf{F}$ is the following collection of classes introduced in VT (2024). Given a system $\Psi = (\psi_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$, consider a sequence of subsets $\mathcal{G} := \{G_j\}_{j=1}^{\infty}$, $G_j \subset \mathbb{Z}^d$, such that

$$G_1 \subset G_2 \subset \cdots \subset G_j \subset G_{j+1} \subset \cdots, \quad \bigcup_{j=1}^{\infty} G_j = \mathbb{Z}^d, \quad G_0 := \emptyset, \tag{4}$$

and the functions representable in the form of absolutely convergent series

$$f = \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}(f)\psi_{\mathbf{k}}, \qquad \sum_{\mathbf{k}\in\mathbb{Z}^d} |a_{\mathbf{k}}(f)| < \infty. \tag{5}$$

Function classes continue

For $\beta \in (0,1]$, $r > 0$, and $b \in \mathbb{R}$ define the class $\mathbf{A}^{r,b}_{\beta}(\Psi, \mathcal{G})$ as the functions f which have representations (5) satisfying the following conditions

$$\Big(\sum_{\mathbf{k} \in G_j \setminus G_{j-1}} |a_{\mathbf{k}}(f)|^{\beta} \Big)^{1/\beta} \le 2^{-rj} j^{b}, \quad j = 1, 2, \dots . \tag{6}$$

Two special cases of these classes have already been studied. Firstly, in VT (2023, 2024) the behavior of $\varrho_m^o(\mathbf{F}, L_p)$ for the classes $\mathbf{A}^r_\beta(\Psi) := \mathbf{A}^{r,0}_\beta(\Psi, \mathcal{G}^c)$ was studied, with $\mathcal{G}^c$ being dyadic cubes,

$$G_j^c := \{\mathbf{k} : \|\mathbf{k}\|_\infty < 2^j\}, \quad j \in \mathbb{N}.$$

History II

The second special case, for the classes $\mathbf{W}^{a,b}_{A_\beta}(\Psi) := \mathbf{A}^{a,b}_\beta(\Psi, \mathcal{G}^h)$ with $\mathcal{G}^h$ being dyadic hyperbolic crosses, was studied in Solodov and Temlyakov (2025). In that case, with $\mathbf{s} = (s_1, \dots, s_d) \in \mathbb{N}^d$, the sets $\mathcal{G}^h$ of hyperbolic crosses are defined as follows

$$\rho(\mathbf{s}) := \left\{\mathbf{k} \in \mathbb{Z}^d : [2^{s_j-1}] \le |k_j| < 2^{s_j}, \quad j = 1, \dots, d\right\},$$

$$G^h_j := \bigcup_{\|\mathbf{s}\|_1 \le j} \rho(\mathbf{s}).$$

Regularity condition RC(θ, θ'). We say that a system $\mathcal{G}$ satisfies regularity condition **RC(θ, θ')** if there exist positive numbers $\theta > \theta' > 0$ such that for any $j \in \mathbb{N}$ we have

$$2^{\theta'} \leq \frac{|G_{j+1}|}{|G_j|} \leq 2^{\theta}, \quad j \in \mathbb{N}, \qquad |G_0| = 1. \tag{7}$$

This condition means that cardinalities of the index sets $\{G_j\}_{j\in\mathbb{N}}$ grow exponentially, with a possibly varying rate which is however bounded from below and from above.

Main result

The following result from Shadrin, Temlyakov, and Tikhonov (2025) gives the rate of the optimal sampling recovery for these classes.

Theorem

Let $2 \le p < \infty$ and let m, v be given natural numbers.
1) Let Ψ be a uniformly bounded Bessel system.
2) Assume in addition that the system $\mathcal{G}$ satisfies condition **RC**(θ, θ').
Then there exist constants $c = c(r, \beta, p, K, \theta, \theta')$ and $C = C(r, \beta, p, \theta, \theta')$ such that we have the bound

$$\varrho_m^o(\mathbf{A}_\beta^{r,b}(\Psi, \mathcal{G}), L_p(\Omega, \mu)) \le C v^{1-1/p-1/\beta} 2^{-rn(v)} n(v)^b \qquad (8)$$

for any m satisfying

$$m \ge cv(\log(2v))^4.$$

Remark

Note, that under a stronger restriction on the system Ψ, when we require that Ψ is a uniformly bounded Riesz system, we can somewhat improve the bound for m to $m \geq cv(\log(2v))^3$ and guarantee that the estimate (8) is provided by a greedy-type algorithm.

Thank you!