On the geometry of WDVV equations and their Hamiltonian formalism in arbitrary dimension

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Contents

Witten–Dijkgraaf–Verlinde–Verlinde equations

(Dubrovin, Encyclopaedia of Math. Phys. 2006)
The problem: in $\mathbb{R}^N$ find a function $F = F(t^1, \ldots, t^N)$ such that

1. $F_{1\alpha\beta} := \dfrac{\partial^3 F}{\partial t^1 \partial t^\alpha \partial t^\beta} = \eta_{\alpha\beta}$ constant symmetric nondegenerate matrix
2. $c_{\alpha\beta}^{\gamma} = \eta^{\gamma\epsilon} F_{\epsilon\alpha\beta}$ structure constants of an associative algebra
3. $F(c^{d_1} t^1, \ldots, c^{d_N} t^N) = c^{d_F} F(t^1, \ldots, t^N)$ quasihomogeneity $(d_1 = 1)$

If $e_1, \ldots, e_N$ is the basis of $\mathbb{R}^N$ then the algebra operation is

$$e_\alpha \cdot e_\beta = c_{\alpha\beta}^{\gamma}(\mathbf{t}) e_\gamma \quad \text{with unity} \quad e_1$$

Why study WDVV?

1. The origin: Topological Quantum Field Theory
   - E. Witten. On the structure of the topological phase of two-dimensional gravity. *Nuclear Physics B*, 340(2):281–332, 1990, DOI: `https://doi.org/10.1016/0550-3213(90)90449-N`.
   - R. Dijkgraaf, H. Verlinde, and E. Verlinde. Topological strings in $d < 1$. *Nuclear Physics B*, 352(1):59–86, 1991, DOI: `https://doi.org/10.1016/0550-3213(91)90129-L`.
2. Mathematical developments: solutions of WDVV are related with Gromov–Witten invariants
   - M. Kontsevich, Yu. Manin. Gromov-Witten Classes, Quantum Cohomology, and Enumerative Geometry, Commun. Math. Phys. 164, 525-562 (1994).
   - I. Strachan. How to count curves: from 19th century problems to 21st century solutions. `https://www.maths.gla.ac.uk/~iabs/Visions.pdf`

Why study WDVV?

3. Solutions correspond to integrable hierarchies (B. Dubrovin)
   - B.A. Dubrovin. Geometry of 2D topological field theories. In *Integrable systems and quantum groups*, volume 1620 of *Lect. Notes Math.*, pages 120–348. Springer, Berlin, Heidelberg, 1996, arXiv: `https://arxiv.org/abs/hep-th/9407018`.
   - B. Dubrovin. Flat pencils of metrics and Frobenius manifolds. In M.-H. Saito, Y. Shimizu, and K. Ueno, editors, *Proceedings of 1997 Taniguchi Symposium "Integrable Systems and Algebraic Geometry"*, pages 42–72. World Scientific, 1998. `https://people.sissa.it/~dubrovin/bd_papers.html`.
   - B.A. Dubrovin. *Encyclopedia of Mathematical Physics*, volume 1 A: A-C, chapter WDVV equations and Frobenius manifolds, pages p. 438–447. SISSA, Elsevier, 2006. ISBN: 0125126611.

Recent research on WDVV

1. Supersymmetric Quantum Mechanics:
   - G. Antoniou and M. Feigin. Supersymmetric V-systems. *Journal of High Energy Physics*, 2019(2):115, Feb 2019, DOI: `https://doi.org/10.1007/JHEP02(2019)115`.
   - A. Galajinsky and O. Lechtenfeld. Superconformal $su(1,1|n)$ mechanics. *Journal of High Energy Physics*, 2016(9):114, Sep 2016, DOI: `https://doi.org/10.1007/JHEP09(2016)114`.
   - N. Kozyrev, S. Krivonos, O. Lechtenfeld, and A. Sutulin. $Su(2|1)$ supersymmetric mechanics on curved spaces. *Journal of High Energy Physics*, 2018(5):175, May 2018, DOI: `https://doi.org/10.1007/JHEP05(2018)175`.
2. Topological quantum field theory
   - O.B. Gomez and A. Buryak. Open topological recursion relations in genus 1 and integrable systems. *Journal of High Energy Physics*, 2021(1):48, Jan 2021, DOI: `https://doi.org/10.1007/JHEP01(2021)048`.

Recent research on WDVV

3. String theory
    - A.A. Belavin and V.A. Belavin. Frobenius manifolds, integrable hierarchies and minimal Liouville gravity. *Journal of High Energy Physics*, 2014(9):151, Sep 2014, DOI: `https://doi.org/10.1007/JHEP09(2014)151`.
    - Xiang-Mao Ding, , Yuping Li, and Lingxian Meng. From r-spin intersection numbers to hodge integrals. *Journal of High Energy Physics*, 2016(1):15, Jan 2016, DOI: `https://doi.org/10.1007/JHEP01(2016)015`.
4. Supersymmetric gauge theory
    - H. Jockers and P. Mayr. Quantum K-theory of Calabi–Yau manifolds. *Journal of High Energy Physics*, 2019(11):11, Nov 2019, DOI: `https://doi.org/10.1007/JHEP11(2019)011`.

Recent research on WDVV

5. Topological Recursion
   - B. Eynard, Topological expansion for the 1-hermitian matrix model correlation functions, JHEP/024A/0904, hep-th/0407261
   - L. Chekhov, B. Eynard, N. Orantin, Free energy topological expansion for the 2-matrix model, JHEP 0612 (2006) 053, math-ph/0603003
   - B. Eynard, A short overview of the "Topological recursion", math-ph/arXiv:1412.3286.
6. Superintegrable systems in Classical Mechanics
   - Vollmer, A. Manifolds with a Commutative and Associative Product Structure that Encodes Superintegrable Hamiltonian Systems. Ann. Henri Poincaré (2025). `https://doi.org/10.1007/s00023-025-01608-5`

WDVV equations

The system of associativity equations, also known as WDVV equations, follows:

$$\begin{aligned} S_{\alpha\beta\gamma\nu} =& \eta^{\mu\lambda}\Big(\frac{\partial^3 F}{\partial t^\lambda \partial t^\alpha \partial t^\beta}\frac{\partial^3 F}{\partial t^\nu \partial t^\mu \partial t^\gamma} - \frac{\partial^3 F}{\partial t^\nu \partial t^\alpha \partial t^\mu}\frac{\partial^3 F}{\partial t^\lambda \partial t^\beta \partial t^\gamma}\Big) \\ =& \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\gamma\nu} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0. \end{aligned}$$

The dependence of the function F on t^1 is 'completely' specified by the requirement $F_{1\alpha\beta} = \eta_{\alpha\beta}$:

$$F = \frac{1}{6}\eta_{11}(t^1)^3 + \frac{1}{2}\sum_{k>1}\eta_{1k}t^k(t^1)^2 + \frac{1}{2}\sum_{k,s>1}\eta_{sk}t^s t^k t^1 + f(t^2, \dots, t^N).$$

so that the WDVV system is an overdetermined system of non-linear PDEs on one unknown function $f = f(t^2, \dots, t^N)$.

Digression: Hamiltonian PDEs

An evolutionary system of PDEs

$$F = u^i_t - f^i(t, x, u^j, u^j_x, u^j_{xx}, \dots) = 0$$

admits a Hamiltonian formulation if there exist A, $\mathcal{H} = \int h\, dx$ such that

$$u^i_t = A^{ij} \left(\frac{\delta \mathcal{H}}{\delta u^j} \right), \quad \text{with} \quad \frac{\delta \mathcal{H}}{\delta u^j} = (-1)^\sigma \partial_\sigma \frac{\partial h}{\partial u^j_\sigma}$$

where $A = (A^{ij})$ is a Hamiltonian operator (Poisson tensor), i.e. a matrix of differential operators $A^{ij} = A^{ij\sigma}\partial_\sigma$, where $\partial_\sigma = \partial_x \circ \cdots \circ \partial_x$ (total x-derivatives σ times), which define a Poisson bracket between functionals.

Solution of WDVV equations and Hamiltonian PDEs

(B. Dubrovin, '90) Let F be a solution of WDVV equations with homogeneity degrees $d_1,\dots d_N$. Let us set

$$c^{\delta\gamma}_{\beta} = \eta^{\delta\alpha}\eta^{\gamma\epsilon}\frac{\partial^3 F}{\partial t^{\epsilon}\partial t^{\alpha}\partial t^{\beta}}.$$

Then, the two operators

$$A_1 = \eta^{ij}\partial_x, \qquad A_2 = g^{ij}_1\partial_x + \Gamma^{ij}_k u^k_x$$

where, after replacing $t^k \to u^k$:

$$g^{ij}_1 = c^{ij}_k d_k u^k$$

are Hamiltonian and compatible: $A_1 + \lambda A_2$ is Hamiltonian for every $\lambda \in \mathbb{R}$, hence they define an integrable system of PDEs of the form $u^i_t = V^i_j u^j_x$.

Solution of WDVV equations and enumerative geometry

Let $N = 3$ and

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad \text{Then:} \quad S_{2233} = f_{yyy} - f_{xxy}^2 + f_{xxx} f_{xyy} = 0.$$

Then, the number of algebraic curves of degree n passing through $3n - 1$ generic points in the projective plane $\mathbb{P}^2$:

$$N(n) = \sum_{i+j=n} \left(i^2 j^2 \binom{3n-4}{3i-2} - i^3 j \binom{3n-4}{3i-1} \right) N(i) N(j)$$

is obtained from a solution of WDVV equation for the Frobenius potential F (free energy) for the projective plane.

Simplest general example: WDVV in the case $N = 3$

If $N = 3$ we have a single equation on $f = f(t^2, t^3) = f(x, y)$: $S_{2233} = 0$. Here $\eta = (\eta_{ij})$ is arbitrary. Explicitly,

$$\begin{gathered}(f_{xyy}\eta^{22} + f_{yyy}\eta^{23} + \eta^{12}\eta_{33})f_{xxx} - f_{xxy}^2\eta^{22} \\ +(-f_{xyy}\eta^{23} + f_{yyy}\eta^{33} - 2\eta^{12}\eta_{23} + \eta^{13}\eta_{33})f_{xxy} - f_{xyy}^2\eta^{33} \\ +(\eta^{12}\eta_{22} - 2\eta^{13}\eta_{23})f_{xyy} + \eta_{22}\eta^{13}f_{yyy} + \eta^{11}(\eta_{22}\eta_{33} - \eta_{23}^2) = 0.\end{gathered}$$

Introducing $u^1 = f_{xxx}$, $u^2 = f_{xxy}$, $u^3 = f_{xyy}$ we have the compatibility conditions

$$u_y^1 = u_x^2, \quad u_y^2 = u_x^3, \quad u_y^3 = (v^3)_x$$

Simplest example: WDVV in the case $N = 3$

$$v^3 = f_{yyy} = \frac{N}{\eta_{22}\eta^{13} + \eta^{23}u^1 + \eta^{33}u^2}$$

$$N = -\eta_{22}\eta_{33}\eta^{11} + \eta_{23}^2\eta^{11} + (2\eta_{23}\eta^{12} - \eta_{33}\eta^{13})u^2$$
$$+(2\eta_{23}\eta^{13} - \eta_{22}\eta^{12})u^3 - \eta_{33}\eta^{12}u^1$$
$$+\eta^{22}(u^2)^2 + \eta^{23}u^2u^3 + \eta^{33}(u^3)^2 - \eta^{22}u^1u^3$$

Solutions of one form of the ‘scalar’ WDVV equation are in ‘bijection’ with solutions of the WDVV ‘system’ of first-order conservation laws.

Higher-order homogeneous Hamiltonian operators

Higher order homogeneous operators were introduced in 1984 by Dubrovin and Novikov. We can consider the second-order and third-order homogeneous operators:

$$\begin{aligned} A_2^{ij} =& g_2^{ij}(\mathbf{u})\partial_x^2 + b_{2\,k}^{ij}(\mathbf{u})u_x^k\partial_x \\ & + c_{2\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{2\,km}^{ij}(\mathbf{u})u_x^k u_x^m, \end{aligned}$$

$$\begin{aligned} A_3^{ij} =& g_3^{ij}(\mathbf{u})\partial_x^3 + b_{3\,k}^{ij}(\mathbf{u})u_x^k\partial_x^2 \\ & + [c_{3\,k}^{ij}(\mathbf{u})u_{xx}^k + c_{3\,km}^{ij}(\mathbf{u})u_x^k u_x^m]\partial_x \\ & + d_{3\,k}^{ij}(\mathbf{u})u_{xxx}^k + d_{3\,km}^{ij}(\mathbf{u})u_x^k u_{xx}^m + d_{3\,kmn}^{ij}(\mathbf{u})u_x^k u_x^m u_x^n. \end{aligned}$$

In canonical form:

$$\begin{aligned} A_2^{ij} =& \partial_x \circ g_2^{ij} \circ \partial_x, \\ A_3^{ij} =& \partial_x \circ (g_3^{ij}\partial_x + c_{3\,k}^{ij}u_x^k) \circ \partial_x, \end{aligned}$$

bi-Hamiltonian structure of WDVV equations

Theorem. For $N = 3$ and for arbitrary η the WDVV system admits a bi-Hamiltonian structure (Vašíček, V. JHEP 2021). For the previous 'simple' η:

$$A_1 = \begin{pmatrix} -\frac{3}{2}\partial_x & \frac{1}{2}\partial_x a & \partial_x b \\ \frac{1}{2}a\partial_x & \frac{1}{2}(\partial_x b + b\partial_x) & \frac{3}{2}c\partial_x + c_x \\ b\partial_x & \frac{3}{2}\partial_x c - c_x & (b^2 - ac)\partial_x + \partial_x(b^2 - ac) \end{pmatrix},$$

$$A_3 = \partial_x \begin{pmatrix} 0 & 0 & \partial_x \\ 0 & \partial_x & -\partial_x a \\ \partial_x & -a\partial_x & (\partial_x b + b\partial_x + a\partial_x a) \end{pmatrix} \partial_x$$

A_1 and A_3 are completely determined by their leading coefficients:

$$g_1^{ij} = \begin{pmatrix} -3/2 & 1/2\, a & b \\ 1/2\, a & b & 3/2\, c \\ b & 3/2\, c & 2(b^2 - ac) \end{pmatrix}, \quad g_3^{ij} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -a \\ 1 & -a & 2b + a^2 \end{pmatrix}$$

Projective invariance of bi-Hamiltonian pair

Theorem Reciprocal transformations of projective type

$$d\tilde{x} = \Delta dx, \quad \tilde{u}^i = T^i(u^j) = (A^i_j u^j + A^i_0)/\Delta$$

with $\Delta = c_i u^i + c_0$ preserve the canonical form of third-order homogeneous operators A_3 (Ferapontov, Pavlov, V. JGP 2014). The leading terms are transformed as

$$g_{3\,ij} \to \frac{\tilde{g}_{3\,ij}}{\Delta^4}$$

where $\tilde{g}_{3\,ij}$ is of the same type as the initial metric; g_3 is identified with a quadratic line complex.

For A_1, we have an indirect proof and a limited understanding.

Digression: Plücker's line geometry

Two infinitesimally close points $V, V + dV \in \mathbb{P}(\mathbb{C}^{n+1})$,

$$V = [v^1, \ldots, v^{n+1}], \quad V + dV = [v^1 + dv^1, \ldots, v^{n+1} + dv^{n+1}]$$

define a line with coordinates

$$p^{\lambda\mu} = v^\lambda dv^\mu - v^\mu dv^\lambda = \det \begin{pmatrix} v^\lambda & v^\mu \\ v^\lambda + dv^\lambda & v^\mu + dv^\mu \end{pmatrix}$$

inside the projective space: $\mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$ (S. Lie coordinates for Plücker embedding).
We regard (u^i), $i = 1, \ldots, n$ as an affine chart on $\mathbb{P}(\mathbb{C}^{n+1})$, so that $u^{n+1} = 1$, $du^{n+1} = 0$ and

$$p^{ij} = u^i du^j - u^j du^i, \quad p^{(n+1)i} = du^i.$$

The algebraic variety of A_3

(Ferapontov, Pavlov, V., JGP 2014, IMRN 2016) The third-order operator A_3 fulfills the condition:

$$\partial_i(g_3)_{jk} + \partial_k(g_3)_{ij} + \partial_j(g_3)_{ki} = 0.$$

It implies that g_3 is a Monge metric: a quadratic form in Plücker's coordinates

$$g_3 = X^T Q X = f_{\lambda\mu,\rho\sigma} p^{\lambda\mu} p^{\rho\sigma}.$$

Intersecting g_3 with the Grassmannian

$$\mathbb{G}(2, \mathbb{C}^{n+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{n+1})$$

we obtain, in the generic case, a quadratic line complex.

More algebraic varieties in A_3

It was proved (Doyle 1992; Potemin & Balandin 1992) that the Monge metric of a third-order homogeneous Hamiltonian operator can be factorized as:

$$g_3 = \varphi_{\alpha\beta}\psi_i^\alpha\psi_j^\beta du^i \otimes du^j,$$
$$\psi_i^\alpha = \psi_{ij}^\alpha u^j + \omega_i^\alpha, \qquad \psi_{ij}^\alpha = -\psi_{ji}^\alpha,$$

where $\alpha, i, j = 1, \ldots, n$. Let us write

$$\psi_i^\alpha du^i = \frac{1}{2}\psi_{ij}^\alpha(u^i du^j - u^j du^i) + \frac{1}{2}\psi_{n+1\,j}^\alpha(u^{n+1}du^i - u^i du^{n+1}).$$

We can interpret

$$\psi^\alpha = \psi_{ab}^\alpha e^a \wedge e^b = \psi_{ab}^\alpha p^{ab}, \qquad a, b = 1, \ldots, n+1,$$

as n linear expressions in the Plücker embedding of $\mathbb{P}^n$: a linear line congruence.

The role of the metric φ

A third-order operator of the form of A_3 is completely determined by a linear line complex, which is the datum of an n-dimensional space of 2-forms

$$A = \mathrm{Span}(\{A^\alpha \mid \alpha = 1, \ldots, n\}) \subset \wedge^2 \mathbb{R}^{n+1}$$

endowed with a scalar product φ which lies in the kernel of the canonical map

$$C\colon S^2(\wedge^2 \mathbb{R}^{n+1}) \to \wedge^4 \mathbb{R}^{n+1}, \quad X \otimes Y \mapsto X \wedge Y$$

restricted to A. In coordinates, if $\varphi = \varphi^{ab,cd} e_a \wedge e_b \otimes e_c \wedge e_d$ we have

$$C(\varphi)_{abcd} = \varphi^{ab,cd} - \varphi^{ac,bd} + \varphi^{ad,bc}$$

NOTE. If η metric on $\mathbb{R}^N$, the natural metric η^0_2 on $\wedge^2 \mathbb{R}^N$ fulfills the equation!

The Hamiltonian equations for φ and ψ

For A_3 to be a Hamiltonian operator it is enough to require that φ is a metric on the n-dimensional space

$$A = \mathrm{Span}(\{A^\alpha = \psi^\alpha_{ab} e^a \wedge e^b \mid \alpha = 1, \ldots, n\}) \subset \wedge^2 \mathbb{R}^{n+1}$$

where a, $b = 1, \ldots, n+1$, which lies in the kernel of C restricted to $S^2(A)$. That amounts to the system of equations:

$$\varphi_{\beta\gamma}(\psi^\beta_{is}\psi^\gamma_{jk} + \psi^\beta_{js}\psi^\gamma_{ki} + \psi^\beta_{ks}\psi^\gamma_{ij}) = 0,$$
$$\varphi_{\beta\gamma}(\omega^\beta_i\psi^\gamma_{jk} + \omega^\beta_j\psi^\gamma_{ki} + \omega^\beta_k\psi^\gamma_{ij}) = 0,$$

where i, $j = 1, \ldots, n$.

WDVV, new results with $N = 4$

How many independent equations are in WDVV system?
If $N = 3$ there is only one equation.

Let $N = 4$, and set $x = t^2$, $y = t^3$, $z = t^4$.

$$\mu f_{yyz}(f_{zzz} - f_{yzz}) + 2f_{yyz}f_{xyz} - f_{yyy}f_{xzz} - f_{xyy}f_{yzz} = 0,$$
$$f_{xxy}f_{yzz} - f_{xxz}f_{yyz} - \mu f_{zzz}f_{xyz} + f_{zzz} + f_{xyy}f_{xzz} + \mu f_{xzz}f_{yzz} - f_{xyz}^2 = 0,$$
$$f_{xxy}f_{yyz} - f_{xxz}f_{yyy} + \mu f_{yyz}f_{xzz} - \mu f_{xyz}f_{yzz} + f_{yzz} = 0,$$
$$f_{xxy}f_{xzz} - \mu f_{xxz}f_{zzz} - 2f_{xxz}f_{xyz} + f_{xxx}f_{yzz} + \mu f_{xzz}^2 = 0,$$
$$f_{xxz}f_{xyy} + \mu f_{xxz}f_{yzz} - f_{yyz}f_{xxx} - \mu f_{xzz}f_{xyz} + f_{xzz} = 0,$$
$$f_{xxy}f_{xyy} + \mu f_{xxz}f_{yyz} - f_{xxx}f_{yyy} - \mu f_{xyz}^2 + 2f_{xyz} = 0.$$

First ‘generic’ case: WDVV with $N = 4$

Choose an independent variable, say x; it possible to find a subsystem of equations that are linear with respect to x-free derivatives:

$$f_{yyy}, \quad f_{yyz}, \quad f_{yzz}, \quad f_{zzz}.$$

This linear subsystem is overdetermined: it consists of 5 equations. They can be solved for the 4 unknowns f_{yyy}, f_{yyz}, f_{yzz}, f_{zzz}. If we introduce new field variables u^k in correspondence with every x-derivative of the third order, i.e.

$$u^1 = f_{xxx}, \quad u^2 = f_{xxy}, \quad u^3 = f_{xxz},$$
$$u^4 = f_{xyy}, \quad u^5 = f_{xyz}, \quad u^6 = f_{xzz}$$

First ‘generic’ case: WDVV with $N = 4$

The linear overdetermined system can be solved. For example, if $\mu = 0$ we have:

$$f_{yyy} = \frac{2u^5 + u^2u^4}{u^1}, \quad f_{yyz} = \frac{u^3u^4 + u^6}{u^1}, \quad f_{yzz} = \frac{2u^3u^5 - u^2u^6}{u^1},$$

$$f_{zzz} = (u^5)^2 - u^4u^6 + \frac{(u^3)^2u^4 + u^3u^6 - 2u^2u^3u^5 + (u^2)^2u^6}{u^1}.$$

It is remarkable that also the remaining nonlinear equation is solved by the above equations. So, the WDVV equations can be written in orthonomic form.
Moreover, the above system is passive: there are no ‘hidden’ integrability conditions (Opanasenko, V. arXiv 2025).

Reducing the WDVV system

Consider the WDVV system:

$$S_{\alpha\beta\gamma\nu} := \eta^{\mu\lambda}(F_{\lambda\alpha\beta}F_{\mu\nu\gamma} - F_{\lambda\alpha\nu}F_{\mu\beta\gamma}) = 0.$$

Theorem (Opanasenko, V. arXiv 2025). Choose an index p, $2 \leqslant p \leqslant N$. Then the WDVV system can be split into:

- a subsystem that is linear with respect to t^p-free derivatives;
- a subsystem that contains nonlinear equations wrt t^p-free derivatives;

The linear system can be uniquely solved with respect to t^p-free derivatives.

Conjecture. The part with nonlinearities vanishes upon the solution of the linear part; WDVV equations can be put in passive orthonomic form Verified up to $N = 5$.

WDVV as a passive orthonomic system

Conjecture. The WDVV subsystem with (possible) nonlinearities vanishes upon the solution of the linear part; WDVV equations can be put in orthonomic form Verified up to $N = 5$.

Theorem. If the Conjecture is true, then the WDVV system in orthonomic form is passive for arbitrary N.

WDVV as a first-order systems of PDEs: $N = 4$

Mokhov and Ferapontov (1996) introduced new letters for third-order derivatives:

$$u^1 = f_{xxx},\ u^2 = f_{xxy},\ u^3 = f_{xxz},\ u^4 = f_{xyy},\ u^5 = f_{xyz},\ u^6 = f_{xzz},$$
$$u^7 = f_{yyy},\ u^8 = f_{yyz},\ u^9 = f_{yzz},\ u^{10} = f_{zzz}.$$

We have the following compatibility relations:

$$\begin{array}{ccc}
u^1_y = u^2_x & u^1_z = u^3_x & u^2_z = u^3_y \\
u^2_y = u^4_x & u^2_z = u^5_x & u^4_z = u^5_y \\
u^3_y = u^5_x & u^3_z = u^6_x & u^5_z = u^6_y \\
u^4_y = u^7_x & u^4_z = u^8_x & u^7_z = u^8_y \\
u^5_y = u^8_x & u^5_z = u^9_x & u^8_z = u^9_y \\
u^6_y = u^9_x & u^6_z = u^{10}_x & u^9_z = u^{10}_y
\end{array}$$

WDVV as a first-order systems of PDEs: $N = 4$

If we express the coordinates $u^7 = f_{yyy}$, $u^8 = f_{yyz}$, $u^9 = f_{yzz}$, $u^{10} = f_{zzz}$ by means of (u^k), $k = 1, \ldots, 6$ using *all* WDVV equations, we have two commuting quasilinear systems of first-order PDEs and a third set of trivial identities:

$$\begin{cases} u^1_y = u^2_x, \\ u^2_y = u^4_x, \\ u^3_y = u^5_x, \\ u^4_y = \left(\frac{2u^5 + u^2u^4}{u^1}\right)_x \\ u^5_y = \left(\frac{u^3u^4 + u^6}{u^1}\right)_x \\ u^6_y = \left(\frac{2u^3u^5 - u^2u^6}{u^1}\right)_x \end{cases}
\qquad
\begin{cases} u^1_z = u^3_x, \\ u^2_z = u^5_x, \\ u^3_z = u^6_x, \\ u^4_z = \left(\frac{u^3u^4 + u^6}{u^1}\right)_x \\ u^5_z = \left(\frac{2u^3u^5 - u^2u^6}{u^1}\right)_x \\ u^6_z = \Big((u^5)^2 - u^4u^6 + \\ \qquad \frac{(u^3)^2u^4 + u^3u^6 - 2u^2u^3u^5 + (u^2)^2u^6}{u^1}\Big)_x \end{cases}$$

WDVV as a first-order systems of PDEs: $N = 4$

What about the residual compatibility conditions? The system

$$\begin{aligned} u^2_z &= u^3_y & \qquad u^4_z &= u^5_y \\ u^5_z &= u^6_y & \qquad u^7_z &= u^8_y \\ u^8_z &= u^9_y & \qquad u^9_z &= u^{10}_y \end{aligned}$$

is identically verified when you restrict it to the two commuting systems on the previous slide, and is equivalent to the fact that the WDVV equations in orthonomic form are passive.

Corollary to the Conjecture. If the Conjecture is true, then WDVV equations can be rewritten as a set of $N - 2$ quasilinear systems of first-order PDEs. The systems are commuting.

NOTE: It is unknown (although very likely) if solutions of WDVV equations are in 'bijection' with 'joint' solutions of the commuting WDVV systems.

WDVV equations as algebraic varieties: preliminaries

Following a construction of Agafonov and Ferapontov (1996-2001) we associate to each system of conservation laws

$$u^i_t = (V^i)_x$$

a congruence of lines in $\mathbb{P}^{n+1}$ with coordinates $[y^1, \ldots, y^{n+2}]$

$$y^i = u^i y^{n+1} + V^i y^{n+2}$$

In other words, the congruence is the family of lines through the points

$$\begin{pmatrix} u^1 & \cdots & u^n & 1 & 0 \\ V^1 & \cdots & V^n & 0 & 1 \end{pmatrix}$$

whose Plücker coordinates are

$$u^i V^j - u^j V^i, \qquad u^i, \qquad V^i, \qquad 1.$$

WDVV equations as algebraic varieties: systems with third-order Hamiltonian structure

Theorem (Ferapontov, Pavlov, V. LMP 2017) The system of conservation laws $u^i_t = (V^i)_x$ is Hamiltonian with respect to A_3 (recall that A_3 is determined by $g_3 = \varphi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$ and $\psi^\alpha_i = \psi^\alpha_{ij}u^j + \omega^\alpha_i$) iff

$$\boxed{V^i = \psi^i_\alpha Z^\alpha = \psi^i_\alpha(\eta^\alpha_i u^i + \xi^\alpha)}$$

$$\varphi_{\beta\gamma}[\psi^\beta_{ij}\theta^\gamma_k + \psi^\beta_{jk}\theta^\gamma_i + \psi^\beta_{ki}\theta^\gamma_j] = 0,$$
$$\varphi_{\beta\gamma}[\psi^\beta_{ik}\xi^\gamma + \omega^\beta_k\eta^\gamma_i - \omega^\beta_i\eta^\gamma_k] = 0.$$

The associated congruence of lines is linear: it holds

$$\boxed{\frac{1}{2}\psi^\alpha_{ij}(u^jV^i - u^iV^j) + \omega^\alpha_i V^i - \eta^\alpha_i u^i - \xi^\alpha = 0.}$$

The Structure Theorem of WDVV equations

Theorem (Opanasenko, V. arXiv 2025) Let $N \geqslant 3$. Given $2 \leqslant \lambda < \mu \leqslant N$, the WDVV equations

$$\{S_{\lambda\beta\mu\nu},\ 1 \leqslant \beta < \nu \leqslant N\}$$

can be written as

$$\frac{1}{2}\psi^{\gamma}_{km}(u^m_\lambda u^k_\mu - u^k_\lambda u^m_\mu) + {}_\lambda\omega^\gamma_k u^k_\mu - {}_\mu\omega^\gamma_m u^m_\lambda - {}_{\lambda\mu}\xi^\gamma = 0,$$

where the equation $S_{\lambda\beta\mu\nu}$ is associated with $\gamma = (\nu, \beta+1)$ and all coefficients are zeroes except:

- for $2 \leqslant \beta < \nu \leqslant N$ and $2 \leqslant a, b \leqslant N$ we set

$$\psi^{(\nu,\beta+1)}_{(\nu,a)(\beta,b)} = \eta^{ab} = -\psi^{(\nu,\beta+1)}_{(\beta,b)(\nu,a)} \qquad \text{if } (\nu, a) \neq (\beta, b);$$

- for $2 \leqslant \beta < \nu < M \leqslant N$ and $\rho \in \{\lambda, \mu\}$ we set

$${}_\rho\omega^{(\nu,\beta+1)}_{(M,\beta)} = -\eta^{M1}\eta_{\rho\nu}, \quad {}_\rho\omega^{(\nu,\beta+1)}_{(M,\nu)} = \eta^{M1}\eta_{\rho\beta};$$

The Structure Theorem, continued

- for $2 \leqslant \beta < \nu \leqslant N$ and $2 \leqslant a < \nu$, with $\rho \in \{\lambda, \mu\}$, we set

$$_{\rho}\omega^{(\nu,\beta+1)}_{(\nu,a)} = \eta^{1a}\eta_{\rho\beta} \text{ with } a \neq \beta,\ \ _{\rho}\omega^{(\nu,\beta+1)}_{(\nu,\beta)} = \eta^{1\beta}\eta_{\rho\beta} - \eta^{1\nu}\eta_{\rho\nu},$$
$$_{\rho}\omega^{(\nu,\beta+1)}_{(\beta,a)} = -\eta^{1a}\eta_{\rho\nu};$$

- for $2 \leqslant \beta < \nu \leqslant N$ we set

$$_{\lambda\mu}\xi^{(\nu,\beta+1)} = -\eta^{11}(\eta_{\lambda\beta}\eta_{\mu\nu} - \eta_{\lambda\nu}\eta_{\mu\beta}); \tag{1}$$

- for $1 < \nu \leqslant N$ and $\beta = 1$ we set

$$_{\lambda}\omega^{(\nu,2)}_{(\nu,\mu)} = 1 \quad \text{and} \quad _{\mu}\omega^{(\nu,2)}_{(\nu,\lambda)} = 1. \tag{2}$$

NOTE: The function of two arguments $(\nu, \beta + 1)$ yields an index ordering, and it is crucial for the induction argument.

WDVV equations as algebraic varieties: $N-2$ linear line congruences

For fixed λ, μ such that $2 \leqslant \lambda < \mu \leqslant N$, the WDVV equations have the form

$$S_{\lambda\beta\mu\nu} = \frac{1}{2}\psi^{\gamma}_{km}(u^m_\lambda u^k_\mu - u^k_\lambda u^m_\mu) + {}_\lambda\omega^\gamma_k u^k_\mu - {}_\mu\omega^\gamma_m u^m_\lambda - {}_{\lambda\mu}\xi^\gamma 1 = 0,$$

where $\gamma = (\nu, \beta + 1)$. These are $N-2$ linear line congruences, one for each choice of λ, μ, each consisting of a system of n linear equations with respect to the Plücker coord. $u^i_\lambda u^j_\mu - u^j_\lambda u^i_\mu$, u^i_λ, u^i_μ, 1.

Is it enough to make them into Hamiltonian systems?

The system $S_{2132} = 0$, $S_{2133} = 0$, $S_{2233} = 0$, $S_{2134} = 0$, $S_{2234} = 0$, $S_{2334} = 0$ ($\lambda = 2$, $\mu = 3$) can be written as

$$\frac{1}{2}\psi^{\gamma}_{km}(u_m v_k - u_k v_m) + {}_2\omega^{\gamma}_{k} v_k - {}_3\omega^{\gamma}_{m} u_m - {}_{23}\xi^{\gamma} = 0,$$

where

$$\Psi^3 = \begin{pmatrix} 0 & -\eta^{22} & -\eta^{23} & 0 & -\eta^{24} & 0 \\ \eta^{22} & 0 & -\eta^{33} & \eta^{24} & -\eta^{34} & 0 \\ \eta^{23} & \eta^{33} & 0 & \eta^{34} & 0 & 0 \\ 0 & -\eta^{24} & -\eta^{34} & 0 & -\eta^{44} & 0 \\ \eta^{24} & \eta^{34} & 0 & \eta^{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \Psi^5 = \begin{pmatrix} 0 & 0 & 0 & -\eta^{22} & -\eta^{23} & -\eta^{24} \\ 0 & 0 & 0 & -\eta^{23} & -\eta^{33} & -\eta^{34} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \eta^{22} & \eta^{23} & 0 & 0 & -\eta^{34} & -\eta^{44} \\ \eta^{23} & \eta^{33} & 0 & \eta^{34} & 0 & 0 \\ \eta^{24} & \eta^{34} & 0 & \eta^{44} & 0 & 0 \end{pmatrix},$$

$$\Psi^6 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\eta^{22} & -\eta^{23} & -\eta^{24} \\ 0 & 0 & 0 & -\eta^{23} & -\eta^{33} & -\eta^{34} \\ 0 & \eta^{22} & \eta^{23} & 0 & \eta^{24} & 0 \\ 0 & \eta^{23} & \eta^{33} & -\eta^{24} & 0 & -\eta^{44} \\ 0 & \eta^{24} & \eta^{34} & 0 & \eta^{44} & 0 \end{pmatrix}, \quad \Psi^1 = \Psi^2 = \Psi^4 = 0,$$

$${}_k\Omega = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -\eta^{12}\eta_{k3} & \eta^{12}\eta_{k2} - \eta^{13}\eta_{k3} & \eta^{13}\eta_{k2} & -\eta^{14}\eta_{k3} & \eta^{14}\eta_{k2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -\eta^{12}\eta_{k4} & -\eta^{13}\eta_{k4} & 0 & \eta^{12}\eta_{k2} - \eta^{14}\eta_{k4} & \eta^{13}\eta_{k2} & \eta^{14}\eta_{k2} \\ 0 & -\eta^{12}\eta_{k4} & -\eta^{13}\eta_{k4} & \eta^{12}\eta_{k3} & \eta^{13}\eta_{k3} - \eta^{14}\eta_{k4} & \eta^{14}\eta_{k3} \end{pmatrix}$$

$${}_{23}\xi = \eta^{11}\left(0, 0, (\eta_{23})^2 - \eta_{22}\eta_{33}, 0, \eta_{23}\eta_{24} - \eta_{22}\eta_{34}, \eta_{24}\eta_{33} - \eta_{23}\eta_{34}\right)^T.$$

The Metric Theorem

Theorem. (Opanasenko, V. arXiv 2025) Up to scaling by a constant multiple, there exists a unique solution $\varphi = \eta_2^0$, with the following ordering:

$$\varphi_{(A,a)(B,b)} = \eta^{AB}\eta^{(a-1)(b-1)} - \eta^{A(b-1)}\eta^{B(a-1)},$$

of the system

$$\mathcal{H}^1_{ijks}\colon \varphi_{\beta\gamma}(\psi^\beta_{is}\psi^\gamma_{jk} + \psi^\beta_{js}\psi^\gamma_{ki} + \psi^\beta_{ks}\psi^\gamma_{ij}) = 0,$$
$$\mathcal{H}^2_{ijk}\colon \varphi_{\beta\gamma}(\omega^\beta_i\psi^\gamma_{jk} + \omega^\beta_j\psi^\gamma_{ki} + \omega^\beta_k\psi^\gamma_{ij}) = 0,$$
$$\mathcal{H}^3_{ijk}\colon \varphi_{\beta\gamma}(\psi^\beta_{ij}\eta^\gamma_k + \psi^\beta_{jk}\eta^\gamma_i + \psi^\beta_{ki}\eta^\gamma_j) = 0,$$
$$\mathcal{H}^4_{ik}\colon \varphi_{\beta\gamma}(\psi^\beta_{ik}\xi^\gamma + \omega^\beta_k\eta^\gamma_i - \omega^\beta_i\eta^\gamma_k) = 0.$$

Order in which the equations $\mathcal{H}$ are solved

$$\varphi = \begin{pmatrix} \eta^{11}\eta^{22} - (\eta^{12})^2 & \eta^{11}\eta^{23} - \eta^{12}\eta^{13} & \eta^{12}\eta^{23} - \eta^{13}\eta^{22} & \varphi_{14} & \varphi_{15} & \varphi_{16} \\ \eta^{11}\eta^{23} - \eta^{12}\eta^{13} & \eta^{11}\eta^{33} - (\eta^{13})^2 & \eta^{12}\eta^{33} - \eta^{13}\eta^{23} & \varphi_{24} & \varphi_{25} & \varphi_{26} \\ \eta^{12}\eta^{23} - \eta^{13}\eta^{22} & \eta^{12}\eta^{33} - \eta^{13}\eta^{23} & \eta^{22}\eta^{33} - (\eta^{23})^2 & \varphi_{34} & \varphi_{35} & \varphi_{36} \\ \varphi_{14} & \varphi_{24} & \varphi_{34} & \varphi_{44} & \varphi_{45} & \varphi_{46} \\ \varphi_{15} & \varphi_{25} & \varphi_{35} & \varphi_{45} & \varphi_{55} & \varphi_{56} \\ \varphi_{16} & \varphi_{26} & \varphi_{36} & \varphi_{46} & \varphi_{56} & \varphi_{66} \end{pmatrix}$$

The Hamiltonian Theorem

Theorem. (Opanasenko, V. arXiv 2025) Let $N \geqslant 3$, and choose an independent variable t^p, $2 \leqslant p \leqslant N$. Then, we obtain $N-2$ Hamiltonian systems of first-order conservation laws, indexed by μ such that $2 \leqslant \mu \leqslant N$ and $\mu \neq p$:

$$(u^I_p)_\mu = (V^I_\mu)_p, \qquad I = 1, \ldots, n, \tag{3}$$

where $V^I_\mu = V^I_\mu(u^J_\lambda)$.

Each of the above systems is Hamiltonian with respect to one and the same third-order Hamiltonian operator A_3

$$A_3^{ij} = \partial_x(g_3^{ij}\partial_x + c^{ij}_k(u^k_p)_x)\partial_x$$

$g_{3\,ij} = \varphi_{\alpha\beta}\psi^\alpha_i\psi^\beta_j$, where ψ^α_i is defined in the Structure Theorem and $\varphi_{\alpha\beta}$ is defined in the Metric Theorem.

Bi-Hamiltonian conjecture

Many of the above first-order systems of conservation laws are endowed with a Ferapontov operator A_1:

$$A_1^{ij} = g_1^{ij}\partial_x + \Gamma^{ij}_k u^k_x + c^{\alpha\beta}(w^i_\alpha)_x \circ \partial_x^{-1} \circ (w^j_\beta)_x,$$

A_1 is compatible with A_3: $A_1 + \lambda A_3$ is Hamiltonian for every $\lambda \in \mathbb{R}$.

Recent results by Lorenzoni, Opanasenko, V. (soon in arXiv) show that compatibility implies that

$$g_1^{ij} = \psi^i_\alpha Z^{\alpha j} + \psi^j_\alpha Z^{\alpha i} - c^{\alpha\beta} w^i_\alpha w^j_\beta$$

where $\psi^i_\alpha Z^{\alpha j}$ are a family (parametrized by j) of fluxes which are Hamiltonian with respect to A_3.

Conjecture. Every WDVV system is bi-Hamiltonian by a pair as above.

Invariance of bi-Hamiltonian structures for WDVV systems

Theorem An invariance transformation of the WDVV equation preserves the form of the Hamiltonian operators in a bi-Hamiltonian first-order WDVV system.

Proof The symmetry group of a third-order WDVV projects to the symmetry group $\mathrm{GL}(N-1,\mathbb{C})$ of a first-order WDVV.

Invariance transformations that involve only two independent variables preserve the form of the Hamiltonian operators [Vašiček, V., 2021].

Any matrix in $\mathrm{GL}(\mathbb{C}^{N-1})$ can be generated by means of 2×2 Gauss' elementary matrices (up to permutations).

Symbolic computations

Within the REDUCE CAS (now free software) we use the packages CDIFF and CDE, freely available at `https://reduce-algebra.sourceforge.io/`.

CDE (by RV) can compute symmetries and conservation laws, local and nonlocal Hamiltonian operators, Schouten brackets of local multivectors, Fréchet derivatives (or linearization of a system of PDEs), formal adjoints, Lie derivatives of Hamiltonian operators, anticommuting variables and super-PDEs.

Cooperation with AC Norman (Trinity College, Cambridge) to improvements and documentation of REDUCE's kernel.

A book, in cooperation with JS Krasil'shchik and AM Verbovetsky: *The symbolic computation of integrability structures for partial differential equations*, is published in the series Texts and Monographs in Symbolic Computation, Springer, 2018.

Thank you!

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